Critical asymptotics for Toeplitz determinants

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Toeplitz determinants

- Toeplitz matrix = matrix which is constant along diagonals

$$\begin{pmatrix} c_0 & c_{-1} & c_{-2} & \cdots & c_{-n+1} \\ c_1 & c_0 & c_{-1} & \ddots & \vdots \\ c_2 & c_1 & \ddots & \ddots & c_{-2} \\ \vdots & \ddots & \ddots & c_0 & c_{-1} \\ c_{n-1} & \cdots & c_2 & c_1 & c_0 \end{pmatrix}$$

- Toeplitz determinant is the determinant of a Toeplitz matrix
- Asymptotics for Toeplitz determinants when the size of the matrices tends to infinity?

Toeplitz determinants

- Consider a symbol $f(e^{i\theta})$ on the unit circle C_1
  - Fourier coefficients

$$c_k = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{-ik\theta}d\theta$$

- Toeplitz determinant for symbol f

$$D_n(f) = \det(c_{j-k})_{j,k=0}^{n-1}$$

  - for fixed f, behavior as $n \to \infty$?
  - same question for f depending on n

Toeplitz determinants

- If $f(\theta)$
  - is "smooth"
  - has no zeros
  - has a continuous logarithm (winding number 0 around the origin)
- Szegő's strong limit theorem: as $n \to \infty$,

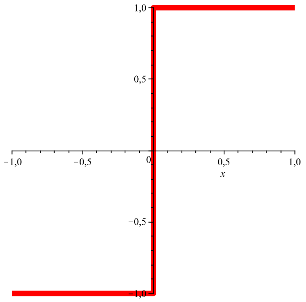
$$\ln D_n(f) = n(\ln f)_0 + \sum_{k=1}^{\infty} k(\ln f)_k(\ln f)_{-k} + o(1),$$

with

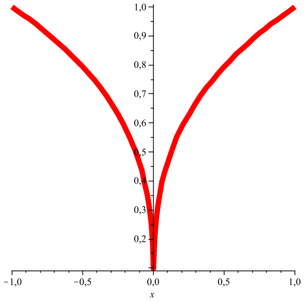
$$(\ln f)_k = \frac{1}{2\pi}\int_0^{2\pi} \ln f(e^{i\theta})e^{-ik\theta}d\theta.$$

Fisher-Hartwig singularities

- Two types of weights for which Szegő asymptotics are not valid
  - jump discontinuities



  - root type singularities



- Example

$$f(e^{i\theta}) = (2-2\cos\theta)^{\alpha} e^{i\beta(\theta-\pi)} e^{V(e^{i\theta})}, \qquad \text{for } 0<\theta<2\pi,$$

with Re $\alpha > -\frac{1}{2}$

  - Fisher-Hartwig singularity at 1

Fisher-Hartwig singularities

- For weights with one Fisher-Hartwig singularity with parameters α (root) and β (jump),

$$\ln D_n(f) = nV_0 + \sum_{k=1}^{\infty} kV_kV_{-k} - (\alpha-\beta)\sum_{k=1}^{\infty} V_k - (\alpha+\beta)\sum_{k=1}^{\infty} V_{-k}$$
$$+ (\alpha^2 - \beta^2)\ln n + \ln \frac{G(1+\alpha+\beta)G(1+\alpha-\beta)}{G(1+2\alpha)} + o(1),$$

as $n \to \infty$, where G is Barnes' G-function, and

$$V_k = \frac{1}{2\pi}\int_0^{2\pi} V(e^{i\theta})e^{-ik\theta}d\theta.$$

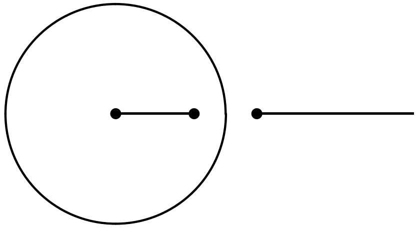
(Fisher-Hartwig '68, Widom '73, Boettcher, Tracy, Ehrhardt-Silbermann '97, Basor-Ehrhardt '01, Deift-Its-Krasovsky)

Transition from Szegő to FH

- weight

$$f(z) = (z - e^{t})^{\alpha+\beta}(z - e^{-t})^{\alpha-\beta} z^{-\alpha+\beta} e^{-i\pi(\alpha+\beta)} e^{V(z)},$$

with V analytic and $t \geq 0$



  - ▸ f analytic on C with winding number zero around the origin for $t > 0$
  - ▸ f has a singularity at 1 for $t = 0$,

$$f(e^{i\theta}) = (2-2\cos\theta)^{\alpha} e^{i\beta(\theta-\pi)} e^{V(e^{i\theta})}, \quad \text{for } 0 < \theta < 2\pi,$$

