Adiabatic invariants and global analysis of ODEs globally in $\mathbb{C}$; applications to P1

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Analysis of ODEs globally in $\mathbb{C}$

- Very few ODEs are integrable (closed form solutions, linearizability through underlying Riemann-Hilbert problems); also, we rarely have convergent expansions valid in wide enough regions. Even when the Painlevé property holds (absence of movable branch points), this alone does not indicate the associated Riemann-Hilbert problem. Finding the global behavior of, and calculating, solutions of ODEs in $\mathbb{C}$ is a challenging task.
- Goal: provide tools to obtain (a finite set) of asymptotic formulas (some Borel summable) in terms of special functions to describe the general solution of nonlinear ODEs in $\mathbb{C}$, integrable or not. Evidently, for nonintegrable equations the analysis is done on a relatively intricate Riemann surface. One can say that solutions approach simpler to describe attractors, and controls in detail the approach.

- The point at ∞ is an irregular singular point for "most" ODEs arising in applications. Thus, no convergent solution representations exist in very wide regions/sectors in $\mathbb{C}$. Classical asymptotic theory provides expansions for small solutions, relatively easy to determine; but small exponential corrections, and together with them degrees of freedom, are typically lost.
- More general than asymptotic expansions, transseries have provided a space of representations closed under most operations of interest, and suitable for the description of all solutions of linear ODEs and all small solutions of nonlinear ones, near irregular singular points; they incorporate all exponentially small corrections.

- Example: series solution of $f' + f = 1/x$ as $x \to \infty$ is $f \sim \tilde{y}_0 = \sum_{k=0}^{\infty} k! x^{-k-1}$; general transseries solution $\tilde{y} = \tilde{y}_0 + Ce^{-x}$. Much more generally, after normalization, generic ODEs near an irregular singular point, (placing it at $x = \infty$) admit transseries of the form $\sum c_{kl} e^{-k\lambda x} x^{-l}$ (first order) or (nth order)

$$\boxed{\sum_{k_1,\dots,k_m;m\le n} (C_1 e^{-\lambda_1 x} x^{a_1})^{k_1} \cdots (C_m e^{-\lambda_m x} x^{a_1})^{k_m} y_{k_1,\dots,k_m}}$$ where $y_{\mathbf{k}}$ are factorially divergent power series (in $1/x$).

- In the nonlinear case, there are infinitely many exponentials and only small exponentials can be allowed for *large* x. Some constants have to be set to zero. Degrees of freedom may be lost in this way too. Transseries approach provides (i) "optimal" normalization (ii) shape of constants of motion.

- $\boxed{\sum_{k_1,\ldots,k_n} (C_1 e^{-\lambda_1 x} x^{a_1})^{k_1} \cdots (C_n e^{-\lambda_n x} x^{a_1})^{k_n} y_{k_1,\ldots,k_n}}$; only small (marginally, oscillatory, if $a_i < 0$) exponentials are allowed for *large* x.
- Note. Its, Garoufalidis, Kapaev calculated transseries expansions with two parameters for P1, asymptotically valid in the narrow region $e^{\pm x} x^{-1/2} \ll 1$ or for $|C| \ll 1$, $x \not\to \infty$.
- But what do we do about divergence?
- Divergence is dealt with using generalized Borel summability; the Borel summation operator is denoted by $\mathcal{LB}$.

