Two-dimensional Euler flows in slowly deforming domains

Adiabatic invariance and geometric angle

J Vanneste

Maxwell Institute and School of Mathematics
University of Edinburgh, UK
www.maths.ed.ac.uk/~vanneste

with D Wirosoetisno (Durham)

Plan

Motivation

Inviscid, incompressible fluids: infinite-dimensional Hamiltonian systems.

What is the effect of slow, Hamiltonian perturbations?

Simplest model:

- 2D incompressible fluid (2D Euler),
- quasi-steady flow,
- bounded domain,
- perturbations introduced by deformations of domain boundary.

Applications: flow control, mixing,. . .

Motivation

Inviscid, incompressible fluids: infinite-dimensional Hamiltonian systems.

What is the effect of slow, Hamiltonian perturbations?

Simplest model:

- 2D incompressible fluid (2D Euler),
- quasi-steady flow,
- bounded domain,
- perturbations introduced by deformations of domain boundary.

Applications: flow control, mixing,. . .

Two issues:

- Eulerian: what is the velocity field $v(x, y, t)$?
- Lagrangian: what is the position $(x(t), y(t))$ of fluid particles?

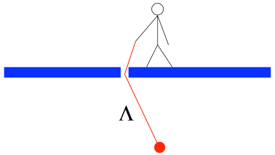
Finite-dimensional analogy: pendulum with variable length $\Lambda(\varepsilon t)$, $\varepsilon \ll 1$.

finite dimensions	fluids
amplitude	Eulerian flow
angle	particle position

Two issues:

- Eulerian: what is the velocity field $v(x, y, t)$?
- Lagrangian: what is the position $(x(t), y(t))$ of fluid particles?

Finite-dimensional analogy: pendulum with variable length $\Lambda(\varepsilon t)$, $\varepsilon \ll 1$.



finite dimensions	fluids
amplitude	Eulerian flow
angle	particle position

2D Euler

Vorticity–streamfunction formulation:

$$\partial_t \omega + [\psi, \omega] = 0, \quad [\psi, \omega] = \psi_x \omega_y - \psi_y \omega_x$$
$$\omega = \Delta \psi.$$

Velocity field: $(u, v) = \nabla^\perp \psi := (-\partial_y \psi, \partial_x \psi)$.

- Fluid-particle position is given by the diffeomorphism g_t,

$$\frac{\mathrm{d}}{\mathrm{d}t} g_t \mathbf{x} = \nabla^\perp \psi(g_t \mathbf{x}, t), \quad |\nabla g_t| = 1.$$

- Vorticity is rearranged by g_t: $\omega(\mathbf{x}, t) = \omega(g_t^{-1}\mathbf{x}, 0)$.
- Steady flows: $[\psi, \omega] = 0 \;\Rightarrow\; \psi = G(\omega)$.

2D Euler

Vorticity–streamfunction formulation:

$$\partial_t \omega + [\psi, \omega] = 0, \quad [\psi, \omega] = \psi_x \omega_y - \psi_y \omega_x$$
$$\omega = \Delta \psi.$$

Velocity field: $(u, v) = \nabla^\perp \psi := (-\partial_y \psi, \partial_x \psi)$.

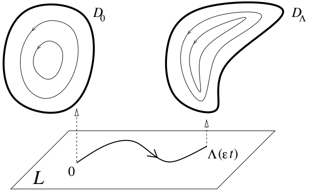
- Fluid-particle position is given by the diffeomorphism g_t,

$$\frac{\mathrm{d}}{\mathrm{d}t} g_t \mathbf{x} = \nabla^\perp \psi(g_t \mathbf{x}, t), \quad |\nabla g_t| = 1.$$

- Vorticity is rearranged by g_t: $\omega(\mathbf{x}, t) = \omega(g_t^{-1}\mathbf{x}, 0)$.
- Steady flows: $[\psi, \omega] = 0 \;\Rightarrow\; \psi = G(\omega)$.

Boundary conditions

Domain D_Λ, with fixed area, is defined by parameters $\Lambda = \Lambda(\varepsilon t)$ with $\varepsilon \ll 1$. Take $\Lambda(0) = 0$.



