Gradient flows and diffusion semigroups in metric spaces under lower curvature bounds

Giuseppe Savaré
http://www.imati.cnr.it/∼savare

Department of Mathematics, University of Pavia, Italy

Edinburgh, *July 19, 2007*

to appear on C.R.A.S.

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2. The (strongest) metric formulation of gradient flows
3. Generation of gradient flows: the variational approach
4. Semi convexity-concavity of the squared distance and curvature

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4. Semi convexity-concavity of the squared distance and curvature
5. Triangle comparison and angle conditions
6. Main results and open problems

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Wasserstein distance: preliminaries

(X, d) (complete, separable) metric space.

- **Wasserstein space:**

$$\mathscr{X} = \mathscr{P}_2(X) := \Big\{\mu \text{ Borel probability measures on } X\Big\} \text{ s.t.}$$
$$\int_X \mathsf{d}^2(x, o)\,\mathrm{d}\mu(x) < +\infty\Big\}.$$

- **Plans-couplings** $\mu_1, \mu_2 \in \mathscr{P}_2(X)$: measures $\boldsymbol{\mu} \in \mathscr{P}_2(X \times X)$ whose marginals are μ_1, μ_2, i.e.

$$\pi^1_\# \boldsymbol{\mu} = \mu_1, \quad \pi^2_\# \boldsymbol{\mu} = \mu_2.$$

$\Gamma(\mu_1, \mu_2)$ is the collection of all couplings of μ_1 and μ_2.

- **Wasserstein distance:**

$$\mathsf{d}^2_{\mathscr{X}}(\mu^1, \mu^2)\mathsf{d}^2_W(\mu^1, \mu^2) := \min\Big\{\int_{X\times X} \mathsf{d}^2(x, y)\,\mathrm{d}\boldsymbol{\mu}(x, y) : \boldsymbol{\mu} \in \Gamma(\mu_1, \mu_2)\Big\}.$$

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- **Relative Entropy:** If $\mu, \gamma \in \mathscr{P}(X)$

$$\mathrm{Ent}(\mu|\gamma) := \begin{cases} \displaystyle\int \rho \log \rho\,\mathrm{d}\gamma & \text{if } \mu = \rho \cdot \gamma \ll \gamma \\ +\infty & \text{otherwise.} \end{cases}$$

Example 1: Gradient flow in the Euclidean space

$X = \mathbb{R}^N$ with the Euclidean distance, $\gamma = e^{-V}\,\mathrm{d}x$

where $V : \mathbb{R}^N \to (-\infty, +\infty]$ is $\boldsymbol{\lambda}$-convex potential, i.e.

$$D^2 V(x)\,\boldsymbol{\xi}\cdot\boldsymbol{\xi} \geq \boldsymbol{\lambda}|\boldsymbol{\xi}|^2 \qquad \forall\, x, \boldsymbol{\xi} \in \mathbb{R}^N.$$

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Jordan-Kinderlehrer-Otto ['98] showed that the gradient flow of ϕ in $\mathscr{P}_2(\mathbb{R}^N)$ is related to the **Fokker-Plank equation**

$$\boxed{\partial_t u - \nabla\cdot\big(\nabla u + u\nabla V\big) = 0 \quad \mu = u\,\mathrm{d}x.}$$

Example 2: Gaussian measures in infinite dimensional Hilbert spaces

$X = H$ is an **Hilbert** space, γ is a **Gaussian** measure.

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the evolution equation in terms of the density $\rho = \mathrm{d}\mu/\mathrm{d}\gamma$,

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Its variational formulation reads

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\int_H \rho_t \zeta \, \mathrm{d}\gamma + \int_H (\nabla \rho_t, \nabla \zeta)_H \, \mathrm{d}\gamma = 0}$$

for every *smooth cylindrical* function $\zeta : H \to \mathbb{R}$.

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More generally, γ could be any **log-concave probability measure** [Ambrosio-S.-Zambotti '07]

$$\gamma(\theta A + (1-\theta)B) \geq \gamma(A)^\theta \gamma(B)^{1-\theta} \quad \forall\, A, B \text{ open subset of } H, \quad \theta \in [0,1].$$

Applications to stochastic perturbations of evolutionary PDE's.

Example 3: Smooth Riemannian manifolds

$X = (M, g)$ is a N-dimensional smooth compact **Riemannian manifold,**
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A common feature: displacement convexity

[McCann '97] A functional $\phi : \mathscr{P}_2(X) \to (-\infty, +\infty]$ is **displacement convex** if each couple of measures $\mu_0, \mu_1 \in D(\phi)$ can be connected by an **optimal transport plan** $\boldsymbol{\mu} \in \Gamma_o(\mu_0, \mu_1)$ such that

$$\phi(\mu_t) \le (1-t)\phi(\mu_0) + t\phi(\mu_1) \qquad \forall\, t \in (0,1).$$

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3. **Riemannian Manifolds**

$$\phi \quad \text{is } \boldsymbol{\lambda}\text{-convex in } \mathscr{X} \quad \Leftrightarrow \quad \mathrm{Ric}_X(\xi, \xi) \ge \boldsymbol{\lambda}|\xi|^2,$$

where Ric is the **Ricci curvature on X**
[von Rennesse-Sturm '04] [Otto-Villani '00],
[Cordero Erausquin-McCann-Schmuckenschläger '01]

A further generalization: metric measure spaces

METRIC MEASURE SPACE (X, d, γ): is a

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Find "minimal" conditions on X which can be also applied to $\mathscr{X} = \mathscr{P}_2(X)$.

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Metric spaces

- (X, d_X) is a (complete, separable) metric space.
- **GEODESIC** (constant speed, minimizing): it is a curve $\mathbf{x} : [0,1] \to X$ s.t.

