Two contour dynamics problems in incompressible flows: the Muskat problem and the QG sharp front

Francisco Gancedo

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The Muskat problem

Equation (Darcy's law)

$$\frac{\mu}{\kappa} v = -\nabla p - (0, 0, \mathrm{g}\,\rho),$$

v velocity, p pressure, μ viscosity, κ permeability, ρ density, and g acceleration due to gravity.

- Muskat (1937)
- Saffman and Taylor (1958)

Equation (Hele–Shaw)

$$\frac{12\mu}{b^2} v = -\nabla p - (0, \mathrm{g}\,\rho),$$

b distance between the plates.

Previous results

- Dombre, Pumir & Siggia (1992).
- Escher & Simmonett (1997).
- Siegel, Caflish & Howison (2004).
- Ambrose (2004).

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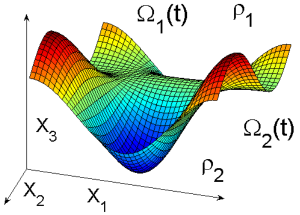
The contour equation

The density ρ is defined by

$$\rho(x_1, x_2, x_3, t) = \begin{cases} \rho_1 & \text{in} \quad \Omega_1(t) \\ \rho_2 & \text{in} \quad \Omega_2(t) \end{cases}$$



with $\rho_1 \neq \rho_2$.

We have the 3DPM contour equation (two dimensional interface) given by

Equation

$$f_t(x,t) = \frac{\rho_2 - \rho_1}{4\pi} PV \int_{\mathbb{R}^2} \frac{(\nabla f(x,t) - \nabla f(x-y,t)) \cdot y}{[|y|^2 + (f(x,t) - f(x-y,t))^2]^{3/2}} dy,$$

$$f(x,0) = f_0(x).$$

$$\left.\begin{array}{l} \rho_t + v \cdot \nabla \rho = 0 \\ v = -\nabla p - (0,0,\rho) \\ \mathrm{div}\, v = 0 \end{array}\right\} \quad \Leftrightarrow \quad \text{3DPM contour equation}$$

The linearized equation

If we neglect the terms of order greater than one

$$f_t = \frac{\rho_1 - \rho_2}{2}(R_1\partial_{x_1} f + R_2\partial_{x_2} f) = \frac{\rho_1 - \rho_2}{2}\Lambda f,$$
$$f(x,0) = f_0(x).$$

Applying the Fourier transform we get

$$\hat{f}(\xi) = \hat{f}_0(\xi)e^{\frac{\rho_1-\rho_2}{2}|\xi|t}.$$

Problem

- $\rho_1 < \rho_2$ *stable case,*
- $\rho_1 > \rho_2$ *unstable case.*

Local well-posedness for the stable case ($\rho_2 > \rho_1$)

Theorem

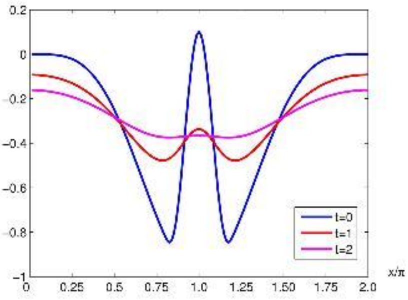
Let $f_0(x) \in H^k$ for $k \geq 4$ and $\rho_2 > \rho_1$. Then there exists a time $T > 0$ so that there is a unique solution to 3DPM contour equation in $C^1([0, T]; H^k(\mathbb{R}^2))$ with $f(x, 0) = f_0(x)$.

Theorem (2-D case, one dimensional interface)

$f_0(x) \in H^k$ for $k \geq 3$ and $\rho_2 > \rho_1$.

Global existence?

- L^∞ decays.
- $\displaystyle\int f(x,t)\,dx = \int f_0(x)\,dx.$
- Global existence for small initial data: $\sum |\xi||f(\xi)| << 1$.



Ill-posedness for the unstable case ($\rho_1 > \rho_2$)

Theorem

Let $s > 3/2$, then for any $\varepsilon > 0$ there exists a solution f of 2DPM contour equation with $\rho_1 > \rho_2$ and $0 < \delta < \varepsilon$ such that $\|f\|_{H^s}(0) \leq \varepsilon$ and $\|f\|_{H^s}(\delta) = \infty$.

The QG equation

Equation

$$\theta_t + u \cdot \nabla\theta = 0,$$

$$u = \nabla^{\perp}\psi, \quad \theta = -(-\Delta)^{1/2}\psi,$$

with $\theta(x,t)$ *the temperature,* $(x,t) \in \mathbb{R}^2 \times \mathbb{R}^+$.

