

Coarsening, viscous flow and transportation

R.V. Kohn

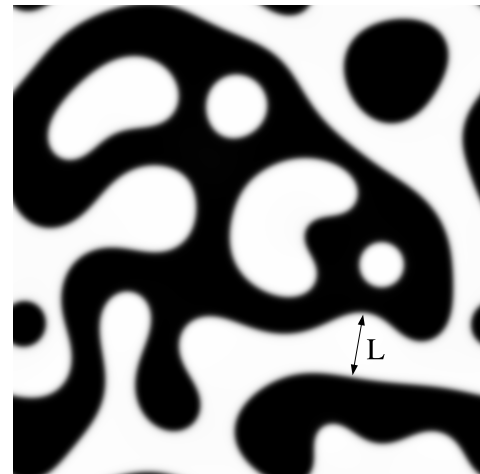
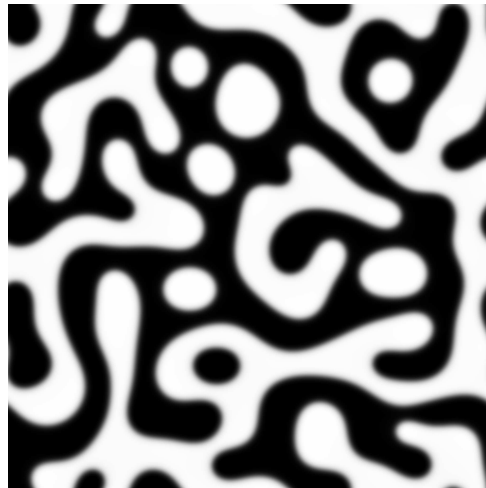
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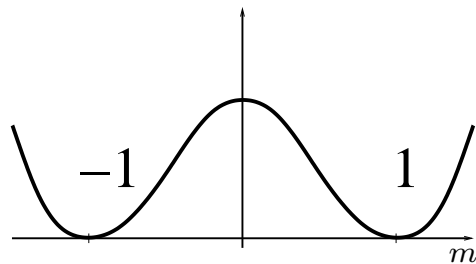
Coarsening of phase distribution



Cahn–Hilliard: a model for demixing

Scalar order parameter $m(x) \in \mathbb{R}$

Energy functional $E(m) = \int \frac{1}{2} |\nabla m|^2 + W(m) dx$



$$W = \frac{1}{4}(1 - m^2)^2$$

Conserved order parameter $\int m dx$

$$\partial_t m - \Delta \frac{\partial E}{\partial m} = 0$$

How does demixing proceed?

Unstable initial data

$m = 0 + \text{random perturbation}$ (mixed).

Absolute minimizers of E

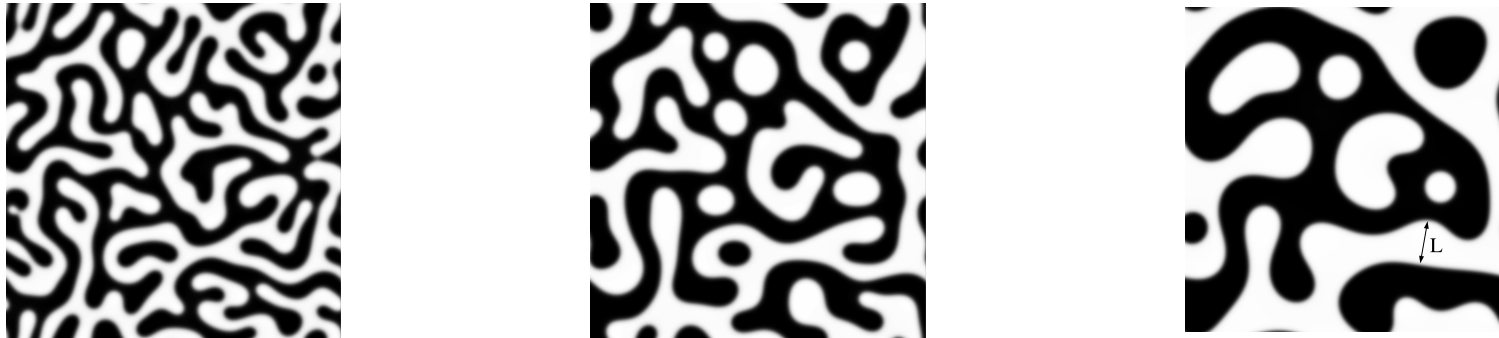
$m \equiv 1$ & $m \equiv -1$ (demixed).

What is the path in a large system?