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- **CONVEXITY**: each couple of points $x_0, x_1 \in D(\phi)$ can be connected by a geodesic $\mathbf{x} \in G(x_0 \to x_1)$ such that

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Gradient flow: evolution variational inequalities

GRADIENT FLOW: an (absolutely continuous) curve $\mathbf{u} : [0, +\infty) \to X$ starting from $\mathbf{u_0}$ such that

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Exponential decay: in the case $\boldsymbol{\lambda} > 0$ ϕ has a unique minimum point $\bar{u}$

$$\mathrm{d}(\mathbf{u}_t, \bar{u}) \le \boxed{e^{-\boldsymbol{\lambda} t}}\mathrm{d}(u_0, \bar{u}),$$

$$\phi(\mathbf{u}_t) - \phi(\bar{u}) \le \boxed{e^{-2\boldsymbol{\lambda} t}}\big(\phi(u_0) - \phi(\bar{u})\big), \quad |\partial\phi|(\mathbf{u}_t) \le \boxed{e^{-\boldsymbol{\lambda} t}}|\partial\phi|(u_0).$$

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This metric approach is modeled on the results for

convex (or $\boldsymbol{\lambda}$-convex) functionals in Euclidean/Hilbert spaces

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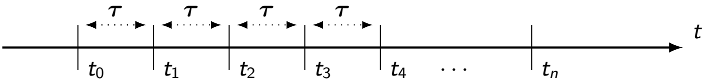
Outline

Minimizing movement method

- Choose a partition of $(0, +\infty)$ of **step size** $\boldsymbol{\tau} > 0$



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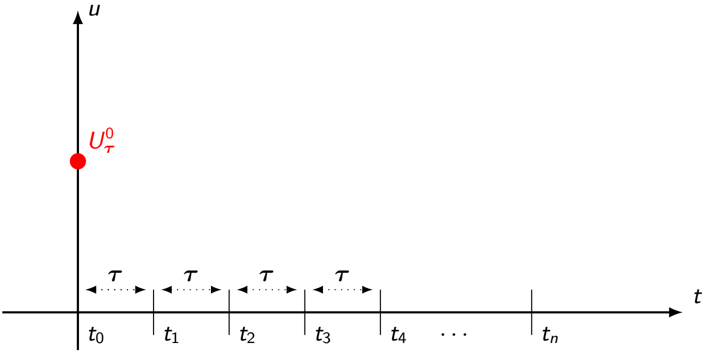

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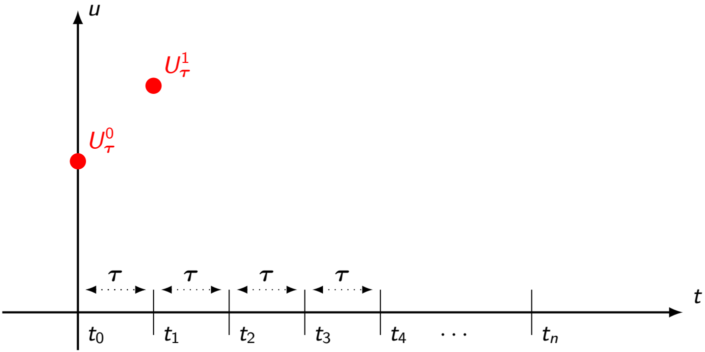

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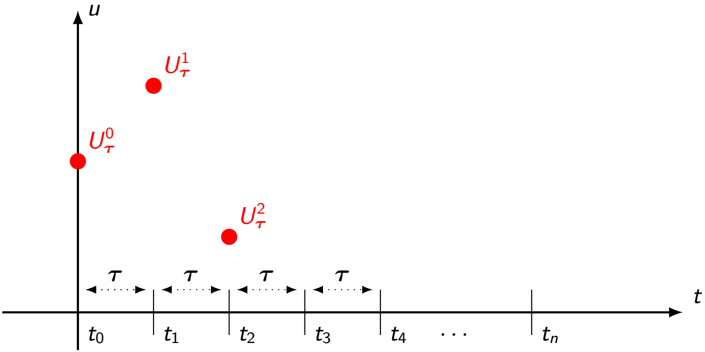

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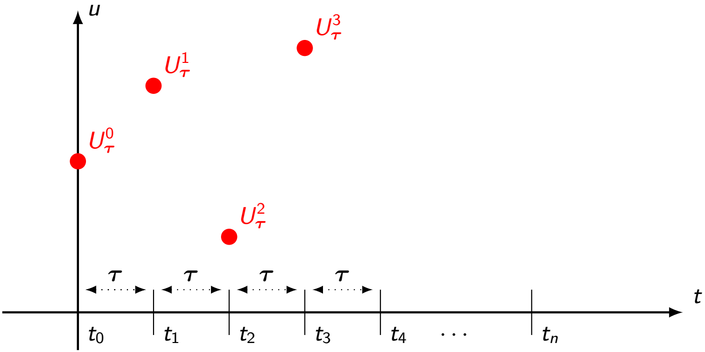

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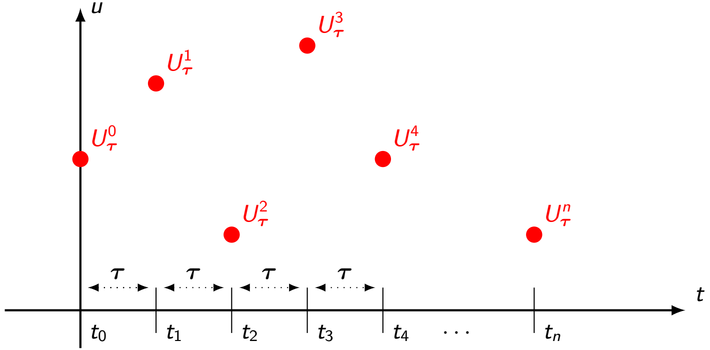

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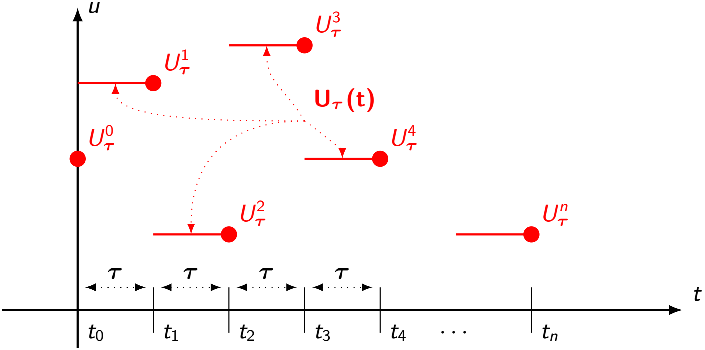

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- $\mathbf{U}_\tau$ is the **piecewise constant** (or **geodesic**) interpolant of $\{U_\tau^n\}_n$.
  We look for **convergence results of $\mathbf{U}_\tau$ as $\tau \downarrow 0$.**

Avoiding compactness by Ekeland variational principle

Let us only assume that

If X is complete and ϕ is lower semicontinuous

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(*Discrete energy inequality*)

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Pass then to the limit in $U_{\tau,\eta}$ as $\tau \to 0, \eta \to 0$.