Transition from Szegő to FH

- Asymptotics as $n \to \infty$ for Toeplitz determinant with weight

$$f(z) = (z - e^{t})^{\alpha+\beta}(z - e^{-t})^{\alpha-\beta} z^{-\alpha+\beta} e^{-i\pi(\alpha+\beta)} e^{V(z)}$$

  - Szegő asymptotics for $t > 0$ fixed,

$$\ln D_n(t) = nV_0 + nt(\alpha+\beta) + \mathcal{O}(1), \quad \text{as } n \to \infty$$

  - Fisher-Hartwig asymptotics for $t = 0$,

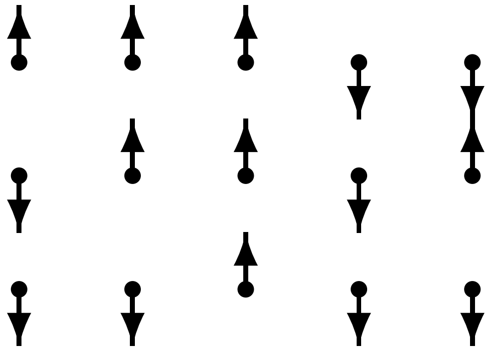
$$\ln D_n(0) = nV_0 + (\alpha^2 - \beta^2)\ln n + \mathcal{O}(1), \quad \text{as } n \to \infty,$$

  with $V_0 = \frac{1}{2\pi}\int_C V(z)\frac{dz}{iz}$.

- what if $t \to 0$?

Application: 2d Ising model

- lattice with an associated spin variable taking values ± 1 at each point of the lattice



- probability measure on spin configurations, depending on temperature T
- 2-spin correlation functions are Toeplitz determinants:

$$< \sigma_{00}\sigma_{kk} >= D_k(f).$$

2d Ising model

- Symbol f is given by

$$f(z;t) = (z - e^t)^{-1/2}(z - e^{-t})^{1/2} z^{-1/2} e^{i\pi/2},$$

$$e^t = \sinh \frac{2\gamma_1}{T} \sinh \frac{2\gamma_2}{T}, \ \alpha = 0, \ \beta = -1/2, \ V(z) = 0$$

  - For $T < T_c$, f is a Szegő weight
  - For $T = T_c$, f is a Fisher-Hartwig weight
  - As $T \nearrow T_c$ or $t \searrow 0$, the phase transition for the 2d Ising model takes place (from finite magnetization to magnetization 0)
- as $n \to \infty$, $t \searrow 0$ such that $2nt \to x$, $\lim n^{1/4} < \sigma_{00}\sigma_{nn} >= F(x)$, where F is expressible in terms of a Painlevé III transcendent *(Wu-McCoy-Tracy-Barouch '76)*

Transition from Szegő to FH

- more general situation: general α, β, V
- what happens if $t \to 0$ simultaneously with $n \to \infty$ (double scaling limit)?

$$f(z) = (z - e^t)^{\alpha+\beta}(z - e^{-t})^{\alpha-\beta} z^{-\alpha+\beta} e^{-i\pi(\alpha+\beta)} e^{V(z)}$$

- Result :
  If $\alpha > -\frac{1}{2}$ and Re $\beta = 0$,

$$\ln D_n(t) = nV_0 + \sum_{k=1}^{\infty} kV_k V_{-k}$$

$$-(\alpha-\beta)\sum_{k=1}^{\infty} k\left[V_k - (\alpha+\beta)\frac{e^{-tk}}{k}\right]\left[V_{-k} - (\alpha-\beta)\frac{e^{-tk}}{k}\right]$$

$$+(\alpha+\beta)nt + \ln\frac{G(1+\alpha+\beta)G(1+\alpha-\beta)}{G(1+2\alpha)} + \Omega(2nt) + o(1),$$

Painlevé V

- Ω is given by

$$\Omega(2nt) = \int_0^{2nt} \frac{\sigma(x) - \alpha^2 + \beta^2}{x} dx + (\alpha^2 - \beta^2) \ln 2nt,$$

where σ is a particular solution to the ODE

$$(x\sigma_{xx})^2 = \left(\sigma - x\sigma_x + 2\sigma_x^2 + 2\alpha\sigma_x\right)^2 - 4\sigma_x^2 (\sigma_x + \alpha + \beta)(\sigma_x + \alpha - \beta)$$

with asymptotics

$$\sigma(x) = \begin{cases} \alpha^2 - \beta^2 + o(1), & x \to 0, \\ x^{-1+2\alpha} e^{-x} \frac{-1}{\Gamma(\alpha-\beta)\Gamma(\alpha+\beta)} \left(1 + \mathcal{O}\left(\frac{1}{x}\right)\right), & x \to +\infty, \end{cases}$$