- Perhaps surprisingly divergent expansions of generic ODEs (& many PDEs) are Borel summable (O.C., Duke Math. J., 1998)
- Consider the system $\boxed{\mathbf{y}' = \mathbf{f}(x, \mathbf{y});\ \ \mathbf{y} \in \mathbb{C}^n \quad (*)}$ with $\mathbf{f}$ analytic at $(\infty, 0)$ and nonresonant: the eigenvalues λ_i of $-\left(\frac{\partial f_i}{\partial y_j}(\infty, 0)\right)_{i,j=1,2,\ldots n}$ are $\mathbb{Q}$-independent.
- The transseries:
  $\tilde{\mathbf{y}} = \tilde{\mathbf{y}}_0 + \sum_{|\mathbf{k}|>0} \mathbf{C}^{\mathbf{k}} \mathrm{e}^{-(\mathbf{k}\cdot\boldsymbol{\lambda})x} x^{\mathbf{k}\cdot\mathbf{m}} \tilde{\mathbf{y}}_{\mathbf{k}};\quad (l \le n),\ \tilde{\mathbf{y}}_{\mathbf{k}}$ formal power series in x^{-1} ;$\tilde{\mathbf{y}}_{\mathbf{k}}$ are generalized Borel summable.
- The general small solution of (*) as $x \to \infty$:
  $\mathbf{y} = \mathbf{y}_0 + \sum_{|\mathbf{k}|>0} \mathbf{C}^{\mathbf{k}} \mathrm{e}^{-(\mathbf{k}\cdot\boldsymbol{\lambda})x} x^{\mathbf{k}\cdot\mathbf{m}} \mathbf{y}_{\mathbf{k}};\ \mathbf{y}_{\mathbf{k}} = \mathcal{LB}\tilde{\mathbf{y}}_{\mathbf{k}}(**)$, convergent in the region $|C_j \mathrm{e}^{-\lambda_j x} x^{m_j}| < \epsilon, \forall j$; $\mathcal{LB}$ is the (generalized) Borel summation operator: general small solution is an "explicit" $\mathbf{F}(x; , \mathbf{z})$, $\mathbf{z} = C^{\mathbf{k}} \mathrm{e}^{-(\mathbf{k}\cdot\boldsymbol{\lambda})x} x^{\mathbf{k}\cdot\mathbf{m}}$, analytic in $\mathbf{z}$ in a polydisk. Complete qualitative and quantitative description.

From small to all

1. The boundary of a transseries region. For simplicity take a transseries with one parameter,

$$\frac{c_{01}}{x}+\frac{c_{11}}{x^2}+\cdots+Ce^{-x}\left(c_{10}+\frac{c_{11}}{x}+\cdots\right)+C^2e^{-2x}\left(c_{20}+\frac{c_{21}}{x}+\cdots\right)\ (*)$$

The region of validity is essentially $|Ce^{-x}|<\epsilon$. What if $|Ce^{-x}|\gtrapprox\epsilon$? Then the natural ordering (by size) of the terms in (*) becomes

$$c_{10}Ce^{-x}+c_{20}C^2e^{-2x}+\cdots+\frac{c_{01}}{x}+Ce^{-x}\frac{c_{11}}{x}+\cdots$$
$$=F_0(Ce^{-x})+\frac{1}{x}F_1(Ce^{-x})+\cdots\quad(1)$$

Surprisingly, perhaps, $F_j(z)$ are *analytic for small* x (but $\sum_k x^{-k}F_k$ is still divergent).

2. First entry into singular regions. The new expansion, in terms of F_j is valid even when y becomes large! It allows for studying formation of singularities.

Theorem (Singular region beyond transs. O C, R D Costin, Inv. Math., 2001)

(i) Take a generic ODEs system near a rank 1 singularity, after normalization ($\mathrm{Re}(x) > 0; \lambda_1 = 1$ *etc.). Let* $\xi(x) = Ce^{-\lambda_1 x}x_1^m$ *and consider a region close to* $i\mathbb{R}$*, where* ξ *is not small anymore. Then,*

$$\mathbf{y}(x;\mathbf{C}) \sim \sum_{m=0}^{\infty} x^{-m}\mathbf{F}_m(\xi(x)) \quad (|x| > R, |\mathbf{F}_0(\xi)| + |\xi| < A(x))$$

where $A(x) \to \infty$ *as* $x \to \infty$*. Region of validity strictly larger than transseries, approaches actual singularities of* $\mathbf{y}$*).*

(ii) If $\mathbf{F_0}$ *is singular at* ξ_s *then* $\mathbf{y}(x)$ *is singular at* $o(1)$ *of* $x_n \in \xi^{-1}(\{\xi_s\}) \cap \mathcal{D}_x$*, as* $x_n \to \infty$*. The points* $\{x_n\}_{n\in\mathbb{N}}$ *form a nearly periodic array*

$$x_n = 2n\pi i + \alpha_1 \ln(2n\pi i) + \ln C_1 - \ln \xi_s + o(1) \quad (*)$$

(Regular singular pattern had been seen in concrete equations.)