Possible applications

BREZIS, CRANDALL, BÉNILAN, PAZY,... $\sim$'70	*Contraction semigroups in Hilbert spaces, quasilinear parabolic P.D.E.'s, variational inequalities,...*
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Otto, Jordan, Kinderlehrer, Walkington, Agueh, Ghossoub, Carrillo-McCann-Villani, Ambrosio-Gigli-S., ... '98∼'06	*Diffusion equations, Wasserstein distance*

In general only **convergence results possibly up to subsequences** are known...

Two different directions...

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2. **The Hilbert-like theory**: it is modeled on the results for

   convex (or $\boldsymbol{\lambda}$-convex) functionals in Euclidean/Hilbert spaces

   and gives the strongest results under restrictive assumptions on the

   - functional ϕ $\rightsquigarrow$ **"convexity"**
   - space $\rightsquigarrow$ **"Euclidean like"**

Here we focus on the second case.

Hilbert space: contraction semigroup and E.V.I.

Uniform Cauchy/error estimates

$$\mathsf{d}^2\big(\mathbf{U}_\sigma(\mathbf{t}), \mathbf{U}_\tau(\mathbf{t})\big) \le \frac{1}{2}(\boldsymbol{\tau}^2 + \boldsymbol{\sigma}^2)|\partial\phi|^2(u_0).$$

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Contraction semigroups, Regularizing effect

Energy identity, Exponential decay...

Main problem

Two main points:

1. **The convergence of the variational method $\mathbf{U}_{\tau,\eta} \to \mathbf{u}$**
   (at least starting from a dense subset D of $D(\phi)$)
2. **$\mathbf{u}$ solves the E.V.I. formulation:**

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The main problem... and the basic requirements

- Find a class of sufficient **general conditions** $\mathscr{C}$ on the metric space X and its distance d yielding **1. and 2.**
- The case of the **Wasserstein space** $\mathscr{X} = \mathscr{P}_2(X)$ should be reduced to analogous properties on the underlying space X, i.e.

$$X \text{ satisfies } \mathscr{C} \quad \Leftrightarrow \quad \mathscr{X} = \mathscr{P}_2(X) \text{ satisfies } \mathscr{C}.$$

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   (at least starting from a dense subset D of $D(\phi)$)
2. **u solves the E.V.I. formulation:**

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\mathsf{d}^2(\mathbf{u}_t, v) \le \phi(v) - \phi(\mathbf{u}_t) \quad \forall\, v \in D(\phi).}$$

All the other properties of the "Euclidean" generation result follow from these ones [Ambrosio-S. ('06)]

The main problem... and the basic requirements

- Find a class of sufficient **general conditions** $\mathscr{C}$ on the metric space X and its distance d yielding **1. and 2.**
- The case of the **Wasserstein space** $\mathscr{X} = \mathscr{P}_2(X)$ should be reduced to analogous properties on the underlying space X, i.e.

$$X \text{ satisfies } \mathscr{C} \quad \Leftrightarrow \quad \mathscr{X} = \mathscr{P}_2(X) \text{ satisfies } \mathscr{C}.$$

- Possibly **avoid compactness** assumptions on X.

Outline

1. Examples of gradient flows in Wasserstein spaces
2. The (strongest) metric formulation of gradient flows
3. Generation of gradient flows: the variational approach
4. **Semi convexity-concavity of the squared distance and curvature**
5. Triangle comparison and angle conditions
6. Main results and open problems

Convexity properties of the distance

The squared distance enters in a crucial way in the minimizing functional

$$\Phi(\tau, u; \mathbf{v}) := \frac{1}{2\tau} \mathrm{d}^2(u, \mathbf{v}) + \phi(\mathbf{v}).$$

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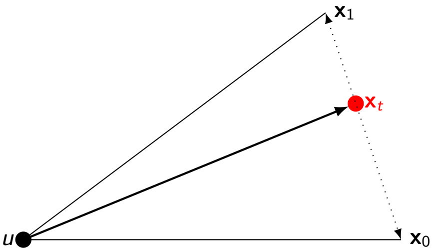

$$\mathrm{d}^2(u, \mathbf{x_t}) = (1 - \mathbf{t})\mathrm{d}^2(u, \mathbf{x}_0) + \mathbf{t}\mathrm{d}^2(u, \mathbf{x}_1) - \mathbf{t(1 - t)}\mathrm{d}^2(\mathbf{x}_0, \mathbf{x}_1).$$

The easiest case: Alexandrov's NPC spaces

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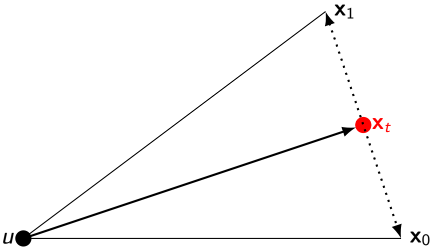

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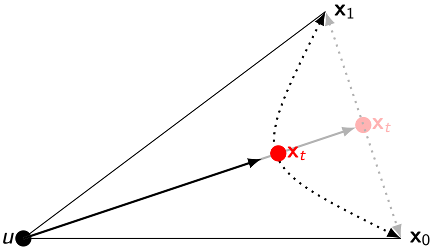
$\Downarrow$ **1-convexity**

$$d^2(u, x_t) \le (1-t) d^2(u, x_0) + t d^2(u, x_1) \quad - \boxed{t(1-t) d^2(x_0, x_1).}$$



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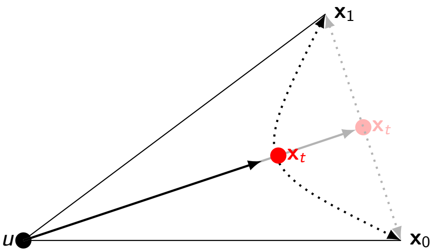

The functional $\Phi(\tau, u; v) := \dfrac{1}{2\tau} d^2(u, v) + \phi(v)$ is $\boxed{\dfrac{1}{\tau} + \lambda}$-convex.

