Title: Numerical solution of the Painlevé equations

Scientific Organisers

- Peter Clarkson (University of Kent, Canterbury, UK), Principal Organiser
- Folkmar Bornemann (Technische Universität München, Germany)
- Percy Deift (Courant Institute, New York, USA)
- Alan Edelman (Massachusetts Institute of Technology, Cambridge, USA)
- Alexander Its (Indiana University Purdue University, Indianapolis, USA)

Workshop Report

1. An explanation of any significant deviation from the original proposal.

The format and content of the workshop was the same as in the original proposal.

2. A short description of the meeting describing the importance of the research reported or undertaken in terms that would be understandable to a general audience.

In recent years the Painlevé equations have emerged as the core of modern special function theory. In the 18th and 19th centuries, the classical special functions such as Bessel, Airy, Legendre and hypergeometric functions were recognized and developed in response to the problems of the day in electromagnetism, acoustics, hydrodynamics, elasticity and many other areas. In the same way, around the middle of the 20th century, as science and engineering continued to expand in new directions, a new class of functions, the Painlevé functions, started to appear in applications. The list of problems now known to be described by the Painlevé equations is large, varied and expanding rapidly. The list includes, at one end, the scattering of neutrons off heavy nuclei, and at the other, the statistics of the zeros of the Riemann-zeta function on the critical line $\mathrm{Re}(z) = 1/2$. In between, amongst many others, there is random matrix theory, the asymptotic theory of orthogonal polynomials, self-similar solutions of soliton equations, combinatorial problems such as Ulam's longest increasing subsequence problem, tiling problems, multivariate statistics in the important asymptotic regime (where the number of variables and samples are comparable and large), and also random growth problems.

Although the Painlevé equations are nonlinear, much is already known about their solutions, particularly their algebraic, analytical and asymptotic properties. This is because the equations are integrable in the sense that they have a Lax-Pair and also a Riemann-Hilbert representation from which the asymptotic behavior of the solutions can be inferred using the non-linear steepest-descent method. The numerical analysis of the Painlevé equations is less developed and presents novel challenges: in particular, in contrast to the classical special functions, where the linearity of the equations greatly simplifies the situation, each problem for the nonlinear Painlevé equations arises essentially anew.

3. A comprehensive report of the workshop. Describe the highlights of the workshop, the involvement of participants and any new collaborations likely to arise.

The workshop brought together researchers from three different fields, namely *Asymptotic Analysis*, *Integrable Systems* and *Numerical Analysis*; the interaction between the latter field and the other two fields has been quite limited. There was much positive interaction between the researchers in the different fields, providing new challenges for the numerical analysts and giving an opportunity for the researchers in asymptotic analysis and integrable systems to describe the numerical problems they would like to be studied together with the details of their analytic structure. The speakers had been selected to include some expository lectures which was important given the wide variety of research interests of the audience, which was appreciated judging by the feedback of the participants. Feedback from the participants of the workshop indicates that the academic value of the workshop was thought to be very high.

On each afternoon there was a two-hour discussion session, led by one of the organisers, with a specific theme. These proved to be extremely useful and were mentioned many times in the feedback from participants as a highlight of the workshop. The discussion sessions gave the opportunity to discuss in detail issues which had been raised in the lectures. Some of the participants, in particular those who did not give talks, contributed to these discussion sessions by giving short, impromptu lectures. In addition to feedback from participants that these discussion sessions were a highlight of the workshop, they were found to be useful ("much more useful than just having more talks"). Further one participant suggested that the structure of the workshop "will serve as a model for many workshops to come".

The opening day of the workshop was devoted to give the participants an introduction to the "*Painlevé Project – goal and method*", in particular the vision of Percy Deift. The discussion session on the second day focussed on "*Asymptotics methods for the Painlevé equations*", which was led by Alexander Its. The discussion sessions on the third and fourth days were centred on "*Random matrices for the Painlevé equations*", since the field of Random matrices has been at the forefront of applications of Painlevé equations and has stimulated much of the work in the field, and were led by Folkmar Bornemann and Alan Edelman.

At the "*Open Problems*" discussion session on the final day of the workshop, chaired by Peter Clarkson, there was a discussion about the future direction of the "Painlevé Project". As the next step of the Painlevé Project, it was decided to establish an e-site, maintained at the National Institute of Standards and Technology (NIST) in Gaithersburg, U.S.A., which now has been done. Interested researchers are asked to send the following information to the site:

1. pointers to new work on the theory of the Painlevé equations (algebraic, analytical, asymptotic or numerical);
2. pointers to new applications of the Painlevé equations;
3. suggestions for possible new applications of the Painlevé equations;
4. requests for specific information about the Painlevé equations.

Letters have been sent to several professional societies giving details of the Painlevé Project and the e-site; we have learnt that information about it will appear shortly in the *Notices of the American Mathematical Society*. Depending on the response to the Painlevé Project, it is proposed to set up a Wiki page for the Painlevé equations, and then ultimately a comprehensive handbook in a style

befitting the digital age. Proposals for mini-symposiums on the "Painlevé Project", with talks given participants of the ICMS workshop under discussion, have been submitted for the *International Congress on Industrial and Applied Mathematics* to be held in Vancouver in 2011.

On one evening during the workshop there was the launch of NIST's "*Digital Library of Mathematical Functions*", which has also been published as the "*NIST Handbook of Mathematical Functions*" [Editors F W J Olver, D W Lozier, R F Boisvert & C W Clark], by Cambridge University Press in 2010. This book is an update and rewrite of Abramowitz & Stegun's "*Handbook of Mathematical Functions*", which is one of the most comprehensive sources of information on special functions, containing definitions, identities, approximations, plots, and tables of values of numerous functions used in virtually all fields of Applied Mathematics. The chapter on "Painlevé transcendents", written by Peter Clarkson, is one of the new chapters that appears in "*Digital Library of Mathematical Functions*" that did not appear in the original "*Handbook of Mathematical Functions*". A presentation and demonstration of the Digital Library was given by Dan Lozier, the General Editor of the DLMF project and a workshop participant, which was followed by a wine reception hosted by Cambridge University Press.

In the weeks since the workshop in Edinburgh, I have heard of new results being obtained on numerical solutions of Painlevé equations, e.g. by Folkmar Bornemann (Munich, Germany) and Sheehan Olver (Oxford) as well as a collaboration between Bengt Fornberg (Boulder, USA) and Andre Weidermann (Stellenbosch, South Africa).

The organisers thank the ICMS for the opportunity to have a workshop on the "Numerical solution of Painlevé equations". We feel that it has given an impetus to get the "Painlevé project" moving forward. The workshop would not have happened for the excellent administrative work done by Audrey Brown whom the organisers thank very much.

Peter Clarkson
12th September 2010