- Constantin, Majda, and Tabak (1994)

We consider weak solutions given by

$$\theta(x_1, x_2, t) = \begin{cases} \theta_1, & \Omega(t) \\ \theta_2, & \mathbb{R}^2 \setminus \Omega(t). \end{cases}$$

The 2-D vortex patch problem

Contour equation

$$w_t + u \cdot \nabla w = 0,$$

$$u = \nabla^{\perp}\psi, \quad w = \Delta\psi,$$

where the vorticity is given by

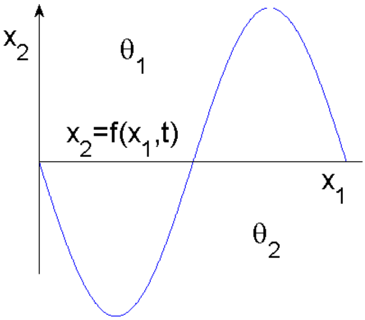
$$w(x_1, x_2, t) = \begin{cases} w_0, & \Omega(t) \\ 0, & \mathbb{R}^2 \setminus \Omega(t). \end{cases}$$

- Chemin (1993)
- Bertozzi and Constantin (1993)

The 2-D patch problem for QG

- Rodrigo (2005), for a periodic C^∞ front, i.e.

$$\theta(x_1, x_2, t) = \begin{cases} \theta_1, & \{f(x_1,t) > x_2\} \\ \theta_2, & \{f(x_1,t) \le x_2\}. \end{cases}$$



The α-patch model

- Córdoba, Fontelos, Mancho and Rodrigo (2005)

Contour equation

$$\theta_t + u \cdot \nabla \theta = 0,$$

$$u = \nabla^{\perp} \psi, \quad \theta = -(-\Delta)^{1-\alpha/2} \psi, \quad 0 < \alpha \leq 1,$$

where the active scalar $\theta(x,t)$ *satisfies*

$$\theta(x_1, x_2, t) = \begin{cases} \theta_1, & \Omega(t) \\ \theta_2, & \mathbb{R}^2 \setminus \Omega(t). \end{cases}$$

The contour equation

Equation

$$x_t(\gamma,t)=\frac{\Theta_\alpha}{2\pi}\int_{\mathbb{T}}\frac{\partial_\gamma x(\gamma,t)-\partial_\gamma x(\gamma-\eta,t)}{|x(\gamma,t)-x(\gamma-\eta,t)|^\alpha}d\eta,\quad 0<\alpha\leq 1,$$
$$x(\gamma,0)=x_0(\gamma).$$

We need that

$$\frac{|x(\gamma,t)-x(\gamma-\eta,t)|}{|\eta|}>0,\quad \forall\,\gamma,\eta\in[-\pi,\pi],$$

therefore we give initial data satisfying this property, and we prove that this condition is satisfied locally in time.

Local well-posedness for $0 < \alpha < 1$

We define

$$F(x)(\gamma,\eta,t) = \frac{|\eta|}{|x(\gamma,t) - x(\gamma-\eta,t)|} \quad \forall\, \gamma,\eta \in [-\pi,\pi],$$

with

$$F(x)(\gamma,0,t) = |\partial_\gamma x(\gamma,t)|^{-1}.$$

Theorem

Let $x_0(\gamma) \in H^k(\mathbb{T})$ for $k \geq 3$ with $F(x_0)(\gamma,\eta) < \infty$. Then there exists a time $T > 0$ so that there is a unique solution to the α-patch model for $0 < \alpha < 1$ in $C^1([0,T]; H^k(\mathbb{T}))$, with $x(\gamma,0) = x_0(\gamma)$.

Existence for $\alpha = 1$; the QG sharp front

We modify the equation as follows:

Contour equation

$$x_t(\gamma,t) = \int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma,t) - \partial_\gamma x(\gamma-\eta,t)}{|x(\gamma,t) - x(\gamma-\eta,t)|} d\eta + \lambda(\gamma,t)\partial_\gamma x(\gamma,t), \quad (1)$$

with

$$\lambda(\gamma,t) = \frac{\gamma+\pi}{2\pi}\int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma,t)}{|\partial_\gamma x(\gamma,t)|^2} \cdot \partial_\gamma\Big(\int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma,t) - \partial_\gamma x(\gamma-\eta,t)}{|x(\gamma,t) - x(\gamma-\eta,t)|} d\eta\Big) d\gamma$$
$$-\int_{-\pi}^{\gamma} \frac{\partial_\gamma x(\eta,t)}{|\partial_\gamma x(\eta,t)|^2} \cdot \partial_\eta\Big(\int_{\mathbb{T}} \frac{\partial_\gamma x(\eta,t) - \partial_\gamma x(\eta-\xi,t)}{|x(\eta,t) - x(\eta-\xi,t)|} d\xi\Big) d\eta. \quad (2)$$

We get

$$|\partial_\gamma x(\gamma,t)|^2 = A(t).$$

- Hou, Lowengrub and Shelley (1997)

Theorem

Let $x_0(\gamma) \in H^k(\mathbb{T})$ for $k \geq 3$ with $F(x_0)(\gamma,\eta) < \infty$. Then there exists a time $T > 0$ so that there is a solution to (1) in $C^1([0,T]; H^k(\mathbb{T}))$ with $x(\gamma,0) = x_0(\gamma)$ and $\lambda(\gamma,t)$ given by (2).

Extra cancellations:

$$\partial_\gamma x(\gamma,t)\cdot\partial_\gamma^2 x(\gamma,t) = 0,$$

and

$$\partial_\gamma x(\gamma,t)\cdot\partial_\gamma^3 x(\gamma,t) = -|\partial_\gamma^2 x(\gamma,t)|^2.$$

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