Generation result as in the Euclidean case

[Mayer '96, Jost, Ambrosio-Gigli-S. '05]

The Wasserstein space is not NPC

In $\mathscr{X} = \mathscr{P}_2(\mathbb{R}^2)$ consider two point masses μ_0 and μ_1 . . .

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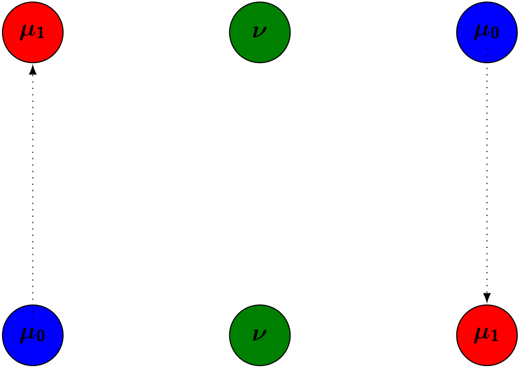

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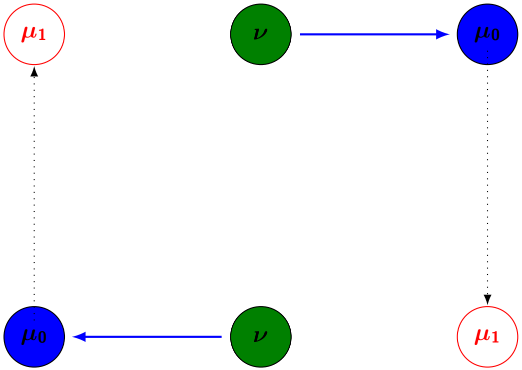

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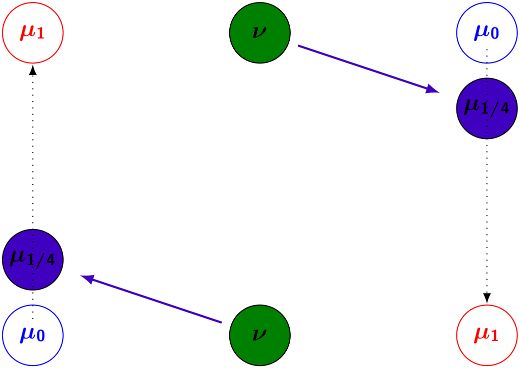

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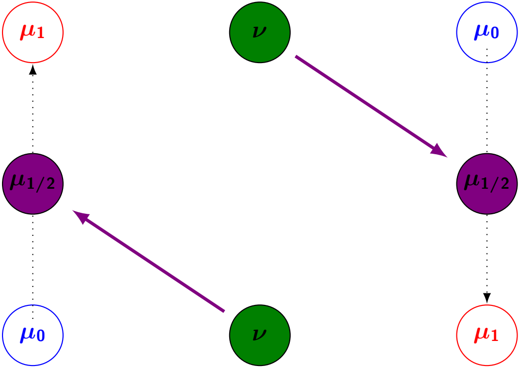

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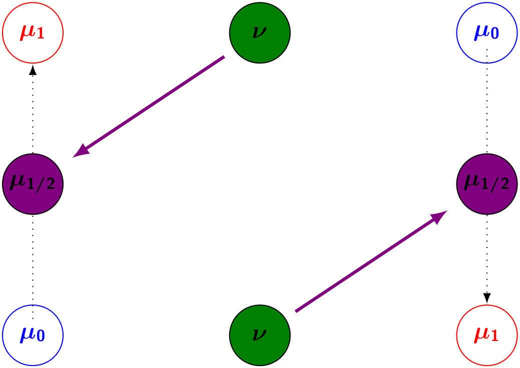

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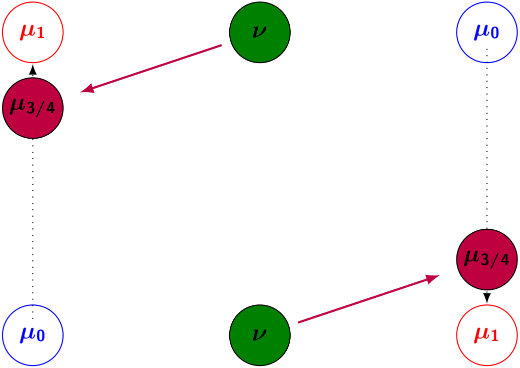

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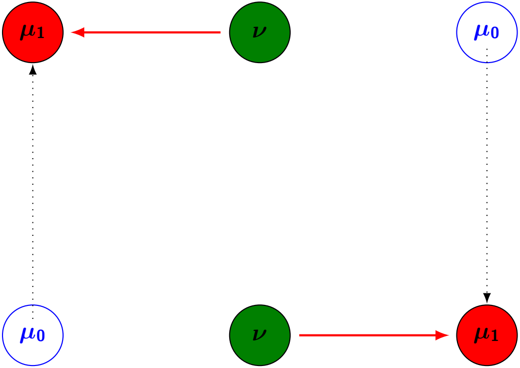

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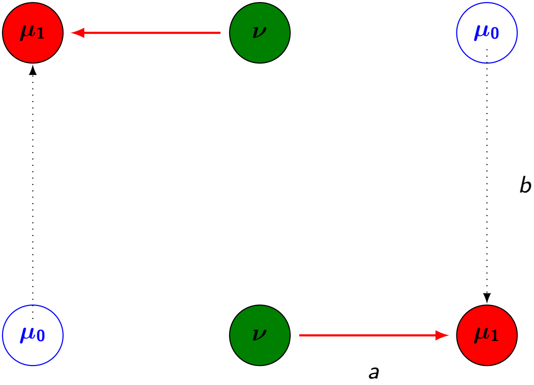
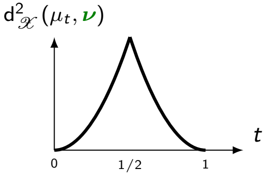


The Wasserstein distance is given by

$$\mathrm{d}^2_X(\nu, \mu_t) = \min\left(a^2 + b^2 t^2, a^2 + b^2(1-t)^2\right)$$

It is not λ-convex, for any λ.



