Numerical Solution of Riemann-Hilbert Problems
Applications to Random Matrices and Painlevé Equations

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Outline

2. Riemann-Hilbert Problems
   - Transformations
   - Singular Integral Equations
3. Discretization of Singular Integral Equations
4. Numerical Solution of the Sine-Kernel RHP
   - Elementary Approaches
   - Baker-Akhiezer Function: Small-\(x\) Solution.
   - Parametrix Analysis: Large-\(x\) Solution
5. Closing Thoughts
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5. Closing Thoughts
Consider the sine kernel $K_x$ on $L^2([-1, 1])$ and the Fredholm determinant:

$$K_x(t, s) := \frac{\sin(x(s - t))}{\pi (s - t)}$$

$$P(x) := \det(I - K_x)$$
The Sine Kernel Determinant

Consider the sine kernel $K_x$ on $L^2([-1, 1])$ and the Fredholm determinant:

$$K_x(t, s) := \frac{\sin(x(s - t))}{\pi(s - t)}$$

$$P(x) := \det(I - K_x)$$

**Theorem (Gaudin and Metha, 1960’s)**

Let $U$ be a Hermitian $N \times N$ matrix drawn from the Gaussian unitary ensemble. Under suitable scaling as $N \to \infty$, the probability of finding no eigenvalues in $\frac{1}{\pi}[-x, x]$ is given by $P(x)$. 
The many assaults on $P(x)$ asymptotia.

Large-$x$ asymptotic expansion.

$$\ln(P(x)) \sim -\frac{x^2}{2} - \frac{1}{4} \log x + B + \frac{a_1}{x} \ldots$$

Selected highlights of the diverse history of $P(x)$ asymptotics:

- **Gaudin (1961)**
  Connections between $K_x$ and spheroidal wavefunctions.

- **Widom (1971)**
  Analysis of $P(x)$ as a non-smooth Toeplitz determinant.

- **Dyson (1976)**
  Recognize $P(x)$ as a problem in inverse scattering.
What is a Riemann-Hilbert Problem?

Let $\Sigma \in \mathbb{C}$ be an oriented contour and $v(t)$ a matrix-valued function on $\Sigma$.

**Definition (RHP)**

A **Riemann-Hilbert Problem** is the problem of determining existence and uniqueness of a sectionally analytic matrix function defined by the data $(\Sigma, v)$

- $m(z)$ is analytic in $\mathbb{C} \setminus \Sigma$
- $m_+(t) = m_-(t)v(t)$, for $t \in \Sigma$
- $m(z) \to \mathbf{I} + O\left(\frac{1}{z}\right)$ as $z \to \infty$.

The Riemann-Hilbert Problem Related to $P(x)$

Define

$$v_m(t, x) := \begin{pmatrix} 0 & e^{2itx} \\ -e^{-2itx} & 2 \end{pmatrix}$$

Theorem (Deift, Its, Zhou (1997) [DIZ])

For $x$ fixed, the RHP $([-1, 1], v_m)$ exists and is unique. Furthermore

$$\frac{d}{dx} \ln(\det(I - K_x)) = i \left( (m_1(x))_{22} - (m_1(x))_{11} \right),$$

where

$$m(z; x) = I + \frac{m_1(x)}{z} + O\left(\frac{1}{z^2}\right), \text{ as } z \to \infty.$$
The Painlevé ODE Related to $P(x)$

Painlevé V is a three-parameter family (rescale $\delta$)

$$-y'' = \left(\frac{3y - 1}{2y(y - 1)}\right)(y')^2 - \frac{1}{x}y' + \frac{(y - 1)^2}{x^2} \left(\alpha y + \frac{\beta}{y}\right) + \frac{\gamma y}{x} + \frac{\delta y(y + 1)}{y - 1}$$

Alternate forms exist.

