Variational models for incompressible Euler equations and measures concentrated on action-minimizing paths

Luigi Ambrosio, Alessio Figalli

Scuola Normale Superiore, Pisa
http://cvgmt.sns.it

Outline

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1. Euler incompressible equations and Arnold geodesics
2. Non-existence results
3. Relaxed solutions
4. Density and relaxation results
5. The pressure field
6. Necessary and sufficient optimality conditions

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Euler incompressible equations

We consider an incompressible fluid moving inside a d-dimensional region D with velocity $\boldsymbol{u}$. The Euler equations for $\boldsymbol{u}$ are

$$\begin{cases} \partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p & \text{in } [0,T] \times D, \\ \mathrm{div}\, \boldsymbol{u} = 0 & \text{in } [0,T] \times D, \\ \boldsymbol{u} \cdot n = 0 & \text{on } [0,T] \times \partial D, \end{cases}$$

where p, the pressure field, is a Lagrange multiplier for the divergence-free constraint.

If $\boldsymbol{u}$ is smooth, it produces a unique flow map g, given by

$$\begin{cases} \dot{g}(t,a) = \boldsymbol{u}(t, g(t,a)), \\ g(0,a) = a. \end{cases}$$

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$$\begin{cases} \dot{g}(t,a) = \boldsymbol{u}(t, g(t,a)), \\ g(0,a) = a. \end{cases}$$

By the incompressibility condition, we get that $g(t,\cdot) : D \to D$ is a measure-preserving diffeomorphism of D:

$$g(t,\cdot)_\# \mu_D = \mu_D \qquad \bigl(\mu_D(g(t,\cdot)^{-1}(E)) = \mu_D(E) \ \ \forall E\bigr),$$

(here and in the sequel $f_\# \mu$ is the push-forward of a measure μ through a map f, and μ_D is the volume measure of the manifold D).

Writing Euler's equations in terms of g, we get

$$\begin{cases} \ddot g(t,a) = -\nabla p\,(t, g(t,a)) & (t,a) \in [0,T] \times D, \\ g(0,a) = a & a \in D, \\ g(t,\cdot) \in \mathrm{SDiff}(D) & t \in [0,T]. \end{cases}$$

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Weak solutions to Euler's equations

In the case $d = 2$, existence can be proved through the *vorticity* formulation: setting $\omega_t(\cdot) = \mathrm{curl}\,\boldsymbol{v}(t,\cdot)$, so that $\boldsymbol{v}(t,\cdot) = \nabla^\perp \Delta^{-1} \omega_t$, the PDE becomes

$$\frac{d}{dt}\omega_t(x) + \mathrm{div}\left(\omega_t(x)\boldsymbol{v}(t,x)\right) = 0.$$

Existence: $\omega_0 \in L^p$, $1 \le p \le \infty$; **Uniqueness:** $\omega_0 \in L^\infty$.

In the case $d > 2$ much less is known: *no* general existence result of distributional solutions is presently available. The canonical approach consists in taking limits as $\varepsilon \downarrow 0$ of solutions to Navier-Stokes equations:

$$\partial_t \boldsymbol{u}^\varepsilon + (\boldsymbol{u}^\varepsilon \cdot \nabla)\boldsymbol{u}^\varepsilon = -\nabla p^\varepsilon + \varepsilon \Delta \boldsymbol{u}^\varepsilon$$

Young measure solutions. (DiPerna-Majda, 1987);
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Arnold's geodesic interpretation

We can (formally) view the space $\mathrm{SDiff}(D)$ of measure-preserving diffeomorphisms of D as an infinite-dimensional manifold with the metric inherited from the embedding in $L^2(D)$, and with tangent space made by the divergence-free vector fields.

Using this viewpoint, Arnold interpreted the previous ODE, and therefore Euler's equations, as a *geodesic* equation on $\mathrm{SDiff}(D)$.

Therefore one can look for solutions of Euler's equations by minimizing

$$\int_0^1 \int_D \frac{1}{2}|\dot{g}(t,x)|^2 \, d\mu_D(x) \, dt$$

among all paths $g(t,\cdot) : [0,1] \to \mathrm{SDiff}(D)$ with $g(0,\cdot) = f$ and $g(1,\cdot) = h$ prescribed (typically, by right invariance, f is taken as the identity map $\boldsymbol{i}$).

Existence: Ebin-Marsden (1970), $g \circ f^{-1} \sim \boldsymbol{i}$.