Positively Curved (PC) spaces

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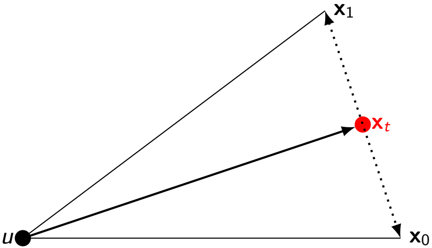

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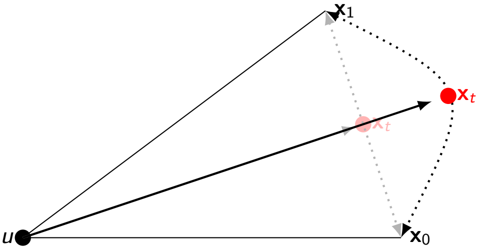

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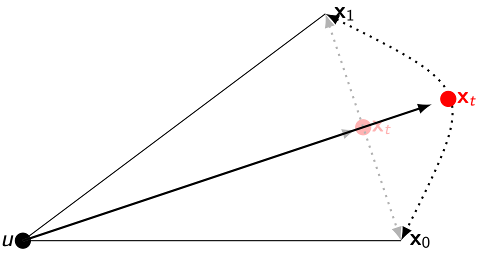

The functional $\Phi(\tau, u; \mathbf{v}) := \frac{1}{2\tau} d^2(u, \mathbf{v}) + \phi(\mathbf{v})$

looses any convexity property

Generation result: an unpublished paper by PERELMAN-PETRUNIN

OHTA ('07) in the compact PC case

by a completely different method.

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- **X is PC if and only if $\mathscr{X} = \mathscr{P}_2(X)$ is PC** [Sturm]

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Unfortunately [STURM]

$$\underline{\mathrm{Curv}}(X) < 0 \quad \Leftrightarrow \quad \underline{\mathrm{Curv}}\big(\mathscr{P}_2(X)\big) = -\infty,$$

so that the property

"X is an Alexandrov space of curvature bounded below"

is not stable w.r.t. the Wasserstein construction.

We need new weaker conditions.

PC spaces: semiconcavity of the distance and angles

PC Alexandrov spaces exhibit two important features:

- The (half) squared distance is **(-1)-concave**,

$$\mathrm{d}^2(u, \mathbf{x_t}) \geq (1-\mathbf{t})\mathrm{d}^2(u, \mathbf{x}_0) + \mathbf{t}\mathrm{d}^2(u, \mathbf{x}_1) - \mathbf{1}\ \boxed{\mathbf{t(1-t)}\mathrm{d}^2(\mathbf{x}_0, \mathbf{x}_1).}$$

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Stability of the semi-concave condition

The squared distance d^2 on X is **$(-K)$-semi-concave** if and only if the Wasserstein distance $d^2_{\mathscr{X}}$ is **$(-K)$-semi-concave** in $\mathscr{X} = \mathscr{P}_2(X)$.

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PC spaces: reference triangle in the plane

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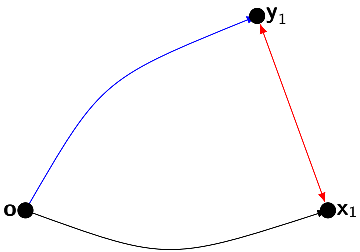

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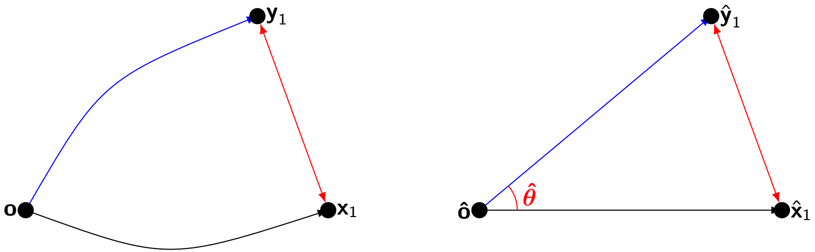

so that

$$d(\mathbf{o}, \mathbf{x}_1) = d(\hat{\mathbf{o}}, \hat{\mathbf{x}}_1), \quad d(\mathbf{o}, \mathbf{y}_1) = d(\hat{\mathbf{o}}, \hat{\mathbf{y}}_1), \quad d(\mathbf{x}_1, \mathbf{y}_1) = d(\hat{\mathbf{y}}_1, \hat{\mathbf{y}}_1).$$

We say that $(\hat{\mathbf{o}}, \hat{\mathbf{x}}_1, \hat{\mathbf{y}}_1)$ is a reference triangle in the plane for $(\mathbf{o}, \mathbf{x}_1, \mathbf{y}_1)$.