Theorem (Jimbo, Miwa, Mori, and Sato (1980))

Let $K_x(s, t)$ be the sine kernel acting on $L^2(-1, 1)$ and define $\theta(x)$

$$\theta(x) := x \frac{d}{dx} \ln \det \left( I - K_x^x \right)$$

Then $\theta$ satisfies the Hirota form of a particular Painlevé V equation

$$(x\theta'')^2 = -4 \left( \theta - x\theta' - (\theta')^2 \right) (\theta - x\theta')$$
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Riemann-Hilbert Transformations

- **Factorization.** \( v = (v_-)^{-1}v_+ \)
- **Conjugation.**

\[
\begin{align*}
m & \text{ satisfies } (v, \Sigma) \\
\tilde{m} & \text{ satisfies } (\tilde{v}, \tilde{\Sigma})
\end{align*}
\]

\[
\Rightarrow R := m\tilde{m}^{-1} \text{ satisfies } (v_R, \Sigma_R)
\]

where \( \Sigma_R \subset \Sigma \cup \tilde{\Sigma} \) and

\[
v_R := (R_-)^{-1}R_+ = \tilde{m}_-v\tilde{v}^{-1}\tilde{m}_-^{-1}.
\]

If \( v(t) = \tilde{v}(t) \), then \( R(t) \) is analytic. (Removable singularities.)

- **Analytic prefactors.** \( \tilde{m}(z) := A(z)m(z) \), then \( \tilde{v} = v \).
Classical Integral Transforms

For an oriented $\Sigma$, define the Hilbert, Cauchy, and limit transforms by

$$H[f](t) := \frac{1}{\pi i} \int_{\Sigma} \frac{f(s)}{s-t} ds, \quad t \in \Sigma$$

$$C[f](z) := \frac{1}{2\pi i} \int_{\Sigma} \frac{f(s)}{s-z} ds, \quad z \in \mathbb{C} \setminus \Sigma$$

$$C_{\pm}[f](t) := \lim_{z \to \pm t} C[f](z)$$

Theorem (Plemelj)

*On appropriate function spaces one has the following operator identities*

$$C_+ + C_- = H$$

$$C_+ - C_- = I$$
Corollary

Let $M$ be a solution to the RHP $(\nu, \Sigma, I)$ and consider a factorization $\nu = (\nu_-)^{-1}\nu_+$. Then $M$ can be represented via

$$M(z) = I + C\left[\mu(\nu_+ - \nu_-)\right], \quad z \in \mathbb{C} \setminus \Sigma,$$

where $\mu$ satisfies the following system of singular integral equations

$$\mu(\nu_+ + \nu_- - 2) = I, \quad t \in \Sigma.$$
Corollary

Let $M$ be a solution to the RHP $(\nu, \Sigma, I)$ and consider a factorization $\nu = (\nu_-)^{-1} \nu_+$. Then $M$ can be represented via

$$M(z) = I + C[\mu(\nu_+ - \nu_-)], \quad z \in \mathbb{C} \setminus \Sigma,$$

where $\mu$ satisfies the following system of singular integral equations

$$\mu \left( \frac{\nu_+ + \nu_-}{2} \right)(t) - H \left[ \mu \left( \frac{\nu_+ - \nu_-}{2} \right) \right](t) = I, \quad t \in \Sigma.$$
Corollary

Let $M$ be a solution to the RHP $(v, \Sigma, I)$ and consider a factorization $v = (v_-)^{-1}v_+$. Then $M$ can be represented via

$$M(z) = I + C[\mu(v_+ - v_-)], \quad z \in \mathbb{C} \setminus \Sigma,$$

where $\mu$ satisfies the following system of singular integral equations

$$\mu \left( \frac{v_+ + v_-}{2} \right)(t) - H \left[ \mu \left( \frac{v_+ - v_-}{2} \right) \right](t) = I, \quad t \in \Sigma.$$

Conversely, any solution of the above generates a solution of the RHP via the Cauchy transform.
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5. Closing Thoughts
Singular Integral Equations and Singularities