- The second order ODE is the so-called Jimbo-Miwa-Okamoto σ-form of the Painlevé V equation

Transition from Szegő to FH

$$\ln D_n(t) = nV_0 + \sum_{k=1}^{\infty} kV_kV_{-k} - (\alpha-\beta)\sum_{k=1}^{\infty} V_k - (\alpha+\beta)\sum_{k=1}^{\infty} V_{-k}$$

$$+(\alpha+\beta)nt+\ln \frac{G(1+\alpha+\beta)G(1+\alpha-\beta)}{G(1+2\alpha)}+\Omega(2nt)+o(1),$$

- special case 1: $n \to \infty$, $t \to 0$ in such a way that $nt \to 0$
  - substituting asymptotics for σ at $+\infty$ leads to FH asymptotics
- special case 2: $n \to \infty$, $t \to 0$, $nt \to \infty$
  - substituting asymptotics for σ at 0 leads to leading order of Szegő asymptotics
  - comparing constants leads to the identity $\Omega(+\infty) = -\ln \frac{G(1+\alpha+\beta)G(1+\alpha-\beta)}{G(1+2\alpha)}$ (total integral identity for σ)

Transition from Szegő to FH

- if $\alpha = 0$, $\beta = -1/2$, the ODE can be reduced to the Painlevé III equation
  - consistent with Ising results
  - if $\alpha > -1/2$, $\beta \in i\mathbb{R}$
    $\rightarrow$ $\sigma(x)$ is real for positive x
    $\rightarrow$ σ does not have poles for $x \in (0, +\infty)$
- for general values of α, β
  - σ can have finitely many poles on $(0, +\infty)$
  - asymptotic expansion holds if Ω is defined by integration of σ over a pole-free contour
  - pole $\leftrightarrow$ zero of Toeplitz determinant
  - different choice of integration contour $\leftrightarrow$ different branch of logarithm

Orthogonal polynomials

Proof of the result based on relation between Toeplitz determinants and orthogonal polynomials *(Deift-Its-Krasovsky)*

- let $f(e^{i\theta})$ be positive on the unit circle and in L^2
- OPs determined uniquely by conditions

$$\frac{1}{2\pi}\int_0^{2\pi} p_n(e^{i\theta})p_m(e^{-i\theta})f(\theta)d\theta = \delta_{nm},$$

or

$$\frac{1}{2\pi i}\int_C p_n(z)p_m(\bar{z})f(z)\frac{dz}{z} = \delta_{nm},$$

Orthogonal polynomials

- Heine’s formula: determinant formula for orthogonal polynomials

$$p_n(z) = \sqrt{\frac{1}{D_{n+1}(f)D_n(f)}} \begin{vmatrix} c_0 & c_{-1} & c_{-2} & \dots & c_{-n} \\ c_1 & c_0 & c_{-1} & \ddots & \vdots \\ c_2 & c_1 & \ddots & \ddots & c_{-2} \\ \vdots & \ddots & \ddots & c_0 & c_{-1} \\ 1 & z & \dots & z^{n-1} & z^n \end{vmatrix}$$

Orthogonal polynomials

- As a consequence, we have

$$\kappa_n(f) = \sqrt{\frac{D_n(f)}{D_{n+1}(f)}}, \qquad D_n(f) = \prod_{j=0}^{n-1} \kappa_j(f)^{-2},$$

  where $\kappa_j > 0$ is leading coefficient of orthonormal polynomial p_j
- asymptotics as $n \to \infty$ for p_n, κ_n are known in many cases
- unfortunately $\kappa_0, \kappa_1, \ldots$ are also needed

Asymptotics for Toeplitz determinants

General approach to obtain asymptotics for Toeplitz determinants for weight f

- Step 1: deform weight f smoothly to a weight for which Toeplitz determinant is known (e.g. uniform weight),

$$f_t(z), \qquad f_1(z) = f, \qquad f_0(z) = 1$$

- Step 2: try to find differential identity for $\frac{d}{dt} \ln D_n(f_t)$ in terms of $p_n, p_{n-1}, \dots, p_{n-k}$ and $\kappa_n, \kappa_{n-1}, \dots, \kappa_{n-j}$
- Step 3: find asymptotics for orthogonal polynomials as $n \to \infty$, uniform in t
- Step 4: integrate differential identity from 0 to 1

Transition from Szegő to FH

Applied to our transition between Szegő and FH

- Step 1: deformation of weight:

$$f_t(z) = (z - e^t)^{\alpha+\beta}(z - e^{-t})^{\alpha-\beta} z^{-\alpha+\beta} e^{-i\pi(\alpha+\beta)} e^{V(z)}$$

- we know asymptotics for $\ln D_n(0)$ (Fisher-Hartwig) and for $\ln D_n(t_0)$ (Szegő)
  - we can integrate from 0 or from t_0