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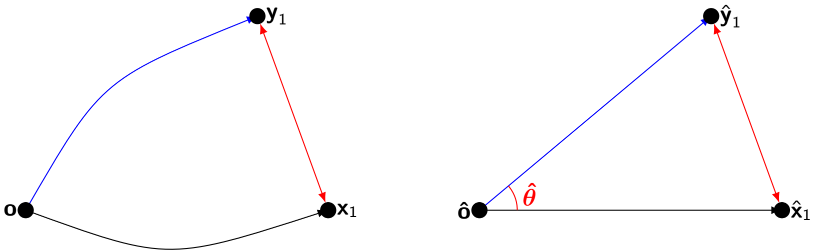

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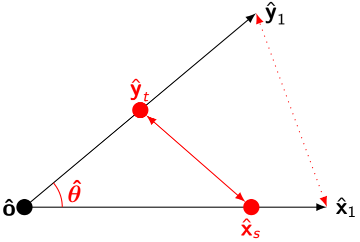
PC spaces: angle comparison

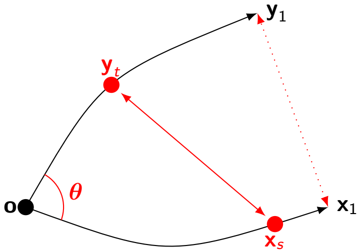


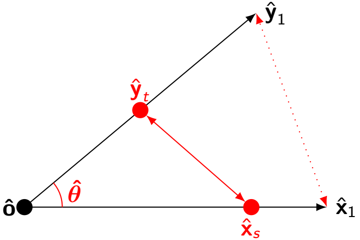
The geometric quantity

$$\alpha(\hat{\mathbf{o}};\hat{\mathbf{x}}_s,\hat{\mathbf{y}}_t) := \frac{\mathrm{d}^2(\hat{\mathbf{x}}_s,\hat{\mathbf{o}}) + \mathrm{d}^2(\hat{\mathbf{y}}_t,\hat{\mathbf{o}}) - \mathbf{d}^2(\hat{\mathbf{x}}_s,\hat{\mathbf{y}}_t)}{2\mathrm{d}(\hat{\mathbf{x}}_s,\hat{\mathbf{o}})\,\mathrm{d}(\hat{\mathbf{y}}_t,\hat{\mathbf{o}})} = \cos\hat{\theta}_{s,t} = \cos\hat{\theta}$$

is independent of **s, t** and defines the **comparison angle** $\hat{\theta}$ between $\hat{\mathbf{x}}, \hat{\mathbf{y}}$.

PC spaces: angle comparison





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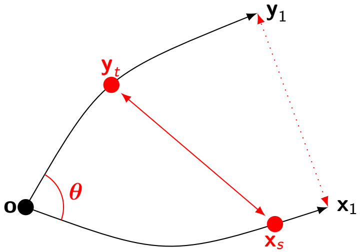
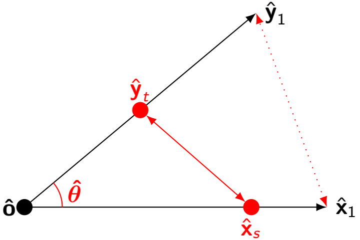
$$\cos\theta = \liminf_{s,t\downarrow 0} \alpha(\mathbf{o},\mathbf{x}_s,\mathbf{y}_t)$$

PC spaces: angle comparison





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X **is PC** iff **the map** $\mathsf{s},\mathsf{t} \mapsto \alpha(\mathbf{o};\mathbf{x}_{\mathsf{s}},\mathbf{y}_{\mathsf{t}})$ **is non decreasing**.

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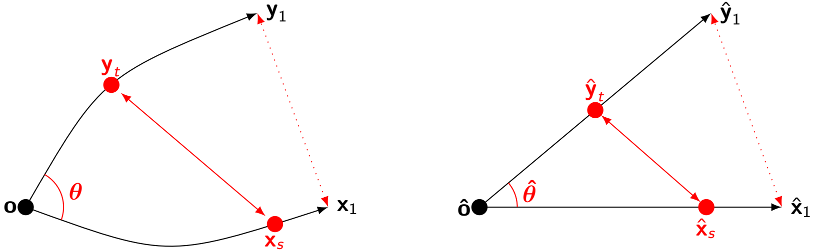
PC spaces: angle comparison



The geometric quantity

$$\alpha(\hat{\mathbf{o}};\hat{\mathbf{x}}_s,\hat{\mathbf{y}}_t) := \frac{d^2(\hat{\mathbf{x}}_s,\hat{\mathbf{o}}) + d^2(\hat{\mathbf{y}}_t,\hat{\mathbf{o}}) - \mathbf{d^2(\hat{x}_s,\hat{y}_t)}}{2d(\hat{\mathbf{x}}_s,\hat{\mathbf{o}})\,d(\hat{\mathbf{y}}_t,\hat{\mathbf{o}})} = \cos\hat{\theta}_{s,t} = \cos\hat{\theta}$$

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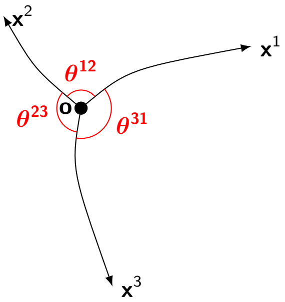
Global condition: $\hat{\theta} \le \theta, \quad \alpha(\mathbf{o};\mathbf{x}_1,\mathbf{y}_1) \ge \cos\theta.$

Angle condition

Consider three geodesics $\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3$ emanating from the same point $\mathbf{o}$.



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$$\hat{\theta}^{12} + \hat{\theta}^{23} + \hat{\theta}^{31} \leq 2\pi$$

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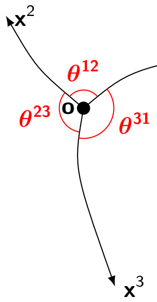

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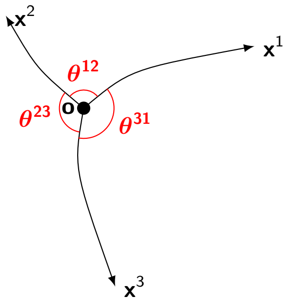

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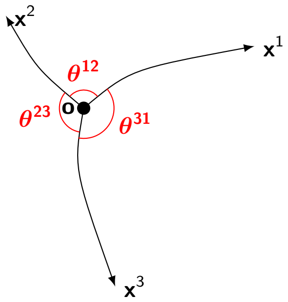

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Angles in general metric spaces