- $aI - bH$ is not analyzable as $I - K$, with $K$ compact.
- $H_{S_1}$ is relatively benign. (Can be computed by FFT!)
- $\Sigma$ a closed curve $\Longrightarrow H_{\Sigma} = H_{S_1} + K$, with $K$ compact.
Discretization of Singular Integral Equations

Singular Integral Equations and Singularities

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- $H_{S_1}$ is relatively benign. (Can be computed by FFT!)
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- Σ an interval, then expect singularities.

$$\int_{-1}^{+1} \frac{1}{\sqrt{1-s^2}} \frac{1}{s-t} \, ds = 0$$

Infinities balance under the singular integral.
Singular Integral Equations and Singularities

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Infinities balance under the singular integral.

- Constant coefficient problem with polynomial right-hand side

\[
\left( aI - \frac{b}{\pi i} \int_{-1}^{1} ds \right) f = (-1)^{\nu} 2^{\nu} P_{n+\nu}^{(\alpha, \beta)}(t), \text{ is solved by} \ldots
\]
Singular Integral Equations and Singularities

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\[
\left( aI - \frac{b}{\pi i} \int_{-1}^{1} \right) f = (-1)^\nu 2^\nu P_{n+\nu}^{(\alpha, \beta)}(t), \text{ is solved by...}
\]

\[
f(t) = (1 + t)^{-\alpha} (1 - t)^{-\beta} P_n^{(-\alpha, -\beta)}(t) + C_{\nu}
\]
Discretization of Singular Integral Equations: Closed Curve

Alternating point quadrature.

Given equispaced \( t_n = nh \) for \( n = 0, \ldots, 2N - 1 \), approximate

\[
\int f(s) \frac{1}{s - t_{2k}} \, ds \sim 2h \sum_{n=0}^{N-1} \frac{1}{t_{2n+1} - t_{2k}} f_{2n+1},
\]

and similarly for \( t = t_{2k+1} \).

**Theorem (Sidi and Israeli, 1988)**

Assume \( K(t, s) \) is periodic and smooth in \( t, s \) with exception of diagonal singularities of the form

\[
K(t, s) = W_1(t, s) \ln |\phi_1(s) - \phi_1(t)| + W_2(t, s) \frac{1}{\phi_2(s) - \phi_2(t)} + W_3(t, s).
\]

Alternating point quadrature converges with exponential accuracy.
Discretization of Singular Integral Equations: Interval

Basic Quadrature on \([-1, 1]\)

Subtraction

\[
\int_{-1}^{1} \frac{\omega(s)f(s)}{s-t_j} \, ds = \int_{-1}^{1} \frac{\omega f(s) - \omega f(t_j)}{s-t_j} \, ds + \omega f(t_j) \ln \left| \frac{1-t_j}{1+t_j} \right|
\]

\[
\approx \sum_{k \neq j} \frac{\omega f(t_k) - \omega f(t_j)}{t_k-t_j} \mu_k + \omega f(t_j) \ln \left| \frac{1-t_j}{1+t_j} \right| + \frac{d}{ds} (\omega(s)f(s)) \bigg|_{s=t_j} \mu_j
\]

\[
= \sum_{k \neq j} \frac{\omega f(t_k)}{t_k-t_j} \mu_k + \omega f(t_j) \left( \sum_{k \neq j} \frac{1}{t_k-t_j} \mu_k + \ln \left| \frac{1-t_j}{1+t_j} \right| \right) + \omega(t_j) \mu_j f'(t_j) + \omega'(t_j) \mu_j f(t_j)
\]
Discretization of Singular Integral Equations: Interval

Basic Quadrature on $[-1, 1]$

Differentiation

\[
\chi_N(t) := (t - t_1)(t - t_2) \ldots (t - t_N)
\]

\[
f(t) \approx \sum_{l=1}^{N} \frac{\chi_N(t)}{(t - s_l)\chi'_N(s_l)} f(s_l)
\]

\[
f'(t) \approx \sum_{l=1}^{N} \frac{\chi'_N(t)(t - s_l) - \chi_N(t)}{(t - s_l)^2\chi'_N(s_l)} f(s_l)
\]