Domain $\partial D_{\Lambda(\varepsilon t)}$ can be defined implicitly: $B(\mathbf{x}; \Lambda(\varepsilon t)) = 0$.
Since ∂D_Λ is a material line

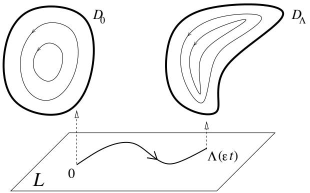
$$\partial_t B + [\psi, B] = 0 \qquad \text{for} \quad B(\mathbf{x}; \Lambda(\varepsilon t)) = 0.$$

Example: deformed disc

$$r = 1 + \delta \sum_m \Lambda_m e^{im\sigma} - \frac{\delta^2}{2} \sum_m |\Lambda_m|^2 + O(\delta^3).$$

Boundary conditions

Domain D_Λ, with fixed area, is defined by parameters $\Lambda = \Lambda(\varepsilon t)$ with $\varepsilon \ll 1$. Take $\Lambda(0) = 0$.



Domain $\partial D_{\Lambda(\varepsilon t)}$ can be defined implicitly: $B(\mathbf{x}; \Lambda(\varepsilon t)) = 0$.

Since ∂D_Λ is a material line

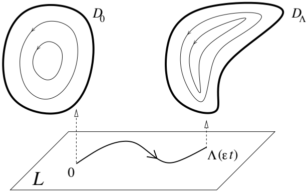
$$\partial_t B + [\psi, B] = 0 \qquad \text{for} \quad B(\mathbf{x}; \Lambda(\varepsilon t)) = 0.$$

Example: deformed disc

$$r = 1 + \delta \sum_m \Lambda_m e^{im\sigma} - \frac{\delta^2}{2} \sum_m |\Lambda_m|^2 + O(\delta^3).$$

Boundary conditions

Domain D_Λ, with fixed area, is defined by parameters $\Lambda = \Lambda(\varepsilon t)$ with $\varepsilon \ll 1$. Take $\Lambda(0) = 0$.



Domain $\partial D_{\Lambda(\varepsilon t)}$ can be defined implicitly: $B(\mathbf{x}; \Lambda(\varepsilon t)) = 0$.

Since ∂D_Λ is a material line

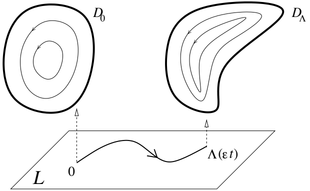
$$\partial_t B + [\psi, B] = 0 \qquad \text{for} \quad B(\mathbf{x}; \Lambda(\varepsilon t)) = 0.$$

Example: deformed disc

$$r = 1 + \delta \sum_m \Lambda_m \mathrm{e}^{im\sigma} - \frac{\delta^2}{2} \sum_m |\Lambda_m|^2 + O(\delta^3).$$

Boundary conditions

Domain D_Λ, with fixed area, is defined by parameters $\Lambda = \Lambda(\varepsilon t)$ with $\varepsilon \ll 1$. Take $\Lambda(0) = 0$.



Domain $\partial D_{\Lambda(\varepsilon t)}$ can be defined implicitly: $B(\mathbf{x}; \Lambda(\varepsilon t)) = 0$.

Since ∂D_Λ is a material line

$$\partial_t B + [\psi, B] = 0 \qquad \text{for} \quad B(\mathbf{x}; \Lambda(\varepsilon t)) = 0.$$

Example: deformed disc

$$r = 1 + \delta \sum_m \Lambda_m \mathrm{e}^{im\sigma} - \frac{\delta^2}{2} \sum_m |\Lambda_m|^2 + O(\delta^3).$$

Leading order

At $t = 0$, in domain D_0, assume that the flow is:

- steady: $\psi = G_0(\omega)$,
- with simple topology,
- H1: sufficiently fast,

$$\oint \frac{\mathrm{d}l}{|\nabla\psi|} \leq c.$$

- H2: Arnold stable:

$$-c_{\mathrm{poi}} < F_0' = 1/G_0' < \infty$$

where c_{poi} is the Poincaré constant of D_0.

Leading order

At $t = 0$, in domain D_0, assume that the flow is:

- steady: $\psi = G_0(\omega)$,
- with simple topology,
- H1: sufficiently fast,

$$\oint \frac{\mathrm{d}l}{|\nabla \psi|} \leq c.$$

- H2: Arnold stable:

$$-c_{\mathrm{poi}} < F_0' = 1/G_0' < \infty$$

where c_{poi} is the Poincaré constant of D_0.

Leading order

At $t = 0$, in domain D_0, assume that the flow is:

- steady: $\psi = G_0(\omega)$,
- with simple topology,
- H1: sufficiently fast,

$$\oint \frac{\mathrm{d}l}{|\nabla\psi|} \leq c.$$

- H2: Arnold stable:

$$-c_{\mathrm{poi}} < F_0' = 1/G_0' < \infty$$

where c_{poi} is the Poincaré constant of D_0.