For a general metric space X we define the **(upper) angle θ** and a **pseudo-scalar product $(\mathbf{x}|\mathbf{y})$** between two geodesics $\mathbf{x}, \mathbf{y}$

$$\cos\theta = \liminf_{s,t\downarrow 0} \alpha(\mathbf{o}, \mathbf{x_s}, \mathbf{y_t}) = \liminf_{s,t\downarrow 0} \frac{d^2(\mathbf{x_s},\mathbf{o}) + d^2(\mathbf{y_t},\mathbf{o}) - d^2(\mathbf{x_s},\mathbf{y_t})}{2d(\mathbf{x_s},\mathbf{o})\,d(\mathbf{y_t},\mathbf{o})}$$

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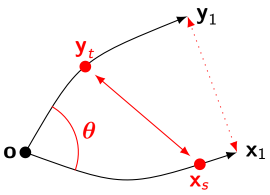

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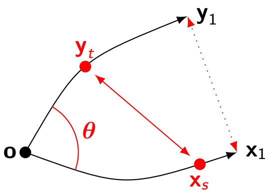
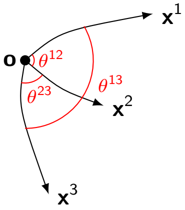


Angles satisfy the triangular inequality

$$\theta^{13} \le \theta^{12} + \theta^{23}$$



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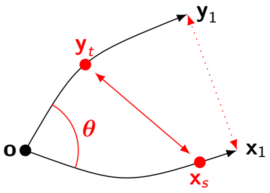

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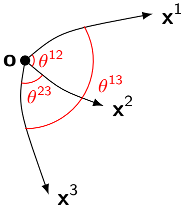

$$\underline{\mathrm{Curv}}(X) > -\infty \quad\Rightarrow\quad \theta^{12} + \theta^{23} + \theta^{31} \le 2\pi.$$

Local angle condition

Definition (LAC)

X satisfies a **local angle condition** if for every triple of geodesics $\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3$ emanating from the same point $\mathbf{o}$

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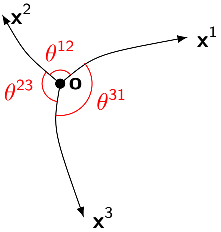

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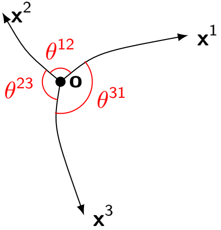

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Equivalent characterizations:

- There exist vectors $\mathbf{v}^1, \mathbf{v}^2, \mathbf{v}^3$ in an Hilbert space H (e.g. $H = \mathbb{R}^3$) s.t.

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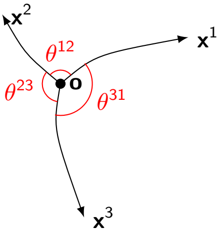

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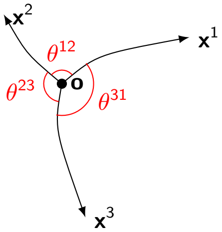

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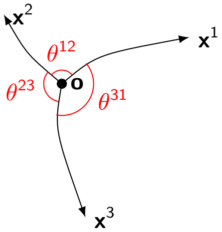

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- A Banach space X satisfies (LAC) iff X is Hilbert.

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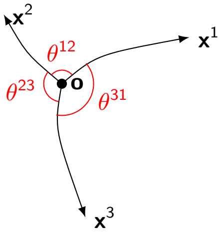

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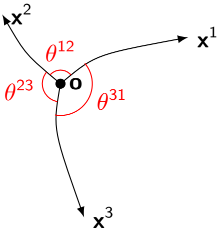

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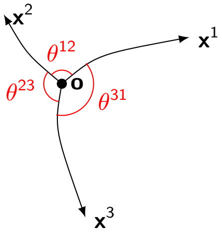

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Outline

Main result

Let X be a complete and separable metric space and let $\phi : X \to (-\infty, +\infty]$ be a proper l.s.c. functional. We assume that

- **X satisfies (LAC);**
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3. for every time interval $(0, T)$ there exists a "universal constant" $C(T)$ also depending on $\mathbf{K}, \lambda$ such that

$$\mathrm{d}^2(\mathbf{u_t}, \mathbf{U}_\tau(\mathbf{t})) \leq (\tau + \eta)\, C(T) |\partial\phi|^2(\mathbf{u_0}) \quad \forall\, t \in (0, T).$$

Applications to metric-measure spaces

Let (X, d, γ) be a metric-measure space s.t.

1. X satisfies **(LAC)**;
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- We consider the space

$$\mathscr{X}_\gamma = \mathscr{P}_2(X|\gamma) := \Big\{\mu \in \mathscr{P}_2(X) : \mathrm{supp}(\mu) \subset \mathrm{supp}(\gamma)\Big\}.$$

- the L^2-Wasserstein distance d_W on $\mathscr{X}_\gamma$
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$$\phi(\mu) = \mathrm{Ent}(\cdot|\gamma) = \int_{\mathrm{X}} \rho \log \rho \,\mathrm{d}\gamma, \qquad \rho = \frac{\mathrm{d}\mu}{\mathrm{d}\gamma}$$

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The entropy functional $\mathrm{Ent}(\cdot|\gamma)$ generates a $\boldsymbol{\lambda}$-contracting gradient flow.

Linearity

A metric space X is **non branching** if two geodesics $\mathbf{x}, \mathbf{y}$ satisfying

$$\mathbf{x}_0 = \mathbf{y}_0, \quad \mathbf{x}_s = \mathbf{y}_s, \quad \text{for some } s \in (0,1)$$

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If X is **non branching** then the gradient flow $\mathbf{S}$ of the entropy functional $\mathrm{Ent}(\cdot|\gamma)$ is **linear**, i.e. for every $\mu, \nu \in \mathscr{X}_\gamma$ and $t > 0$

$$\mathbf{S}_t(\alpha\mu + \beta\nu) = \alpha\mathbf{S}_t(\mu) + \beta\mathbf{S}_t(\nu) \quad \forall\, \alpha, \beta \geq 0, \quad \alpha + \beta = 1.$$

Kernels and representation formulae

We set

$$\nu_{x,t} := \mathbf{S}_t(\delta_x) \ll \gamma, \quad \vartheta_{x,t} := \frac{\mathrm{d}\nu_{x,t}}{\mathrm{d}\gamma}$$