Assume $(t_k, \mu_k)$ is $N$-point classical Gauss-Legendre

\[
\int_{-1}^{1} f(s) ds = \sum_{k=1}^{N} f(s_k)\mu_k, \quad f \in \mathcal{P}_{2N-1}.
\]

$\chi'_N(t_j)$ can be evaluated by recurrence relations.
Discretization of Singular Integral Equations: Interval

Basic Quadrature on $[-1, 1]$

Discretization of $H$ on $[-1, 1]$

\[
\frac{1}{\pi i} \int_{-1}^{1} \frac{\omega(s)f(s)}{s - t_j} \, ds \approx H^{(N)}_\omega f := \left[ \tilde{H}^{(N)} + \Lambda_\mu D + \Lambda_\mu \omega' \right] f
\]
Discretization of Singular Integral Equations: Interval

Basic Quadrature on $[-1, 1]$

Discretization of $H$ on $[-1, 1]$

$$\frac{1}{\pi i} \int_{-1}^{1} \frac{\omega(s)f(s)}{s-t_j} \, ds \approx H_{\omega}^{(N)}f := \left[ \tilde{H}^{(N)} + \Lambda_{\mu} D + \Lambda_{\mu\omega'} \right] f$$

where $\tilde{H}^{(N)}$ is the subtraction approximation,

$$\left( \tilde{H}^{(N)} \right)_{jk} = \begin{cases} \frac{\omega(t_k)}{\pi i (t_k - t_j)} \mu_k & j \neq k \\ \frac{\omega(t_j)}{\pi i} \left( - \sum_{k \neq j} \frac{1}{t_k - t_j} \mu_k + \ln \left| \frac{1 - t_j}{1 + t_j} \right| \right) & j = k \end{cases}$$

and...
Discretization of Singular Integral Equations: Interval

Basic Quadrature on \([-1, 1]\)

Discretization of \(H\) on \([-1, 1]\)

\[
\frac{1}{\pi i} \int_{-1}^{1} \frac{\omega(s)f(s)}{s-t_j} \, ds \approx H^{(N)}_\omega f := \left[ \tilde{H}^{(N)} + \Lambda_\mu D + \Lambda_\mu \omega' \right] f
\]

\(D\) is differentiation of the Lagrange interpolation,

\[
(D)_{jk} = \begin{cases} 
\frac{(1-t_k^2)P_{N-1}(t_j)}{(t_j-t_k)(1-t_j^2)P_{N-1}(t_k)} & j \neq k \\
\frac{t_j^2}{(1-t_j^2)} & j = k
\end{cases}
\]

and...
Discretization of Singular Integral Equations: Interval
Basic Quadrature on $[-1, 1]$

Discretization of $H$ on $[-1, 1]$

$$\frac{1}{\pi i} \int_{-1}^{1} \frac{\omega(s)f(s)}{s - t_j} \, ds \approx H^{(N)}_\omega f := \left[ \tilde{H}^{(N)} + \Lambda_\mu D + \Lambda_\mu \omega' \right] f$$

$\Lambda_*$ are diagonal matrices, e.g.

$$\Lambda_\mu = \text{diag}(\mu_1, \ldots, \mu_N).$$
Endpoint singularities are to be expected.
Dyadic refinement near $\pm 1$.
On diagonal blocks, shift and scale the rule described above.
On off-diagonal blocks discretize as if “smooth”.  

\[
\begin{align*}
\tau_{K-5} & \quad \tau_{K-4} & \quad \tau_{K-3} & \quad \tau_{K-2} \\
\vdots & \quad \vdots & \quad \vdots & \quad \vdots \\
t_n & \quad t_{n+1} & \quad \bullet & \quad +1
\end{align*}
\]
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When have we solved a numerical problem?