We shall denote by $\delta(f,h)$ the Arnold distance in $\mathrm{SDiff}(D)$.

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This variational problem differs from the classical Euler's one, because the initial and final diffeomorphims, and not the initial velocity, are prescribed.

Nevertheless, the investigation of this problem leads to difficult and still not completely understood questions (typical of Calculus of Variations) namely:

(a) Non-attainment and Lavrentiev (gap) phenomena;
(b) Necessary and sufficient optimality conditions;
(c) Regularity of the pressure field;
(d) Regularity of (relaxed) curves with minimal length.

Main contributions: Brenier, Shnirelman.

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Non-existence results

Shnirelman found examples of maps $g \in \mathrm{SDiff}([0,1]^2)$ which cannot be connected to $\boldsymbol{i}$ by a path with finite action, i.e. $\delta(\boldsymbol{i}, g) = +\infty$.

Furthermore, he proved the existence of $h \in \mathrm{SDiff}([0,1]^3)$ of the form

$$h(x_1, x_2, x_3) = (g_1(x_1, x_2), g_2(x_1, x_2), x_3), \quad \text{with} \quad g \in \mathrm{SDiff}([0,1]^2)$$

for which $\delta(\boldsymbol{i}, h)$ is not attained.

These negative results motivate somehow the analysis of relaxed solutions. In this connection, notice that the relaxation of the Arnold distance

$$\delta_*(h) := \inf \left\{ \liminf_{h \to \infty} \delta(\boldsymbol{i}, h_n) : \int_D |h_n - h|^2 \, d\mu_D \to 0 \right\}$$

is still not known in the 2-dimensional case.

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Furthermore, he proved the existence of $h \in \mathrm{SDiff}([0,1]^3)$ of the form

$$h(x_1, x_2, x_3) = (g_1(x_1, x_2), g_2(x_1, x_2), x_3), \quad \text{with} \quad g \in \mathrm{SDiff}([0,1]^2)$$

for which $\delta(\boldsymbol{i}, h)$ is not attained.

These negative results motivate somehow the analysis of relaxed solutions. In this connection, notice that the relaxation of the Arnold distance

$$\delta_*(h) := \inf\left\{\liminf_{h\to\infty} \delta(\boldsymbol{i}, h_n) : \int_D |h_n - h|^2 \, d\mu_D \to 0\right\}$$

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Relaxed solutions

Two levels of relaxation can be imagined: the first one is to relax the smoothness constraint, and to allow $g(t,\cdot)$ to be *measure-preserving maps* (not necessarily injective):

$$S(D) := \left\{ g : D \to D : \ \mu_D(g^{-1}(A)) = \mu_D(A) \ \forall A \in \mathcal{B}(D) \right\}.$$

We will see that a second level is necessary, giving up the idea that $g(t,\cdot)$ is a map, but allowing it to be a *measure preserving plan* (roughly speaking, a multivalued map):

$$\Gamma(D) := \{\eta \in \mathscr{P}(D \times D) : \ \eta(A \times D) = \mu_D(A) = \eta(D \times A) \ \ \forall A \in \mathcal{B}(D)\}.$$

The space $S(D)$ "embeds" into $\Gamma(D)$ considering

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From the Lagrangian viewpoint, it is natural to follow the path of each particle, and to relax the smoothness and injectivity constraints, allowing fluid paths to split, forward or backward in time.

These remarks led in 1989 Brenier to the following model: let

$$\Omega(D) := C\left([0,1];D\right), \qquad e_t(\omega) := \omega(t),\ \ t \in [0,1].$$

Then, we minimize the action functional

$$\mathscr{A}(\eta) := \int_{\Omega(D)} \frac{1}{2}\int_0^1 |\dot\omega|^2\,dt\,d\eta(\omega), \qquad \eta \in \mathscr{P}(\Omega(D))$$

$$(e_0, e_1)_\#\eta = (\boldsymbol{i} \times h)_\#\mu_D, \qquad (e_t)_\#\eta = \mu_D\ \ \forall t \in [0,T].$$

Classical flows $g(t,a)$ induce generalized ones via the relation $\eta = (\Phi_g)_\#\mu_D$, with

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In this relaxed model some obstructions of the original one disappear: for instance, in the case $D = [0,1]^d$ or $D = \mathbb{T}^d \sim \mathbb{R}^d/\mathbb{Z}^d$, it is always possible to connect $\boldsymbol{i}$ to h by a path with finite action, less than $\sqrt{d}$.