Leading order

At $t = 0$, in domain D_0, assume that the flow is:

- steady: $\psi = G_0(\omega)$,
- with simple topology,
- H1: sufficiently fast,

$$\oint \frac{\mathrm{d}l}{|\nabla \psi|} \leq c.$$

- H2: Arnold stable:

$$-c_{\mathrm{poi}} < F_0' = 1/G_0' < \infty$$

where c_{poi} is the Poincaré constant of D_0.

Asymptotic expansion

For $t > 0$, the domain is deformed slowly: $D_\Lambda(\tau)$, $\tau = \varepsilon t$.

What are ψ and ω for $\tau = O(1)$, ie $t = O(\varepsilon^{-1})$?

Expand in powers of ε:

$$\omega = \omega_\Lambda + \varepsilon\,\omega^{(1)} + \varepsilon^2 \omega^{(2)} + \cdots, \qquad \psi = \psi_\Lambda + \varepsilon\,\psi^{(1)} + \varepsilon^2 \psi^{(2)} + \cdots.$$

At leading order, the flow is steady:

$$[\psi_\Lambda, \omega_\Lambda] = 0 \quad \Rightarrow \quad \psi_\Lambda = G_\Lambda(\omega_\Lambda).$$

At the next order, vorticity is transported:

$$\partial_\tau \omega_\Lambda + [\psi^{(1)}, \omega_\Lambda] + [\psi_\Lambda, \omega^{(1)}] = 0 \quad \Rightarrow \quad \partial_\tau \omega_\Lambda + [\phi, \omega_\Lambda] = 0,$$

where $\phi = [1 - G'_\Lambda \Delta]\,\psi^{(1)}$.

Asymptotic expansion

For $t > 0$, the domain is deformed slowly: $D_\Lambda(\tau)$, $\tau = \varepsilon t$.

What are ψ and ω for $\tau = O(1)$, ie $t = O(\varepsilon^{-1})$?
Expand in powers of ε:

$$\omega = \omega_\Lambda + \varepsilon\,\omega^{(1)} + \varepsilon^2\omega^{(2)} + \cdots\,, \qquad \psi = \psi_\Lambda + \varepsilon\,\psi^{(1)} + \varepsilon^2\psi^{(2)} + \cdots\,.$$

At leading order, the flow is steady:

$$[\psi_\Lambda, \omega_\Lambda] = 0 \quad \Rightarrow \quad \psi_\Lambda = G_\Lambda(\omega_\Lambda).$$

At the next order, vorticity is transported:

$$\partial_\tau\omega_\Lambda + [\psi^{(1)}, \omega_\Lambda] + [\psi_\Lambda, \omega^{(1)}] = 0 \quad \Rightarrow \quad \partial_\tau\omega_\Lambda + [\phi, \omega_\Lambda] = 0,$$

where $\phi = [1 - G'_\Lambda \Delta]\,\psi^{(1)}$.

Asymptotic expansion

For $t > 0$, the domain is deformed slowly: $D_\Lambda(\tau)$, $\tau = \varepsilon t$.

What are ψ and ω for $\tau = O(1)$, ie $t = O(\varepsilon^{-1})$?
Expand in powers of ε:

$$\omega = \omega_\Lambda + \varepsilon\,\omega^{(1)} + \varepsilon^2\omega^{(2)} + \cdots\,, \qquad \psi = \psi_\Lambda + \varepsilon\,\psi^{(1)} + \varepsilon^2\psi^{(2)} + \cdots\,.$$

At leading order, the flow is steady:

$$[\psi_\Lambda, \omega_\Lambda] = 0 \quad \Rightarrow \quad \psi_\Lambda = G_\Lambda(\omega_\Lambda).$$

At the next order, vorticity is transported:

$$\partial_\tau\omega_\Lambda + [\psi^{(1)}, \omega_\Lambda] + [\psi_\Lambda, \omega^{(1)}] = 0 \quad \Rightarrow \quad \partial_\tau\omega_\Lambda + [\phi, \omega_\Lambda] = 0,$$

where $\phi = [1 - G'_\Lambda\Delta]\,\psi^{(1)}$.

Asymptotic expansion

For $t > 0$, the domain is deformed slowly: $D_\Lambda(\tau)$, $\tau = \varepsilon t$.

