

How do we check the Painlevé prop. for  $u(x)$ ?

Recall that the solution of a RHP is governed by a sing. int. eqn.  $\mu = (1 - C_w)^{-1} I$ . Here  $C_w$  depends on  $x$  in an analytic fashion and it then follows by analytic Fredholm theory (the same result needed for Weyl's ess. spec. thm!) that the only sing's of  $(1 - C_w)^{-1}$  are poles of finite order that can accumulate only at  $\infty$ .

Thus solutions of PII cannot have movable ess. sing's!

Another example: Boussinesq equation

$$u_{tt} = u_{xxxx} + \text{nonlinear}$$

Linearized eqn.: forward and backwards heat eqn.  
Thus, at the linear level, have a stable- unstable manifold decomp. in phase space.

By RHP, obtain the same result in the fully nonlinear case (Beals-D.-Tomei).

Need RH techniques.

### (c) Asymptotics

From the RHP for NLS we see that soln  $q(x, t)$  of NLS can be written in the form

$$q(x, t) = \mathcal{F}\left(r(\cdot)e^{i\theta(\cdot)}\right), \quad \theta(\cdot) = x(\cdot) - t(\cdot)^2,$$

where  $\mathcal{F}$  is some (nonlinear) functional of  $re^{i\theta}$ . If  $r$  is "small", as  $t \rightarrow \infty$

$$\mathcal{F}\left(re^{i\theta}\right) \sim \text{const.} \int_{\mathbb{R}} r(z)e^{i\theta} dz = \text{const.} \int_{\mathbb{R}} r(z)e^{i(xz-tz^2)} dz$$

and we are dealing with a classical steepest descent problem. What we need is a steep. desc. method for the fully non-linear functional. More later!

## (d) Perturbation theory

In analysing perturbations of NLS

$$iq_t + q_{xx} - 2|q|^2q - \varepsilon W(|q|)q = 0 \quad (7)$$

$$q(x, t = 0) = q_0(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty \quad (8)$$

( $W(|q|) \sim |q|^l$ ,  $l > 2$  as  $|q| \rightarrow \infty$ ) the scattering map  $S : q \mapsto r$  plays the same role as the linear Fourier transform in the analysis of more standard situations, e.g.,  $iq_t + q_{xx} - \varepsilon W(|q|)q = 0$ . To be successful, the estimates one gets for  $S$  must be as fine as one gets for the Fourier transform, and this is precisely what one can obtain via RHP. In particular, get  $L^p$  estimates,  $p > 2$ , as  $t \rightarrow \infty$ , and find that (7) is also integrable!

#### 4. How do RHP's arise?

One systematic way is from integrable eqn's. Here there is a linear operator  $L$ , say, assoc. with the eqn. as in the case of KdV, NLS, Boussinesq, as above. One singles out eigensolutions  $\psi$  (so-called Beals-Coifman solutions) of the operator, e.g. for  $z \in \mathbb{C} \setminus \mathbb{R}$  in case NLS

$$\frac{d\psi}{dx} = i\frac{z}{2}\sigma_3\psi + \begin{pmatrix} 0 & q(x) \\ \bar{q}(x) & 0 \end{pmatrix} \psi$$

$$\psi(x, z)e^{-i\frac{z}{2}\sigma_3x} \rightarrow I \quad \text{as } x \rightarrow \infty$$

$$\psi(x, z)e^{-i\frac{z}{2}\sigma_3x} \text{ is bounded as } x \rightarrow -\infty$$

Then one immediately shows that for fixed  $x$  these  $\psi = \psi(x, z)$  solve a RHP on a contour  $\Sigma$  determined by  $L$ . This is one way RHP's arise.

Sometimes RHP's appear just out of the blue. This was the case for the above RHP's for OP's and Painlevé equations. In this serendipitous world we find:

- (i) random matrix theory: it just so happens that the interesting statistics for the standard ensembles can be expressed in terms of OP's: hence RH applies;
- (ii) Toeplitz, Hankel determinants: classical relations: Toeplitz, Hankel  $\leftrightarrow$  OP's. But combinatorial problems often have solutions in terms of Toeplitz or Hankel det's: hence OP's, hence RH applies. True, in particular, for Ulam's longest increasing subsequence problem (Gessel: Baik-D.-Johansson)

There is, however, another useful systematic in the business: theory of integrable operators.

Special cases: Tracy, McCoy et al. (60s)

Elements of general theory: Sakhnovic (late 60s)

But full general theory of such operators is due to Its, Izergin, Korepin and Slavnov in early 90s.