Criteria for claiming victory (in order):

1. Discretization must be well-conditioned.
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   Discretization with respect to problem parameters.
   Error with respect to discretization.
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1. Discretization must be well-conditioned.
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Take this with a grain of salt...
Basic Formulation

- Solve \([-1, 1], \nu_m(t; x)\) where
  \[
  \nu_m(t; x) = e^{itx\sigma_3} v_0 e^{-itx\sigma_3}
  \]
  \[
  v_0 = \begin{pmatrix}
  0 & 1 \\
  -1 & 2
  \end{pmatrix}
  \]
  \[
  \sigma_3 = \begin{pmatrix}
  1 & 0 \\
  0 & -1
  \end{pmatrix}
  \]

- Experiment with rotating a fixed factorization

  \[
  \nu_-(t; x) = \text{Rot}(\theta) v_0^{-\frac{1}{2}} e^{-itx\sigma_3}
  \]
  \[
  \nu_+(t; x) = \text{Rot}(\theta) v_0^{\frac{1}{2}} e^{-itx\sigma_3}
  \]

where \(\text{Rot}(\theta)\) is 2-by-2 rotation matrix.
Convergence with respect to discretization.

\[ \text{Exponential Convergence for } x=2 \]

\[ \text{Error} \]

\[ \text{Discretization, N} \]

- \( \text{err}_N := \left| \frac{d}{dx} \ln \left( P^{(N)}(2) \right) - \frac{d}{dx} \ln \left( P^{\infty}(2) \right) \right| \).

- Dyadic refinement near endpoints can be effective.
Conditioning is a horror!

- Did not change substantially with $\theta$. 
Did not change substantially with $\theta$.

Find another approach.
Riemann-Surface Analysis

For $\Sigma = [−1, 1]$ the surface is genus 0.

- Define $g(z) = \sqrt{z - 1} \sqrt{z + 1}$.

\[
g_+(t) + g_-(t) = 0 \quad t \in \Sigma
\]
\[
g_+(t) - g_-(t) = i\sqrt{1 - t^2} \quad t \in \Sigma
\]
\[
g(z) - z \approx O\left(\frac{1}{z}\right) \quad z \to \infty
\]

- Define $f(z; x) := f_0(z)e^{ix(z-g(z))}\sigma_3$.

- Compute jump matrix

\[
v_f := (f)^{-1}_-(f)_+ = \begin{pmatrix} 0 & 1 \\ -1 & 2e^{-2x}\sqrt{1-t^2} \end{pmatrix}.
\]

Oscillatory dependence on $x$ is now exponentially damped.
Small-$x$ vs. Large-$x$

- Two constant matrix limits. For $-1 < t < 1$,

\[
\begin{pmatrix}
0 & 1 \\
-1 & 2
\end{pmatrix} \rightarrow v_f(t; x) = \begin{pmatrix}
0 & 1 \\
-1 & 2e^{-2x\sqrt{1-t^2}}
\end{pmatrix} \rightarrow v_\infty := \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\]

$f_0(z)$ and $f_\infty(z)$ can be found explicitly. (Diagonalization.)

- Different singularities near $t = \pm 1$.

\[
f_0(t) \sim \ln(t \pm 1)
\]
\[
f_\infty(t) \sim (t \pm 1)^{\frac{1}{4}}
\]

- Logarithmic singularity wins.

\[
f(z; x) = A(z; x)f_0(z)e^{-ixg(z)}\sigma_3
\]

where $A(z; x)$ is entire in $z$. 

Conjugation and Small-\(x\) Residual Problem

- Build in the logarithmic singularity for \(|x| \text{ "small"}

\[ R_0(z; x) := f(z; x) f_0(z)^{-1}. \]

- “Discontinuity” as \(t \to \pm 1\) gone.
Condition number for $R_0$

Condition Number for $R_0$ is Controlled

Conditioning as a function of $x$ is much improved.
Define \( f_p(z; x) \) sectionally: 
\[
D^- \cup D^+ \cup \left( C \setminus \left( \overline{D^-} \cup \overline{D^+} \cup [-1, 1] \right) \right)
\]

- \( f_p(z; x) \) will match \( f(z; x) \) uniformly up to \( O \left( \frac{1}{x} \right) \)
- Solve a small residual problem.

\[ \sigma_1 H_p(-z; x) \sigma_1 \]

\[ f_\infty(z) \]

\[ D^- \quad \text{and} \quad D^+ \]

\[ H_p(z; x) \]

\[ \pm 1 \]
Parametrix Analysis

Fasten your seat belts...