What are ψ and ω for $\tau = O(1)$, ie $t = O(\varepsilon^{-1})$?
Expand in powers of ε:

$$\omega = \omega_\Lambda + \varepsilon\,\omega^{(1)} + \varepsilon^2 \omega^{(2)} + \cdots, \qquad \psi = \psi_\Lambda + \varepsilon\,\psi^{(1)} + \varepsilon^2 \psi^{(2)} + \cdots.$$

At leading order, the flow is steady:

$$[\psi_\Lambda, \omega_\Lambda] = 0 \quad \Rightarrow \quad \psi_\Lambda = G_\Lambda(\omega_\Lambda).$$

At the next order, vorticity is transported:

$$\partial_\tau \omega_\Lambda + [\psi^{(1)}, \omega_\Lambda] + [\psi_\Lambda, \omega^{(1)}] = 0 \quad \Rightarrow \quad \partial_\tau \omega_\Lambda + [\phi, \omega_\Lambda] = 0,$$

where $\phi = [1 - G'_\Lambda \Delta]\, \psi^{(1)}$.

Computing ψ_Λ et ω_Λ: formulation in terms of a diffeomorphism g_Λ

réarranging ω_Λ:

- $|\nabla g_\Lambda| = 1$,
- $g_\Lambda D_0 = D_\Lambda$,
- $\omega_\Lambda = \omega_0 \circ g_\Lambda^{-1} = (g_\Lambda^{-1})^* \omega_0$.

It follows that

$$\psi_\Lambda = G_\Lambda(\omega_\Lambda) \;\Rightarrow\; \omega_\Lambda = \Delta G_\Lambda(\omega_\Lambda) \;\Rightarrow\; \omega_0 = g_\Lambda^* \Delta (g_\Lambda^{-1})^* (G_\Lambda \circ \omega_0),$$

a nonlinear PDE for g_Λ and G_Λ.

For sufficiently small domain deformations and for $\psi_0 \in C^{k,\alpha}$, $k \geq 3$: contraction mapping argument shows

- existence of a solution g_Λ, unique modulo translation along contours of ω_0 (V & Wirosoetisno 2005),
- $g_\Lambda, \psi_\Lambda, \omega_\Lambda$ depend on Λ instantaneously.

Computing ψ_Λ et ω_Λ: formulation in terms of a diffeomorphism g_Λ

réarranging ω_Λ:

- $|\nabla g_\Lambda| = 1$,
- $g_\Lambda D_0 = D_\Lambda$,
- $\omega_\Lambda = \omega_0 \circ g_\Lambda^{-1} = (g_\Lambda^{-1})^* \omega_0$.

It follows that

$$\psi_\Lambda = G_\Lambda(\omega_\Lambda) \;\Rightarrow\; \omega_\Lambda = \Delta G_\Lambda(\omega_\Lambda) \;\Rightarrow\; \omega_0 = g_\Lambda^* \Delta (g_\Lambda^{-1})^* (G_\Lambda \circ \omega_0),$$

a nonlinear PDE for g_Λ and G_Λ.

For sufficiently small domain deformations and for $\psi_0 \in C^{k,\alpha}$, $k \geq 3$: contraction mapping argument shows

- existence of a solution g_Λ, unique modulo translation along contours of ω_0 (V & Wirosoetisno 2005),
- $g_\Lambda, \psi_\Lambda, \omega_\Lambda$ depend on Λ instantaneously.

Computing ψ_Λ et ω_Λ: formulation in terms of a diffeomorphism g_Λ

réarranging ω_Λ:

- $|\nabla g_\Lambda| = 1$,
- $g_\Lambda D_0 = D_\Lambda$,
- $\omega_\Lambda = \omega_0 \circ g_\Lambda^{-1} = (g_\Lambda^{-1})^* \omega_0$.

It follows that

$$\psi_\Lambda = G_\Lambda(\omega_\Lambda) \;\Rightarrow\; \omega_\Lambda = \Delta G_\Lambda(\omega_\Lambda) \;\Rightarrow\; \omega_0 = g_\Lambda^* \Delta (g_\Lambda^{-1})^* (G_\Lambda \circ \omega_0),$$

a nonlinear PDE for g_Λ and G_Λ.

For sufficiently small domain deformations and for $\psi_0 \in C^{k,\alpha}$, $k \geq 3$: contraction mapping argument shows

- existence of a solution g_Λ, unique modulo translation along contours of ω_0 (V & Wirosoetisno 2005),
- $g_\Lambda, \psi_\Lambda, \omega_\Lambda$ depend on Λ instantaneously.

Computing ψ_Λ et ω_Λ: formulation in terms of a diffeomorphism g_Λ réarranging ω_Λ:

- $|\nabla g_\Lambda| = 1$,
- $g_\Lambda D_0 = D_\Lambda$,
- $\omega_\Lambda = \omega_0 \circ g_\Lambda^{-1} = (g_\Lambda^{-1})^* \omega_0$.