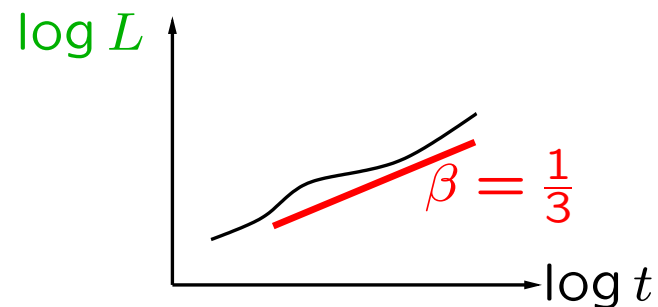
periodic boundary condition in $(0, \Lambda)^n$ with $\Lambda \gg 1$

Numerics: statistically self-similar coarsening

Spatial configuration characterized by single length L

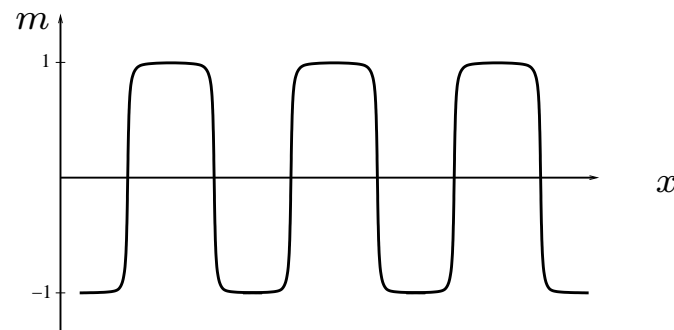


Coarsening rate characterized by single exponent β



Goal: Understand some aspects via PDE methods

Lower bounds — too many stationary points of E

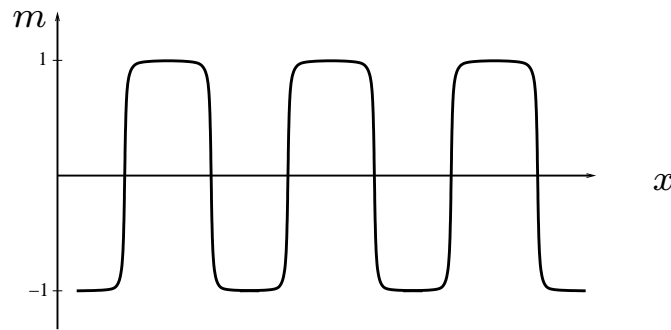


Upper bounds — via lower bounds on E

Goal: A priori bounds on rate of decay of E

Bounds on coarsening rate?

Lower bounds — too many stationary points of E



Upper bounds — via lower bounds on E

Goal: A priori bounds on rate of decay of E

An abstract framework

(\mathcal{M}, g) Riemannian manifold

E functional on \mathcal{M}

$\dot{x} = -\text{grad}_g E(x)$ gradient flow

metric tensor $g_x(\delta x, \delta x) \rightsquigarrow$ induced distance $d(x_0, x_1)$

infinitesimal

global

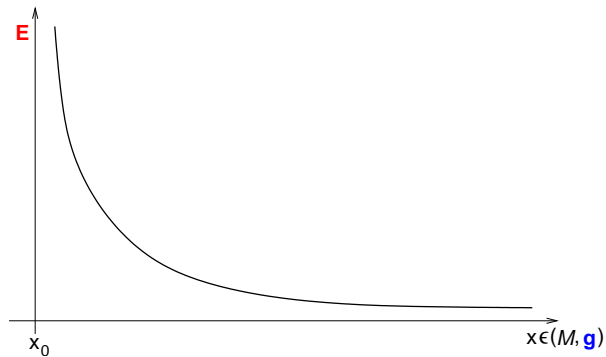
Relating energy landscape to gradient dynamics

Theorem (Kohn & O.). Assume for some $x^* \in \mathcal{M}$, $\alpha > 0$

$$E(x) d(x^*, x)^\alpha \gtrsim 1 \quad \text{for } E(x) \leq 1$$

Then for all $\sigma \in (1, \frac{\alpha+2}{\alpha})$

$$\int_0^T E(x(t))^\sigma dt \gtrsim \int_0^T (t^{-\frac{\alpha}{\alpha+2}})^\sigma dt \quad \text{for } T \gg d(x^*, x(0))^{\alpha+2} \\ \text{and } E(x(0)) \leq 1$$



geometric exponent α

\rightsquigarrow dynamic exponent $\frac{\alpha}{\alpha+2}$