- **Representation** For every $\mu \in \mathscr{X}_\gamma$ with $\mu_t = \mathbf{S}_t(\mu)$ we have

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- **Wasserstein contraction**:

$$W_p(\mathsf{S}_t(\mu), \mathsf{S}_t(\nu)) \leq e^{-\boldsymbol{\lambda} t} W_p(\mu, \nu) \qquad \forall\, p \in [1, 2].$$

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- **The adjoint semigroup $\mathscr{S}_t^*$** satisfies the Feller property

$$\mathscr{S}_t^*\big(L^\infty(\gamma)\big) \subset C_b^0(X), \qquad \mathscr{S}_t^*\big(\mathrm{Lip}(X)\big) \subset \mathrm{Lip}(X)$$
$$\mathrm{Lip}\big(\mathscr{S}_t^*(\psi);X\big) \le e^{-\lambda t}\mathrm{Lip}\big(\psi;X\big)$$

Stability of gradient flows under Measured GH-convergence

Let $(X^k, \mathrm{d}^k, \gamma^k)$ be a sequence of metric-measure spaces converging to $(X^\infty, \mathrm{d}^\infty, \gamma^\infty)$ under **measured Gromov-Hausdorff convergence** [Sturm; Lott-Villani]
Thus we can find a family of (semi)-distances $\hat{\mathrm{d}}^k$ on the disjoint union $\hat{X}^k = X^k \bigsqcup X^\infty$ such that

$$\lim_{k\to\infty} \mathrm{d}^2_{\hat{\mathscr{X}}^k}(\gamma^k, \gamma^\infty) = 0 \quad \hat{\mathscr{X}}^k = \mathscr{P}_2(X^k).$$

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Convergence of gradient flows [Ambrosio-S.-Zambotti]

If $\mu^k_t = \mathbf{S}^{\mathbf{k}}_{\mathbf{t}}[\mu^k]$ is the solution of the gradient flow on X^k and the initial data μ^k converges to μ^∞ according to $(\star)$, then μ^k_t converges to μ^∞_t for every $t > 0$.

Open problems

- Simmetry of the Entropy gradient flow, Dirichlet form
- Tangent cones/spaces, continuity equation
- Berstein-like estimates
- Weaker conditions than semi-concavity of the squared distance
- Singular distances, Wiener spaces
- . . .

Proof (1): Discrete E.V.I.

Recall that the continuous E.V.I. is

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\mathsf{d}^2\big(\mathbf{u}_t, v\big) \le \phi(v) - \phi(\mathbf{u}_t) \quad \forall\, v \in D(\phi).$$

Suppose that the solution U_τ^n of the variational scheme satisfies the **discrete E.V.I.**

$$\boxed{\frac{1}{2\tau}\Big(\mathsf{d}^2(U_\tau^{n+1}, V) - \mathsf{d}^2(U_\tau^n, V)\Big) \le \phi(V) - \phi(U_\tau^{n+1}) + A\mathsf{d}^2(U_\tau^{n+1}, V) + \mathscr{R}_\tau^{\mathbf{n}}}$$

where the **residual error** $\mathscr{R}_\tau^{\mathbf{n}}$ fulfills

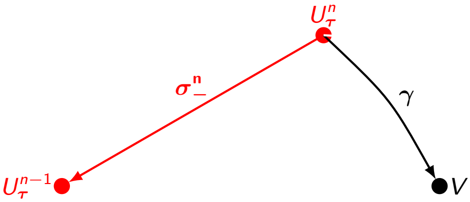
$$\tau\sum_{n=1}^{N}\mathscr{R}_\tau^{\mathbf{n}} \le o(1) \quad N\tau \ge T.$$

Then the **Minimizing Movement scheme is convergent.** [Ambrosio-Gigli-S.]

Proof (2): First variation, angle condition, and semiconcavity

U_τ^n minimizes the functional

$$U \mapsto \frac{\mathrm{d}^2(U, U_\tau^{n-1})}{2\tau} + \phi(U).$$



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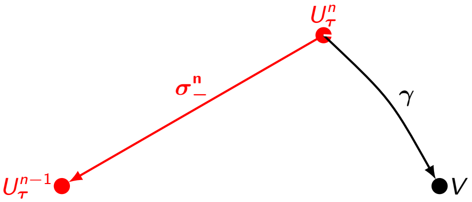

$$\boxed{\frac{1}{\tau}(\sigma_-^{\mathbf{n}}|\gamma) \le \phi(V) - \phi(U_\tau^n)}$$

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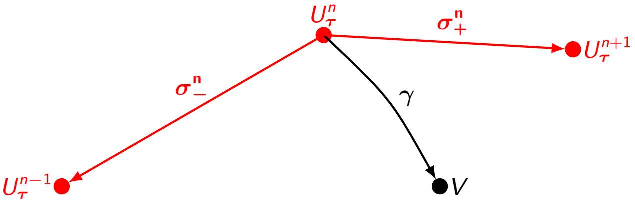

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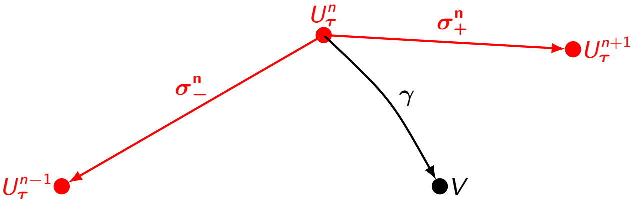

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Proof (3): discrete convexity of the functional

$$\phi(U_\tau^{n+1}) - 2\phi(U_\tau^{n}) + 2\phi(U_\tau^{n-1}) \geq 0$$

$$\sum_{n=1}^{+\infty} \Big(\phi(U_\tau^{n+1}) - 2\phi(U_\tau^{n}) + \phi(U_\tau^{n-1})\Big) \leq \phi(U_\tau^{0}) - \phi(U_\tau^{1}) \leq \boldsymbol{\tau} |\partial\phi|^2(U_\tau^{0})$$