It follows that

$$\psi_\Lambda = G_\Lambda(\omega_\Lambda) \;\Rightarrow\; \omega_\Lambda = \Delta G_\Lambda(\omega_\Lambda) \;\Rightarrow\; \omega_0 = g_\Lambda^* \Delta (g_\Lambda^{-1})^* (G_\Lambda \circ \omega_0),$$

a nonlinear PDE for g_Λ and G_Λ.

For sufficiently small domain deformations and for $\psi_0 \in C^{k,\alpha}$, $k \geq 3$: contraction mapping argument shows

- existence of a solution g_Λ, unique modulo translation along contours of ω_0 (V & Wirosoetisno 2005),
- $g_\Lambda, \psi_\Lambda, \omega_\Lambda$ depend on Λ instantaneously.

Computing ψ_Λ et ω_Λ: formulation in terms of a diffeomorphism g_Λ réarranging ω_Λ:

- $|\nabla g_\Lambda| = 1$,
- $g_\Lambda D_0 = D_\Lambda$,
- $\omega_\Lambda = \omega_0 \circ g_\Lambda^{-1} = (g_\Lambda^{-1})^* \omega_0$.

It follows that

$$\psi_\Lambda = G_\Lambda(\omega_\Lambda) \;\Rightarrow\; \omega_\Lambda = \Delta G_\Lambda(\omega_\Lambda) \;\Rightarrow\; \omega_0 = g_\Lambda^* \Delta (g_\Lambda^{-1})^* (G_\Lambda \circ \omega_0),$$

a nonlinear PDE for g_Λ and G_Λ.

For sufficiently small domain deformations and for $\psi_0 \in C^{k,\alpha}$, $k \geq 3$: contraction mapping argument shows

- existence of a solution g_Λ, unique modulo translation along contours of ω_0 (V & Wirosoetisno 2005),
- $g_\Lambda, \psi_\Lambda, \omega_\Lambda$ depend on Λ instantaneously.

Deformed disc

Example: axisymmetric flow

$$\psi^{(0)} = \psi^{(0)}(r),$$

is the deformed disc:

$$r = 1 + \delta \sum_m \Lambda_m \mathrm{e}^{\mathrm{i}m\sigma} - \frac{\delta^2}{2} \sum_m |\Lambda_m|^2 + O(\delta^3).$$

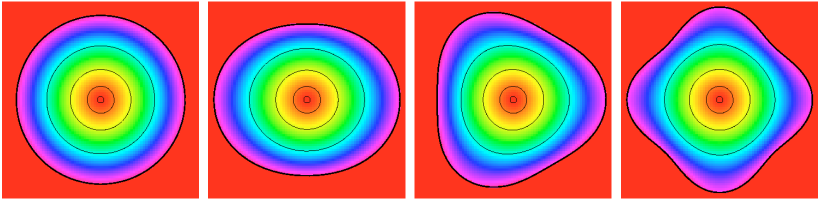
Deformed disc

Example: axisymmetric flow

$$\psi^{(0)} = \psi^{(0)}(r),$$

is the deformed disc:

$$r = 1 + \delta \sum_m \Lambda_m \mathrm{e}^{\mathrm{i}m\sigma} - \frac{\delta^2}{2} \sum_m |\Lambda_m|^2 + O(\delta^3).$$



$\psi^{(0)} = r^{1/2}$, $\delta = 0.05$.

Infinitesimal deformation

Let $\mathbf{d}$ denote the differential in parameter space. Consider

$$\frac{\mathrm{d}}{\mathrm{d}\tau} g_\Lambda = \mathbf{d} g_\Lambda \cdot \dot{\Lambda} =: (\boldsymbol{\nabla}^\perp \Phi \circ g_\Lambda) \cdot \dot{\Lambda}.$$

This defines $\Phi = \Phi_1 \mathrm{d}\Lambda_1 + \Phi_2 \mathrm{d}\Lambda_2 + \cdots$:

- function-valued 1-form over parameter space,
  $\Phi : T\mathcal{L} \to C(D_\Lambda)$, ie, $\Phi \cdot \dot{\Lambda} : D_\Lambda \to \mathbb{R}$,
- the function represents a vector field through $\boldsymbol{\nabla}^\perp$,
- Φ is a connection form.

Infinitesimal deformations

The differential of $\omega_0 = g_\Lambda^* \Delta (g_\Lambda^{-1})^* (G_\Lambda \circ \omega_0)$ gives

$$(\Delta - F'_\Lambda)\,[\psi_\Lambda, \Phi] + \Delta(\mathbf{d} G_\Lambda \circ \omega_\Lambda) = 0.$$

For given ψ_Λ and ω_Λ satisfying H1–H2, this equation has a solution Φ unique, modulo addition of an arbitrary 1-form $\Pi \circ \omega_\Lambda$.