Let  $\Sigma$  be an oriented contour in  $\mathbb{C}$ . We say that an operator  $K$  is integrable if it has a kernel of the form

$$K(z, z') = \frac{\sum_{j=1}^N f_j(z)g_j(z')}{z - z'}, \quad z, z' \in \Sigma \quad (9)$$

for some  $N$  and  $f_j, g_j \in L^\infty(\Sigma)$ ,

$$(Kf)(z) = \int_{\Sigma} K(z, z')f(z') dz'$$

Operators  $K$  form an algebra, but also have the following remarkable property: if  $K = \frac{\sum f_j g_j}{z - z'}$  is integrable, then so is  $(1 - K)^{-1} - 1$  and if we write

$$(1 - K)^{-1} = I + \frac{\sum_{j=1}^N F_j(z)G_j(z')}{z - z'},$$

then  $F_j, G_k$  can be obtained by solving a (naturally assoc.) RHP on  $\Sigma$ :

Let  $m = m(z)$  solve the normalized RHP  $(\Sigma, v)$  where

$$v = I - 2\pi i f \cdot g^T,$$

$f = (f_1, \dots, f_N)^T, g = (g_1, \dots, g_N)^T$ . Then

$$F = m_{\pm} f \quad \text{and} \quad G = (m_{\pm}^T)^{-1} g$$

Often one is faced with having to compute a determinantal quantity  $\alpha = \det(1 - K)$ , and it turns out that  $K$  is integrable. Then

$$\begin{aligned} \log \alpha &= \log \det(1 - K) = \int_0^1 \frac{d}{dt} \log \det(1 - tK) dt \\ &= \int_0^1 \frac{d}{dt} \operatorname{tr} \log(1 - tK) dt = - \int_0^1 \operatorname{tr} \left( \frac{tK}{1 - tK} \right) \frac{dt}{t} \end{aligned}$$

But  $(1 - tK)^{-1}tK = (1 - tK)^{-1} - I$  is then expressible in terms of a RHP, to which steepest descent method can be applied. In this way one can prove, for example, the celebrated Strong Szegő Limit Thm for Toeplitz determinants, and also evaluate asymp's of Hankel dets.

Here is another example: Consider the spin- $\frac{1}{2}$ XY model in a (critical) magnetic field with Hamiltonian

$$H = -\frac{1}{2} \sum_{l \in \mathbb{Z}} (\sigma_l^x \sigma_{l+1}^x + \sigma_l^z)$$

Then it turns out that the auto-correlation function of the first spin component at inverse temperature  $\beta$

$$\chi(t) = \langle \sigma_0^x(t) \sigma_0^x \rangle_\beta$$

can be written (McCoy, Perk, Shrock) in the form

$$\chi(t) = e^{-t^2/2} \det(1 - K_t)$$

where

$$K_t(z, z') = \varphi(z) \frac{\sin[it(z - z')]}{\pi(z - z')}$$

for  $-1 \leq z, z' \leq 1$ , and  $\varphi(z) = \tanh(\beta\sqrt{1 - z^2})$ . Clearly integrable operator, and following above route, can show (D.-Zhou) that as  $t \rightarrow \infty$

$$\chi(t) = \exp\left\{\frac{t}{\pi} \int_{-1}^1 \log |\tanh \beta s| ds + O(\log t)\right\}$$

## Wiener - Hopf Theory

$$(a) \quad f(x) - \int_0^{\infty} k(x-y) f(y) dy = g(x), \quad x > 0$$

— solvable by RHP: classical  
↑  
scaler

(b)

$$f(x) - \int_1^x k(x-y) f(y) dy = g(x), \quad -1 < x < 1$$

— also solvable by (2x2) - RHP

$$\text{eg } k(x-y) = \sqrt{\frac{-iy}{\pi}} e^{i\omega(x-y)^2}$$

— from Laser Theory

Ques: spec  $K$  as  $\omega \rightarrow \infty$ ?

Can use steepest descent method for RHP

## 5. Some words about the **steepest descent method**.

All the phenomena of the classical steep. desc. meth. are present in the non-linear method for RHP:

- for NLS, have  $e^{i\theta} = e^{i(xz-tz^2)}$ ; only 1 stationary phase pt.  $z = \frac{x}{2t}$
- for MKdV and KdV, have  $e^{i\theta} = e^{i(4xz+tz^3)}$ ; now two stat. phase pts  $z = \pm\sqrt{\frac{-x}{12t}}$  (if  $x/12t < 0$ ). The 2 stat. phase pts do not interact if we are in the space-time region

$$-M < \frac{x}{t} < -\frac{1}{M}, \quad M > 0$$

But in the region where  $x/t \rightarrow 0$  we have the non-linear analog of caustics: here the solution of MKdV, or KdV, looks like a self-similar version of PII.

So far, theory proceeds in analogy with linear st. desc. method.

- but new phenomena, beyond the scope of linear theory, start to appear. The new phenomena have the property that in place of stat. phase pts., or coalescing stat. phase pts. as in caustic situations, one now has

lines of stationary phase, each pt. contributing equally to the leading asymp's of the problem

This implies that instead of linear (or modulated linear) type oscillations, e.g. in the long-time behavior of soln's of MKdV

$$q(x, t) \sim \frac{\lambda(x/t)}{t^{1/2}} \sin\left(t\left(\frac{x}{t}\right)^2 + \nu\left(\frac{x}{t}\right) \log t\right)$$

one now has genuinely non-linear oscillations, expressed e.g. in terms of the Jacobi *sn* or *cn* functions in place of sin, etc. Situation arises e.g. in

- (i) universality for RMT
- (ii) semi-classical limit of NLS
- (iii) continuum limit of Toda lattice, etc.

**6. Numerics:** use RHP to compute solutions of Painlevé equations,...