Singular region analysis

- $F_k(\xi)$ ($\xi = Ce^{-x}x^a$) are relatively easy to determine.
- Practical method: in the ODE system, plug in the expansion $\tilde{\mathbf{y}}(x,C) = \sum_{m=0}^{\infty} x^{-m}\mathbf{F}_m(\xi)$, treat ξ and x as if they were "independent" variables and collect the powers of x^{-1}.
- One obtains differential equations for $\mathbf{F}_j$ simpler than the original ODE.
- For $\boxed{\text{P1 (Boutroux-normalized)}}$ we get $\xi^2 F_0'' + \xi F_0' = F_0 + \frac{1}{2}F_0^2$

  Becomes autonomous with $s = \ln\xi$. Upshot:

$$F_0 = \frac{\xi}{(\xi/12-1)^2}$$

Can show that all the F_k are *rational* functions.

$$\boxed{\text{P2}} \quad F_0 = \frac{\xi}{1-\xi^2/9}$$

The territory of the large

- The $\mathbf{F}_j$ representation may also fail, if $\mathbf{F}_j$ become large; for P_1 F_1 is bounded, but $F_j \sim \xi^{2k-1} \sim e^{|(2k-1)x|}$.
- We now note that in

$$y(x;C) \sim \sum_{m=0}^{\infty} x^{-m} F_m(Ce^{-x}) \quad (|x| > R, |F_0(\xi)| + |\xi| < A(x))$$

  C, expressed as a function of x and y is a conserved quantity (constant of motion, constant along trajectories).
- Solving for C as a function of (y, x) yields constants of motion which are valid in the regions which are *complementary* to the transseries and borderline regions. In a sector near infinity of any width, in $\mathbb{C}$ or on a Riemann surface, only finitely many of these constants are needed.

Note. Existence of such constants does not necessarily imply integrability: they can be multivalued, and valid only sectorially. But this does not affect controllability of solutions.

Analysis of general solutions; first order ODEs

Consider first the simplest case, scalar first order equations. Typically, they can be brought to the form

$$y' = P_0(y) + Q(y, 1/x) = \sum_{k=0}^{\infty} \frac{P_k(y)}{x^k} \tag{2}$$

where P_0 is a polynomial (analytic would be OK) and $Q(y, z)$ is entire in y and analytic in z for small z, and $O(y^2, yz^2, z)$ for small y and z.

- We show that there are finitely many constants of motion whose union of domains covers the phase space.
- Assuming x_0 is regular and $y_0 = y(x_0)$, one calculates, say numerically, the solution in a compact set, while outside it, asymptotic formulae give accurate description.

The transseries are easy to calculate [DMJ]. At edges of validity of transseries we have singular expansions [Inv. Math], obtained by regrouping the terms of the transseries,
$u = G_0(C_1 e^{-x} x^{\beta}) + x^{-1} G_1(C_1 e^{-x} x^{\beta}) + \cdots$. What lies beyond?

- Solving for $C_1 \sim e^x x^{-\beta} G_0^{-1}(u) + \cdots$ and taking the log,

$$C_2 = C_2(u, x) = x - \beta \log x + F_0(u) + x^{-1} F_1(u) + O(1/x^2)(*)$$

is a constant of motion.

- The F_j are obtained from $\frac{dC}{dx} := C_x + C_u u' = 0$, solving order by order (in $1/x$). The F_j can be calculated by quadratures since they come from a (nearby) autonomous system.
- There are clear similarities to adiabatic invariants in Hamiltonian systems. Note: C_2 controls all solutions, even those without any transseries (in fact, in the regions *complementing* transseries): there are no smallness requirements (in fact there are "no-smallness" ones).
- The form (*), with $-\beta \log x$, etc. is essential. In fact, it is unique up to trivial transformations.

Theorem (OC, M. Huang, 2010; constants of motion)

(i) If $P_0(u)$ is small (under some generic assumptions) u is given by a transseries in a half plane, and $C(u(x),x)$, obtained by inversion of the transseries is a constant of motion.
(ii) In any region
$\mathcal{D}=\{(u,x):|P_0(u)|>\epsilon>0,|x|>R,\arg(x)=\varphi(|x|)\}$, *there is a constant of motion (the region might be part of a Riemmann surface as well, not only of $\mathbb{C}$.). The constant of motion can be calculated order in $(1/x)$ by order by quadratures. ϵ can be chosen arbitrarily small if R is chosen correspondingly large.*
(iii) The different regions overlap and the constants of motion match each other.