Transition from Szegő to FH

- Step 2: differential identity

$$\frac{d}{dt}\ln D_n(t) = -(\alpha+\beta)e^t\left(Y^{-1}Y'\right)_{22}(e^t)+(\alpha-\beta)e^{-t}\left(Y^{-1}Y'\right)_{22}(e^{-t})$$

where

$$Y(z) = \begin{pmatrix} \kappa_n^{-1}p_n(z) & \kappa_n^{-1}p_n^{-1}\int_{C_1}\frac{p_n(\xi)}{\xi-z}\frac{f(\xi)d\xi}{2\pi i\xi^n} \\ -\kappa_{n-1}z^{n-1}p_{n-1}(z^{-1}) & -\kappa_{n-1}\int_{C_1}\frac{p_{n-1}(\xi^{-1})}{\xi-z}\frac{f(\xi)d\xi}{2\pi i\xi} \end{pmatrix}$$

  - Y is solution of the Riemann-Hilbert problem for orthogonal polynomials

Riemann-Hilbert problem

- Search for a 2×2 matrix-valued function Y satisfying the conditions

(a) Y is analytic in $\mathbb{C} \setminus C_1$

(b) $Y_+ = Y_- \begin{pmatrix} 1 & z^{-n} f(z) \\ 0 & 1 \end{pmatrix}$ for $z \in C_1$

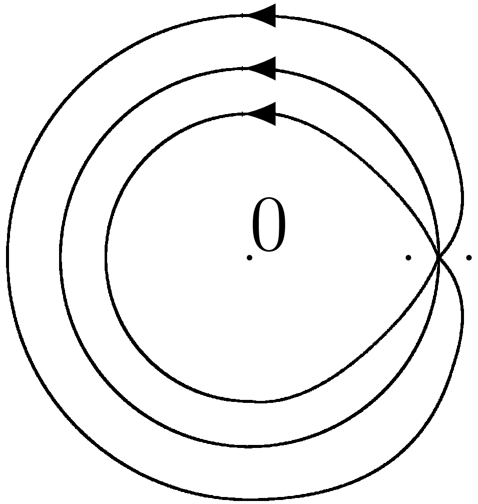
(c) $Y(z) = (I + \mathcal{O}(1/z)) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}$ as $z \to \infty$

- Unique solution

$$Y(z) = \begin{pmatrix} \kappa_n^{-1} p_n(z) & \kappa_n^{-1} p_n^{-1} \int_{C_1} \frac{p_n(\xi)}{\xi - z} \frac{f(\xi) d\xi}{2\pi i \xi^n} \\ -\kappa_{n-1} z^{n-1} p_{n-1}(z^{-1}) & -\kappa_{n-1} \int_{C_1} \frac{p_{n-1}(\xi^{-1})}{\xi - z} \frac{f(\xi) d\xi}{2\pi i \xi} \end{pmatrix}$$

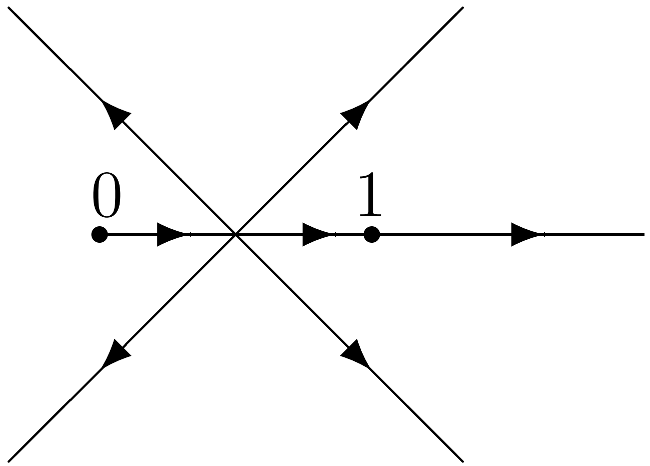
Transition from Szegő to FH

- Step 3: asymptotics for orthogonal polynomials
  - asymptotic analysis of the RH problem for orthogonal polynomials: Deift-Zhou steepest descent method
  - series of transformations including contour deformation
  - local analysis near 1 required: this is the step where the Painlevé V equation appears



Local parametrix

- Local parametrix constructed using a model RH problem for the Painlevé V equation



- PV-jump contour can be seen as a magnified version of the OP jump contour near 1
- singular points $0, 1$ of PV-RH problem correspond to singularities $e^{\pm t}$ of the symbol

Open problems

- Other phase transitions for Toeplitz determinants occur e.g. in Heisenberg spin chain models *(Abanov-Franchini)*
  - what if two or more singularities approach each other on the unit circle?
  - arc where the symbol approaches zero?