- Given $0 < r < 1$, let $D_r^\pm := \{ z \mid |z \mp 1| < r \}$.
- For $z \in D_r^+$ define

$$\beta(z) := \left( \frac{z + 1}{z - 1} \right)^{\frac{1}{4}}$$
Parametrix Analysis

Fasten your seat belts...

- Given $0 < r < 1$, let $D_r^{±1} := \{ z \mid |z ± 1| < r \}$.
- For $z \in D_r^+$ define

$$
\beta(z) := \left( \frac{z + 1}{z - 1} \right)^{1/4}
$$

$$
\check{H}_{\nu}^{(1,2)}(z) := \frac{1}{\sqrt{\frac{2}{\pi z}}} e^{±i\left( z - \left( \frac{\nu}{2} + \frac{1}{4} \right) \frac{\pi}{4} \right)} H_{\nu}^{(1,2)}(z)
$$
Parametrix Analysis

Fasten your seat belts...

- Given $0 < r < 1$, let $D_r^{\pm 1} := \{ z | |z \mp 1| < r \}$.
- For $z \in D_r^+$ define
  \[
  \beta(z) := \left( \frac{z + 1}{z - 1} \right)^{\frac{1}{4}}
  \]
  \[
  \tilde{H}^{(1,2)}_{\nu}(z) := \frac{1}{\sqrt{2 \pi z}} e^{\mp i(z - \left( \frac{\nu}{2} + \frac{1}{4} \right) \pi)} H^{(1,2)}_{\nu}(z)
  \]
  \[
  H_p(z; x) := \frac{1}{2} \begin{pmatrix}
  \beta(z) \tilde{H}_0^{(1)}(xg(z)) + \frac{1}{\beta(z)} \tilde{H}_1^{(1)}(xg(z)) & i \left( \beta(z) \tilde{H}_0^{(2)}(xg(z)) - \frac{1}{\beta(z)} \tilde{H}_1^{(2)}(xg(z)) \right) \\
  -i \left( \beta(z) \tilde{H}_0^{(1)}(xg(z)) - \frac{1}{\beta(z)} \tilde{H}_1^{(1)}(xg(z)) \right) & \beta(z) \tilde{H}_0^{(2)}(xg(z)) + \frac{1}{\beta(z)} \tilde{H}_1^{(2)}(xg(z))
  \end{pmatrix}
  \]

- Reflect for $z \in D_r^-$
- Two amazing facts
  \[
  v_p(t; x) = v_f(t; x), \quad t \in (-1, -1 + r) \cup (1 - r, 1)
  \]
  \[
  H_p(t; x) \sim f_\infty(t) + O \left( \frac{1}{x} \right), \quad t \in \partial D_r^+ \text{ as } x \to \infty
  \]

Reflect for $z \in D_r^-$

Two amazing facts

\[
  v_p(t; x) = v_f(t; x), \quad t \in (-1, -1 + r) \cup (1 - r, 1)
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  H_p(t; x) \sim f_\infty(t) + O \left( \frac{1}{x} \right), \quad t \in \partial D_r^+ \text{ as } x \to \infty
  \]
Jump Matrices for $f_p(z; x)$ and $f(z; x)$

\[ \Sigma_{f_p} \]

\begin{align*}
&\begin{array}{c}
\partial D^- \\
-1 \\
+ \\
\end{array} & & & &
\begin{array}{c}
\partial D^+ \\
+ \\
+ \\
\end{array}
\end{align*}