Example: Abel's equation, $u' = u^3 - z$

The Abel equation $y' = y^3 - z$ equation is not (known to be) integrable in any classical sense. And there probably is no well behaved global constant of motion. But there still are, as mentioned, finitely many constants covering all of $\mathbb{C}$, so that the solution can be fully described.

The normalized form of this equation is obtained by the transformation $y(x) = Bx^{1/5}u(x)$ where $x = At^{5/3}$ with $A = -B^2/5, 15/B^5 = -1/9$ is

$$u' + 3u^3 - \frac{1}{9} + \frac{1}{5x}u = 0 \qquad (3)$$

Example: Abel's equation, normalized. $u' + 3u^3 - \frac{1}{9} + \frac{1}{5x}u = 0$

In regions where $u(x)$ is not too small, the constant is given asymptotically by an expansion of the general form mentioned before ($\beta = 1/5$ here),

$$C(u, x) = -x + \frac{1}{5}\log x + F_0(u) + \frac{F_1(u)}{x} + \frac{F_2(u)}{x^2} + O(x^{-3})$$

where

$$F_0(u) = \sqrt{3}\arctan\left(\frac{6u+1}{\sqrt{3}}\right) - \log(3u-1) + \frac{1}{2}\log(9u^2+3u+1)$$

$$F_1(u) = \frac{1}{10}\left(\frac{54u^2}{1-27u^3} - 4\sqrt{3}\arctan\left(\frac{6u+1}{\sqrt{3}}\right)\right) + \frac{1}{25}$$

This provides an asymptotic formula for the general solution, in regions where they are not close to roots of the polynomial; in the opposite case, the solutions have Borel summable transseries. The two regions **match** in a narrow subregion.

- The form of u in the non-transseries region is

$$u = \frac{1}{3}\exp\left[\left(-C - x + \frac{1}{5}\log x + \left(\sqrt{3} - \frac{2\sqrt{3}}{5x}\right)\arctan\left(\frac{6u+1}{\sqrt{3}}\right)\right.\right.$$
$$\left.\left. -\log(3u-1) + \frac{1}{2}\log(9u^2+3u+1)\right) + \frac{1}{x}\left(\frac{27u^2}{5(1-27u^3)} + \frac{1}{25}\right) + \dots\right] + \frac{1}{3} \tag{4}$$

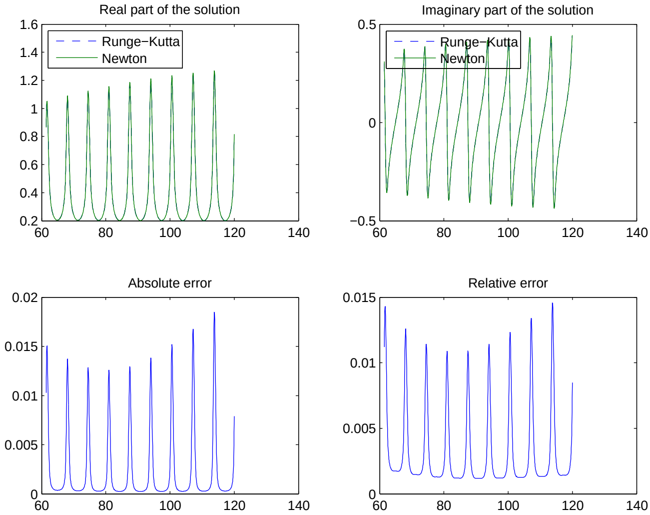

Figure: Comparison of solutions from the Runge-Kutta method and from Newton's method using the formal constant of motion.

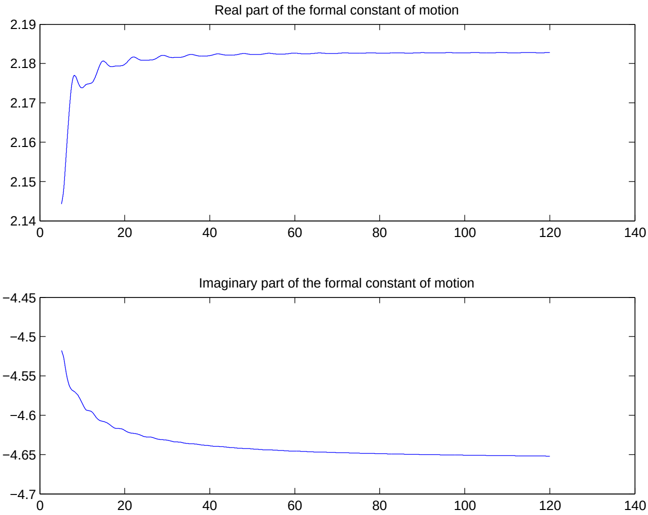

Figure: Formal constant of motion with F_0 and F_1.