The curvature associated with Φ satisfies

$$\mathbf{d}\Phi + \tfrac{1}{2}[\Phi \wedge \Phi] = w \circ \omega_\Lambda,$$

where w is an arbitrary 2-form. This expresses the instantaneous dependence of g_Λ on Λ (modulo translation along contours of ω_Λ).

Infinitesimal deformations

The differential of $\omega_0 = g_\Lambda^* \Delta (g_\Lambda^{-1})^* (G_\Lambda \circ \omega_0)$ gives

$$(\Delta - F'_\Lambda)\,[\psi_\Lambda, \Phi] + \Delta(\mathbf{d}G_\Lambda \circ \omega_\Lambda) = 0.$$

For given ψ_Λ and ω_Λ satisfying H1–H2, this equation has a solution Φ unique, modulo addition of an arbitrary 1-form $\Pi \circ \omega_\Lambda$.

The curvature associated with Φ satisfies

$$\mathbf{d}\Phi + \tfrac{1}{2}[\Phi \wedge \Phi] = w \circ \omega_\Lambda,$$

where w is an arbitrary 2-form. This expresses the instantaneous dependence of g_Λ on Λ (modulo translation along contours of ω_Λ).

Fluid-particle position

The position of fluid particles is governed by the Hamiltonian system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{\partial\psi}{\partial y} \quad \text{et} \quad \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\partial\psi}{\partial x},$$

For $t = O(\varepsilon^{-1})$, the Hamiltonian can be taken as

$$\psi = \psi_{\Lambda} + \varepsilon\psi^{(1)}.$$

Necessary to compute $\psi^{(1)}$.

The information needed is encoded in Φ!

Fluid-particle position

The position of fluid particles is governed by the Hamiltonian system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{\partial \psi}{\partial y} \quad \text{et} \quad \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\partial \psi}{\partial x},$$

For $t = O(\varepsilon^{-1})$, the Hamiltonian can be taken as

$$\psi = \psi_\Lambda + \varepsilon \psi^{(1)}.$$

Necessary to compute $\psi^{(1)}$.

The information needed is encoded in Φ!

Recall that

$$\partial_\tau \omega_\Lambda + [(1 - G'_\Lambda \Delta)\, \psi^{(1)}, \omega_\Lambda] = 0,$$

and

$$\mathbf{d}\omega_\Lambda + [\Phi, \omega_\Lambda] = 0 \;\Rightarrow\; \partial_\tau \omega_\Lambda + [\Phi, \omega_\Lambda] \cdot \dot{\Lambda} = 0,$$

Identifying gives

$$[1 - G'_\Lambda \Delta]\, \psi^{(1)} = (\Phi + \Pi \circ \omega_\Lambda) \cdot \dot{\Lambda} =: \Phi^\star \cdot \dot{\Lambda},$$

where the 1-form Π remains to be determined.

The connection $\Phi^\star$

- differs from other Φ by a function of ω_Λ,
- is natural through its link with $\psi^{(1)}$,
- is unique, independent of the choice for Φ.

Recall that

$$\partial_\tau \omega_\Lambda + [(1 - G'_\Lambda \Delta)\, \psi^{(1)}, \omega_\Lambda] = 0,$$

and

$$\mathbf{d}\omega_\Lambda + [\Phi, \omega_\Lambda] = 0 \;\Rightarrow\; \partial_\tau \omega_\Lambda + [\Phi, \omega_\Lambda] \cdot \dot{\Lambda} = 0,$$

Identifying gives

$$[1 - G'_\Lambda \Delta]\, \psi^{(1)} = (\Phi + \Pi \circ \omega_\Lambda) \cdot \dot{\Lambda} =: \Phi^\star \cdot \dot{\Lambda},$$

where the 1-form Π remains to be determined.

The connection $\Phi^\star$

- differs from other Φ by a function of ω_Λ,
- is natural through its link with $\psi^{(1)}$,
- is unique, independent of the choice for Φ.