Moreover, standard compactness/lower semicontinuity arguments provide existence of generalized flows with minimal action.

However, it is not clear how this model could be used to connect a general map $f \in S(D)$ to $h \in S(D)$ (by right invariance, this is clear only if f is invertible).

These remarks led us to a more general model, which allows to connect $\eta = \eta_a \otimes \mu_D$ to $\gamma = \gamma_a \otimes \mu_D$. The idea, which appears first in Brenier's *Eulerian-Lagrangian* model (CPAM, 1999) is to "double" the state space, adding to the Eulerian state space D a Lagrangian state space $A \sim D$.

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Eulerian-Lagrangian model

Let

$$\Omega^*(D) := \Omega(D) \times A, \qquad \pi(\omega, a) = a$$

Then, we consider probability measures $\boldsymbol{\eta} = \boldsymbol{\eta}_a \otimes \mu_D$ in $\Omega^*(D)$ having μ_D as second marginal. Again, we minimize the action

$$\mathscr{A}(\boldsymbol{\eta}) := \int_{\Omega^*(D)} \frac{1}{2} \int_0^1 |\dot{\omega}|^2 \, dt \, d\boldsymbol{\eta}(\omega, a)$$

with the incompressibility constraint $(e_t)_\# \boldsymbol{\eta} = \mu_D$ for all t and the endpoint constraints:

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Denoting by $\overline{\delta}^2(\eta, \gamma)$ the minimal action, it turns out that one can define natural operations of *reparameterization*, *restriction* and *concatenation* in this class of flows. These imply that $(\overline{\delta}, \Gamma(D))$ is a metric space. Indeed, it is *complete* and a *length* space, whose convergence is stronger than weak convergence.

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SCUOLA NORMALE SUPERIORE

Analogies with optimal transport theory

It is easy to check that the optimal transport problem in D with $c(x,y) = d_D^2(x,y)$ can be also formulated by minimizing

$$\int_{\Omega(D)} \int_0^1 |\dot{\omega}|^2 \, dt \, d\boldsymbol{\eta}(\omega),$$

with the constraints $(e_0)_\# \boldsymbol{\eta} = \mu$, $(e_1)_\# \boldsymbol{\eta} = \nu$.

More generally, we know from the work of Bernard-Buffoni that one can consider *action-minimizing* measures

$$\min \left\{ \int_{\Omega(D)} \int_0^1 L(t,\omega,\dot{\omega}) \, dt \, d\boldsymbol{\eta}(\omega) : \ \boldsymbol{\eta} \in \mathcal{K} \right\}$$

with various constraints $\mathcal{K}$, and recover for instance Mather's theory.

In these models *no* interaction between the paths occurs. Several more recent models of *branched* optimal transportation (Buttazzo, Morel, Solimini, Xia,...) include some form of interaction and can still be formulated in this language (the so-called *traffic plans*, Bernot-Caselles-Morel).

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Motivation for the extension to $\Gamma(D)$

Even for deterministic initial and final data, there exist example of minimizing geodesics η that are *not deterministic* in between: $(e_0, e_t)_\# \eta \in \Gamma(D) \setminus S(D)$, $t \in (0,1)$.

Example

We want to connect in $D = B_1(0) \subset \mathbb{R}^2$ the map (x_1, x_2) to $-(x_1, x_2)$. Two classical solutions:

$$[0,\pi] \ni t \mapsto (x_1 \cos \pm t + x_2 \sin \pm t, x_1 \sin \pm t + x_2 \cos \pm t)$$

On the oher hand, one can consider the family of maps $\omega_{x,\theta}$ connecting x to $-x$

$$\omega_{x,\theta}(t) := x \cos t + \sqrt{1 - |x|^2}(\cos\theta, \sin\theta) \sin t \qquad \theta \in (0,\pi)$$

and define $\eta := \omega_{x,\theta}\left(\frac{1}{\pi^2}\mathscr{L}^2 \lfloor D \times \mathscr{L}^1 \lfloor (0,\pi)\right)$. It turns out that η is optimal as well, and non-deterministic in between.

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Density and relaxation results

It is well-known (L.C.Young) that $S(D)$, when viewed as a subset of $\Gamma(D)$ through the embedding $g \mapsto (\boldsymbol{i} \times g)_{\#}\mu_D$, is dense.