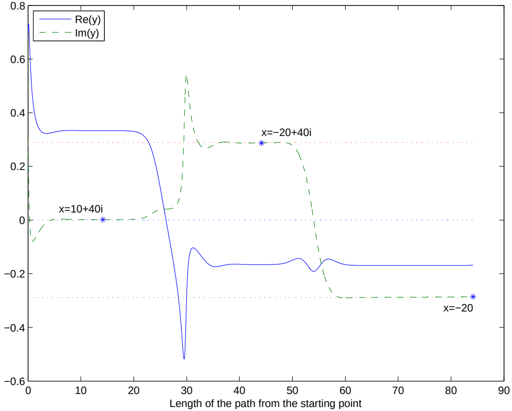

Figure: Monodromy at ∞ of general solution. horizontal axis=arclength. Dotted horizontal lines are the imaginary parts of the three roots. There are two periodic arrays of singularities in every transition region. In integrable systems, connection formulas follow (in nonintegrable equations, the solutions are multivalued).

The Painlevé equation P1

- $y'' = y^2 + x$. Goal: provide asymptotics to all orders of the general solution, in terms of constants of motion. Solutions can be obtained, with high accuracy, from these.
- Asymptotics: Extensive research, substantial results by Clarkson, Deift, Its, Kapaev, Kitaev... (isomonodromy–Riemann Hilbert), Kruskal-Nalini (direct asymptotics -multiscale analysis).
- (Modified) Boutroux normalized equation:
$$u'' + \frac{u'}{x} - u - \frac{u^2}{2} - \frac{392}{625x^4} = 0 \ (*).$$
- The first constant of motion can be obtained as before, from the transseries, $C_1(x,s,u) = x + F_0(s,u) + x^{-1}F_1(s,u)...$ where $s = u'^2 - u^3/3 + u^2$, inserted in (*) and requiring $C_x + C_u u' + C_s s' = 0$. (The representation (x,u,s) has technical advantages over (x,u,u'), since now F_j satisfy PDEs, and one of the characteristic variables is s; using this reduces the problem to ODEs.)

We let $R(u,s)=\sqrt{u^3/3+u^2+s}$ and get

$$\frac{\partial F_0}{\partial u}=-\frac{1}{R} \tag{5}$$

$$\frac{\partial F_1}{\partial u}=R\int_{u_0}^{u}\frac{1}{R^3(u',s)}du' \tag{6}$$

- Thus $F_0=L(s,u)=\int_{u_0}^{u}1/R(u',s)du'$,
- **A useful identity.** Let $Q=\frac{2}{3}s^{-1}(3s+4)^{-1}R^{-1}P(s,u)$, $P=6(s+su+3u)+3u^2-4u^3(u^2+1)$; then,

$$\frac{\partial Q}{\partial u}=\frac{1}{R^{3/2}}-\rho(s)R; \qquad \rho(s):=\frac{5}{3s\,(3\,s+4)}$$

- Thus, $F_1=-\frac{27\rho(s)}{10}P_1(s,u)+\frac{\rho(s)}{2}J^2(s,u)+g_1(s)$, $g_1(s)$ is determined by consistency conditions at next order (staggered pattern persists). Here $J=\int R$, $P_1=\int P$.

- $C_1(x,s,u) = x - L(s,u) + \sum_{k=1}^{\infty} \frac{F_k(s,u)}{x^k}$
- (provisional form) $C_1(x,s,u) = x - L(s,u) + x^{-1}\left(\frac{27\rho(s)}{10} P_1(s,u) + \frac{\rho(s)}{2} J^2(s,u) + g_1(s)\right) + O(x^{-2})$