To determine $\psi^{(1)}$ and $\Phi^\star$, invoke vorticity conservation to 2nd order:

$$\int_{\omega_\Lambda+\varepsilon\omega^{(1)}+\cdots<\Omega} \mathrm{d}^2\mathbf{x} = \int_{\omega_\Lambda<\Omega} \mathrm{d}^2\mathbf{x} \;\Rightarrow\; \oint_{\omega_\Lambda=\Omega} \omega^{(1)} \frac{\mathrm{d}l}{|\nabla\psi_\Lambda|} = 0.$$

Hence

$$\oint_{\omega_\Lambda=\Omega} \Delta\psi^{(1)} \frac{\mathrm{d}l}{|\nabla\psi_\Lambda|} = 0.$$

Trajectories

Once $\psi^{(1)}$ is determined, one can solve

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{\partial \psi}{\partial y} \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\partial \psi}{\partial x}, \quad \psi = \psi_\Lambda + \varepsilon \psi^{(1)}.$$

Doubly perturbed Hamiltonian system:

- slow variation of the parameters $\psi_\Lambda = \psi_\Lambda(\Lambda(\varepsilon t))$,
- additive perturbation $\varepsilon \psi^{(1)}$.

Standard analysis: use action–angle variables, $(x, y) \mapsto (I, \theta)$, with

- $I = I(\omega_\Lambda)$: area inside vorticity contour ω_Λ,
- θ conjugate variable.

This gives:

$$\frac{\mathrm{d}I}{\mathrm{d}t} = -\varepsilon \frac{\partial}{\partial \theta}(\cdots) \quad \Rightarrow \quad \Delta I = O(\varepsilon):$$

vorticity conservation.

Trajectories

Once $\psi^{(1)}$ is determined, one can solve

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{\partial \psi}{\partial y} \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\partial \psi}{\partial x}, \quad \psi = \psi_\Lambda + \varepsilon \psi^{(1)}.$$

Doubly perturbed Hamiltonian system:

- slow variation of the parameters $\psi_\Lambda = \psi_\Lambda(\Lambda(\varepsilon t))$,
- additive perturbation $\varepsilon \psi^{(1)}$.

Standard analysis: use action–angle variables, $(x, y) \mapsto (I, \theta)$, with

- $I = I(\omega_\Lambda)$: area inside vorticity contour ω_Λ,
- θ conjugate variable.

This gives:

$$\frac{\mathrm{d}I}{\mathrm{d}t} = -\varepsilon \frac{\partial}{\partial \theta}(\cdots) \quad \Rightarrow \quad \Delta I = O(\varepsilon):$$

vorticity conservation.

Trajectories

Once $\psi^{(1)}$ is determined, one can solve

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{\partial\psi}{\partial y} \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\partial\psi}{\partial x}, \quad \psi = \psi_\Lambda + \varepsilon\psi^{(1)}.$$

Doubly perturbed Hamiltonian system:

- slow variation of the parameters $\psi_\Lambda = \psi_\Lambda(\Lambda(\varepsilon t))$,
- additive perturbation $\varepsilon\psi^{(1)}$.

Standard analysis: use action–angle variables, $(x, y) \mapsto (I, \theta)$, with

- $I = I(\omega_\Lambda)$: area inside vorticity contour ω_Λ,
- θ conjugate variable.

This gives:

$$\frac{\mathrm{d}I}{\mathrm{d}t} = -\varepsilon\frac{\partial}{\partial\theta}(\cdots) \quad \Rightarrow \quad \Delta I = O(\varepsilon):$$

vorticity conservation.

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\partial \psi_\Lambda}{\partial I} + \varepsilon \frac{\partial}{\partial I}\{\cdots\} \cdot \dot{\Lambda} \quad \Rightarrow \quad \Delta\theta = \Delta\theta_{\mathrm{dyn}} + \Delta\theta_{\mathrm{geo}}$$

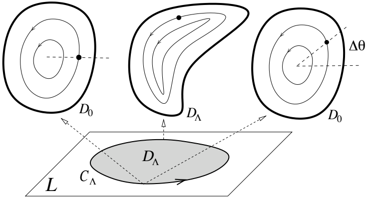
where

- the dynamic angle is

$$\Delta\theta_{\mathrm{dyn}} = \frac{1}{\varepsilon}\frac{\partial}{\partial I}\int_0^\tau \psi_{\Lambda(\tau')}(I)\,\mathrm{d}\tau' = O(\varepsilon^{-1}),$$

- the geometric angle $\Delta\theta_{\mathrm{geo}}$ is defined unambiguously for cyclic deformations of D_Λ.

Geometric angle

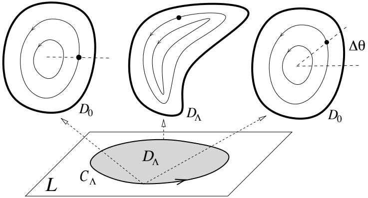


Averaging gives:

$$\Delta\theta_{\mathrm{geo}} = \frac{\mathrm{d}}{\mathrm{d}I} \int_{\mathcal{S}_\Lambda} \mathbf{d}\Phi^\star + \tfrac{1}{2}[\Phi^\star \wedge \Phi^\star].$$

- $\Delta\theta_{\mathrm{geo}}$ is the integral in parameter space of the curvature of the natural connection $\Phi^\star$ (cf Hannay & Berry).
- $\Delta\theta_{\mathrm{geo}}$ depends only on ω_Λ and hence on I, and is independent of the speed of deformation.