Also, in the case $D = [0,1]^d$, $d \geq 2$, Brenier-Gangbo proved that

$$S(D) = \overline{\mathrm{SDiff}(D)}^{L^2(\mu_D)}.$$

A natural question is the study of the relation between the distance $\overline{\delta}$ and the relaxation δ_* of the Arnold distance, namely

$$\delta_*(h) := \inf\left\{\liminf_{h\to\infty} \delta(\boldsymbol{i}, h_n) : \int_D |h_n - h|^2 \, d\mu_D \to 0\right\} \qquad h \in S(D).$$

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Theorem (Density of smooth flows (Shnirelman, 1994))

Assume $D = [0,1]^d$, $d > 2$. *For any generalized incompressible flow* $\boldsymbol{\eta}$ *between* $\boldsymbol{i}$ *and* $h \in \mathrm{SDiff}(D)$ *there exist smooth flows* g_k *connecting* $\boldsymbol{i}$ *to* h *satisfying:*

(a) $\mathscr{A}(g_k) \to \mathscr{A}(\boldsymbol{\eta})$;

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Furthermore, the Arnold distance is topologically equivalent to the L^2 *distance:*

$$\frac{1}{\sqrt{2}}\|f-g\|_{L^2(\mu_D)} \le \delta(f,g) \le C\|f-g\|^{\alpha}_{L^2(\mu_D)}.$$

These two facts imply that, when $d > 2$, no gap phenomenon occurs, i.e. $\delta_*(h) = \overline{\delta}(\boldsymbol{i}, h)$ for all $h \in S(D)$.

We extended this result to non-deterministic final data γ: we proved indeed that $\delta_*(\gamma) = \overline{\delta}(\boldsymbol{i},\gamma)$ for all $\gamma \in \Gamma(D)$, where

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The pressure field

Brenier proved in 1993 that, even though geodesics are not unique in general, there is a *unique*, up to an additive time-dependent constant (given the initial and final conditions), pressure field.

The pressure field arises if one relaxes the incompressibility constraint, considering *almost incompressible* flows ν. Denoting by ρ^ν the density produced by the flow

$$(e_t)_\# \boldsymbol{\nu} = \rho^{\boldsymbol{\nu}} \mu_D,$$

we say that ν is almost incompressible if $\|\rho^\nu - 1\|_{C^1} \leq 1/2$.

Theorem (pressure as a Lagrange multiplier)

Let η be optimal between η and γ. There exists $p \in (C^1)^*$ such that

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$$\frac{d}{d\varepsilon}\boldsymbol{X}^t(\varepsilon,x) = \boldsymbol{w}(t,\boldsymbol{X}^t(\varepsilon,x)), \qquad \boldsymbol{X}^t(0,x) = x$$

and perturb (smoothly) the paths ω by $\omega(t) \mapsto \boldsymbol{X}^t(\varepsilon,\omega(t))$. These perturbations induce a perturbation $\boldsymbol{\eta}_\varepsilon$ of $\boldsymbol{\eta}$, which is almost incompressible. Then, first variation gives

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$$\frac{d}{d\varepsilon}\boldsymbol{X}^t(\varepsilon,x) = \boldsymbol{w}(t,\boldsymbol{X}^t(\varepsilon,x)), \qquad \boldsymbol{X}^t(0,x) = x$$

and perturb (smoothly) the paths ω by $\omega(t) \mapsto \boldsymbol{X}^t(\varepsilon,\omega(t))$.
These perturbations induce a perturbation $\boldsymbol{\eta}_\varepsilon$ of $\boldsymbol{\eta}$, which is almost incompressible. Then, first variation gives

$$\int_{\Omega^*(D)}\int_0^1 \dot\omega(t)\cdot\frac{d}{dt}\boldsymbol{w}(t,\omega(t))\,dt\,d\eta(\omega,a) + \langle p, \operatorname{div}\boldsymbol{w}\rangle = 0$$

and this equations uniquely determines p, independently of the chosen minimizer η.