Second constant: order reduction, or better, by noting that the ODE should be tamer in the "hodograph" representation

$$\frac{ds}{du} = -\frac{2\sqrt{u^3/3 + u^2 + s(u)}}{x(u)} \qquad (7)$$

$$\frac{dx}{du} = \frac{1}{\sqrt{u^3/3 + u^2 + s(u)}} \qquad (8)$$

We start from this system, and write it in an integral contractive form; with $s(u) = D_\infty + \delta(u)$

$$\delta(u) = \frac{\delta_0 x_0}{x(u)} - \frac{2J(u, D_\infty)}{x(u)} - \frac{1}{4x(u)} \int_{u_0}^{u} \frac{\delta^2(t) G_0(t, \delta(t))}{R(t)^3} dt \qquad (9)$$

A similar equation is satisfied by $\xi := x - L(u, D_\infty)$ (quite a bit of algebra here...); the constants are obtained rigorously by inverting the system asymptotically for the initial conditions as functions of x, s, u. Second constant is

$$C_2 = J(s)(x - L(u,s)) + L(s)J(u,s) + \frac{H_1(u,s)}{x} + \frac{H_2(u,s)}{x^2} + O(1/x^3)$$

$H_1 = -\frac{27\rho(s)J(s)}{10} P_1(s,u) + h_1(s)$, etc.

All asymptotic forms of the constants are direction-dependent (e.g., u- cycle is fixed). Note: it is the asymptotic expansion, and not the constant, which depends on angle. (Similar to the classical Stokes phenomenon-change in asymptotic behavior of, say, entire functions.)

In second order equations, even if starting at large x we may need an intermediate step before reaching the asymptotic regime.

Initial stages: the discrete evolution and constant

$$\frac{ds}{du} = -\frac{2\sqrt{u^3/3 + u^2 + s(u)}}{x(u)} \tag{10}$$

$$\frac{dx}{du} = \frac{1}{\sqrt{u^3/3 + u^2 + s(u)}} \tag{11}$$

- Fix for now a "cycle", a closed curve surrounding nontrivially the roots of $u^3/3 + u^2 + s(u)$; we integrate along (possibly non-integer) multiples of a cycle.
- Consider the closed loop integrals below (when the loop is not closed we write $J(s,u), L(s,u)$)

$$J(s) := \oint \sqrt{u^3/3 + u^2 + s}\,du; \quad L(s) := \oint (u^3/3 + u^2 + s)^{-1/2} du$$

We let (s_n, x_n) be the value of (s, x) after n cycles. We assume for simplicity that there are no crossings of the roots in the process (one loop would be the monodromy map, Poincaré map); we naturally set $J_n = J(s_n)$ and $L_n = L(s_n)$. One can show that, for large enough starting point x_0, we have, with Q_0 a number,

- $J(s) := \oint \sqrt{u^3/3 + u^2 + s}du$; $L(s) := \oint (u^3/3+u^2+s)^{-1/2}du$

$$s_{n+1} = s_n - 2\frac{J_n}{x_n} + \frac{J_n L_n}{x_n^2} + O\left(\frac{1}{x_n^3}\right) \quad (12)$$

$$x_{n+1} - x_n = L_n + \frac{Q_0 J_n + \frac{1}{2}\rho(s_n) J_n^2}{x_n} + O(1/x_n^2) + O(1/x_n^2) \quad (13)$$

- Heuristics: The recurrence relation predicts a slow evolution (relative to the relevant quantities).
- Thus, to leading order $s_{n+1} - s_n \approx \frac{ds}{dn} \cdot 1, x_{n+1} - x_n \approx \frac{dx}{dn}$ (think Euler-Maclaurin). By substitution from (12),(13) and division, we get (note that $J' = \frac{1}{2}L$)

$$\frac{ds}{dx} \approx -\frac{2J}{Lx} \Rightarrow \frac{J'}{J}ds = -dx \Rightarrow J_n x_n \approx J_0 x_0$$

a first, discrete, constant of motion to which corrections can be calculated similarly. (For a rigorous argument: calculate $J_{n+1}x_{n+1} - J_n x_n$.)

Conclusions

We are developing a theory, which appears to be applicable to a wide class of equations to determine the global behavior of the general solution.

- The general solution may have regions where it approaches a constant. Those are describe in great detail by transseries. [DMJ]
- There is a borderline region where transseries start failing and singularities start forming [Inv. Math.]
- Away from those regions, there exist constants of motion which describe the singular behavior in the rest of the phase space. For second order equations, a typical solution may have no transseries region.
- For all first order equations and many second order ones, including P1 and P2, the constants can be obtained asymptotically by quadratures.