<table>
<thead>
<tr>
<th>$t$</th>
<th>$v_{f_p}$</th>
<th>$v_f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\partial D^+$</td>
<td>$v_p(t; x)$</td>
<td>$1$</td>
</tr>
<tr>
<td>$(1 - r, 1)$</td>
<td>$v_f(t; x)$</td>
<td>$v_f(t; x)$</td>
</tr>
<tr>
<td>$(-1 + r, 1 - r)$</td>
<td>$v_\infty$</td>
<td>$v_f(t; x)$</td>
</tr>
<tr>
<td>$(-1, -1 + r)$</td>
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</tr>
<tr>
<td>$\partial D^-$</td>
<td>$v_p(t; x)$</td>
<td>$1$</td>
</tr>
</tbody>
</table>
Define a residual problem

\[ R_p(z; x) := f(z; x)f_p(z; x)^{-1} \]

Two consequences:

\((-1, -1 + r)\) and \((1 - r, 1)\) conjugate away

Remaining jump matrices are \(I + O\left(\frac{1}{x}\right)\)

Large \(x\) asymptotic expansion follows.
The Triple Points

- \( v_{R_p}^{(1)} \neq v_{R_p}^{(2)} \neq v_{R_p}^{(3)} \)
- \( R_p \) circuit around the \( t = 1 - r \) results in a consistency condition

\[
(v_{R_p}^{(1)})^{-1} v_{R_p}^{(2)} (v_{R_p}^{(3)})^{-1} = I
\]

- Define a factorization on such that \( v_\pm(t; x) \) are continuous!
The Triple Point Factorization

1. Smooth continuation $v_1(t; x) \rightarrow \tilde{v}_1(z; x)$. 
The Triple Point Factorization

1. Smooth continuation $v_1(t; x) \rightarrow \tilde{v}_1(z; x)$.
2. Build continuation into factorization for $v_2(t; x)$. 
The Triple Point Factorization

1. Smooth continuation $\nu_1(t; x) \rightarrow \tilde{\nu}_1(z; x)$.

2. Build continuation into factorization for $\nu_2(t; x)$.

3. Homotope $\tilde{\nu}_1(t; x)$ to $I$ by $\nu_3(t; x)$.
Large-\(x\) Residual Problem with Factorization

Condition Number for \(R_p\) is Decreasing(?)

![Graph showing decreasing condition number for \(R_p\)]
Large-x Residual Problem with Factorization

$\log_{10}(\text{cond}(M_{R_p}))$ for Complex $z$
Outline


2. Riemann-Hilbert Problems
   - Transformations
   - Singular Integral Equations

3. Discretization of Singular Integral Equations

4. Numerical Solution of the Sine-Kernel RHP
   - Elementary Approaches
   - Baker-Akhiezer Function: Small-$x$ Solution.
   - Parametrix Analysis: Large-$x$ Solution

5. Closing Thoughts
Future Work on This Problem

1. Resolve question of condition number for $\Re(z) > 0$.
2. Return to $m_0(z; x) = f_0(z)e^{izx\sigma_3}$ solving $([-1, 1], v_m, e^{izx\sigma_3})$.
   - Write $m(z; x) = \phi(z; x)m_0(z; x)$.
   - Define $\phi(z; x)$ with jump on $r \cdot S_1$ with $r > 1$.
   - The resulting singular integral equations can be written

$$\mu(t; x)a(t; x) - \frac{1}{\pi i} \int_0^{2\pi} \mu(s; x) \cot \left( \frac{s - t}{2} \right) ds = I$$
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     \[
     \mu(t; x)a(t; x) - \frac{1}{\pi i} \int_0^{2\pi} \mu(s; x) \cot \left( \frac{s - t}{2} \right) ds = I
     \]
   - Some sort of electrostatics problem?
Numerical Evaluation of Special Functions

The DLMF is about to have a “birthday party”.

\[ \log_{10}(|\Gamma(z)|) \]
Numerical Evaluation of Special Functions

Real($P_v(z)$) vs. Imag($P_v(z)$)