Geometric angle



Averaging gives:

$$\Delta\theta_{\mathrm{geo}} = \frac{\mathrm{d}}{\mathrm{d}I}\int_{\mathcal{S}_\Lambda} \mathbf{d}\Phi^\star + \tfrac{1}{2}[\Phi^\star \wedge \Phi^\star].$$

- $\Delta\theta_{\mathrm{geo}}$ is the integral in parameter space of the curvature of the natural connection $\Phi^\star$ (cf Hannay & Berry).
- $\Delta\theta_{\mathrm{geo}}$ depends only on ω_Λ and hence on I, and is independent of the speed of deformation.

Deformed disc

To leading order in δ,

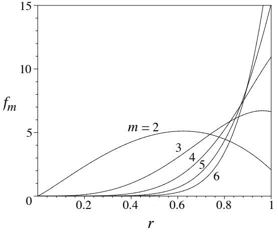
$$\Delta\theta_{\mathrm{geo}} = \delta^2 \sum_{m>0} f_m(r)\mathcal{A}_m + O(\delta^3),$$

where

$$\mathcal{A}_m = -\frac{\mathrm{i}}{2}\int_{\mathcal{D}_\Lambda} \mathbf{d}\Lambda_m \wedge \mathbf{d}\Lambda_m^*$$

is the area enclosed by Λ_m in $\mathbb{C}$.

The $f_m(r)$ can be calculated by solving 2nd-order ODEs.



$$\psi^{(0)} = r^{1/2}.$$

Conclusions

- A procedure to compute the quasi-steady 2D flows in slowly deforming domains.
- To leading order,
  - Eulerian fields ψ et ω depend on domain shape instantaneously, independent of deformation history,
  - particle positions depend on deformation history in a geometric manner.
- Key variable: connection $\Phi : T\mathcal{L} \to C(\mathbb{R}^2)$, defined up to arbitrary function of vorticity, and its natural version, $\Phi^\star$, related to 2nd order flow fields.

Conclusions

- A procedure to compute the quasi-steady 2D flows in slowly deforming domains.
- To leading order,
  - Eulerian fields ψ et ω depend on domain shape instantaneously, independent of deformation history,
  - particle positions depend on deformation history in a geometric manner.
- Key variable: connection $\Phi : T\mathcal{L} \to C(\mathbb{R}^2)$, defined up to arbitrary function of vorticity, and its natural version, $\Phi^\star$, related to 2nd order flow fields.

Conclusions

- A procedure to compute the quasi-steady 2D flows in slowly deforming domains.
- To leading order,
  - Eulerian fields ψ et ω depend on domain shape instantaneously, independent of deformation history,
  - particle positions depend on deformation history in a geometric manner.
- Key variable: connection $\Phi : T\mathcal{L} \to C(\mathbb{R}^2)$, defined up to arbitrary function of vorticity, and its natural version, $\Phi^\star$, related to 2nd order flow fields.

Conclusions

- A procedure to compute the quasi-steady 2D flows in slowly deforming domains.
- To leading order,
  - Eulerian fields ψ et ω depend on domain shape instantaneously, independent of deformation history,
  - particle positions depend on deformation history in a geometric manner.
- Key variable: connection $\Phi : T\mathcal{L} \to C(\mathbb{R}^2)$, defined up to arbitrary function of vorticity, and its natural version, $\Phi^{\star}$, related to 2nd order flow fields.

Open problems

- What happens when *H2* ceases to hold:

$$\oint \frac{\mathrm{d}l}{|\nabla \psi|} = O(\varepsilon^{-1})?$$

  Formation of separatrices: non-parallel critical layers.

- 3D version?

- Geometric interpretation:
  - G = group of area-preserving diffeomorphisms of $\mathbb{R}^2$,
  - $H \subset G$ = diffeomorphisms leaving D_0 invariant,
  - $H_0 \subset H$ = diffeomorphisms leaving ω_0,
  - $G/H = \mathcal{L}$ = set of possible domain shapes,
  - G/H_0 = set of possible rearrangements of the vorticity $\omega_\Lambda = g_\Lambda^* \omega_0$.
  - G and G/H_0 are fibre bundles with base G/H.
- The computation of ω_Λ defines a section of G/H_0.
- $\Phi^\star$ is a connection on G.