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As $\boldsymbol{w}$ is arbitrary, the first variation also leads to a weak formulation of Euler's equations

$$\partial_t \overline{\boldsymbol{v}}_t(x) + \operatorname{div}\left(\overline{\boldsymbol{v} \otimes \boldsymbol{v}}_t(x)\right) + \nabla_x p(t,x) = 0.$$

Here

$$\overline{\boldsymbol{v}}_t \mu_D = (e_t)_\#(\dot\omega(t)\boldsymbol{\eta}), \qquad \overline{\boldsymbol{v} \otimes \boldsymbol{v}}_t \mu_D = (e_t)_\#(\dot\omega(t) \otimes \dot\omega(t)\boldsymbol{\eta}).$$

In general, however, $\overline{\boldsymbol{v} \otimes \boldsymbol{v}}_t \neq \overline{\boldsymbol{v}}_t \otimes \overline{\boldsymbol{v}}_t$ (due to branching and multiple velocities), and this precisely marks the difference between genuine distributional solutions to Euler's equation and "generalized" ones.

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SCUOLA NORMALE SUPERIORE

Necessary and sufficient optimality conditions

The basic remark is that *any* function $q \in L^1([0,1] \times D)$ with $\int_D q(t,\cdot)\, d\mu_D = 0$ induces a null-lagrangian for the minimization problem (with the incompressibility constraint): indeed

$$\int_{\Omega^*(D)} \int_0^1 q(t,\omega(t))\, dt\, d\boldsymbol{\eta}(\omega,a) = \int_0^1 \int_D q(t,x)\, d\mu_D(x)\, dt = 0$$

for any generalized incompressible flow $\boldsymbol{\eta}$. If we denote by

$$c_q^{0,1}(x,y) := \inf\left\{ \int_0^1 \frac{1}{2}|\dot\omega|^2 - q(t,\omega)\, dt : \ \omega(0)=x,\ \omega(1)=y \right\}$$

the value function for the Lagrangian $\mathcal{L}_q := \int \frac{1}{2}|\dot\gamma|^2 - q(t,\gamma)$, we have also

$$\int_{\Omega^*(D)} \int_0^1 \frac{1}{2}|\dot\omega|^2 - q(t,\omega)\, dt\, d\boldsymbol{\eta}(\omega,a) \geq \int_D c_q^{0,1}(a,h(a))\, d\mu_D(a)$$

for any incompressible flow η between $\boldsymbol{i}$ and h. Moreover, equality holds only iff η-almost every (ω,a) is a $c_q^{0,1}$-minimizing path.

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SCUOLA NORMALE SUPERIORE

Theorem (Brenier, 1989)

Assume that we have a smooth solution to Euler's equations in $[0,T]\times D$, whose pressure field p satisfies

$$(*)\qquad T^2 \sup_{t\in[0,T]} \sup_{x\in D} |\nabla_x^2 p(t,x)| < \pi^2.$$

Then, the measure $\boldsymbol{\eta}$ induced by $\boldsymbol{u}$ via the flow map is optimal.

This follows by the fact that the paths $t\mapsto g(t,a)$ corresponding to classical solutions to Euler's equations satisfy $\ddot\omega(t) = -\nabla p(t,\omega)$, and (*) implies that stationary paths for the action are also minimal for $\mathcal{L}_p$.

How far are these conditions from being *necessary*? From now on, for simplicity we consider the case $D=\mathbb{T}^d$ only, and set $\mu_D=\mu_{\mathbb{T}}$.

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The pressure field satisfies $\nabla p \in \mathcal{M}_{\rm loc}\left((0,1) \times \mathbb{T}^d\right)$.

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Theorem (A-Figalli, 2007)

The pressure field belongs to $L^2_{\mathrm{loc}}\left((0,1); BV(\mathbb{T}^d)\right)$.

Necessary and sufficient optimality conditions

The two main degrees of freedom in optimal transport problems are:

- In moving mass from x to y, the path that should be followed;
- The amount of mass that should be moved from x to y.

In our case, both things will depend on $\mathcal{L}_p$. But, since p defined only up to negligible sets, the value of the Lagrangian $\mathcal{L}_p$ on a path ω is *not* invariant in the Lebesgue equivalence class. We had to:

- define a *precise* representative $\bar{p}$ in the Lebesgue equivalence class of p; it turns out that the correct definition is

$$\bar{p}(t,x) := \liminf_{\varepsilon \downarrow 0} p(t,\cdot) * \phi_\varepsilon(x).$$

- consider, in the minimization problem, only paths ω satisfying

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where

$$Mp(t,x) := \sup_{r\in(0,1)} \frac{1}{\omega_r r^d} \int_{B_r(x)} |p|(t,y)\,dy$$

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Theorem

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for any $\lambda \in \mathscr{P}(\mathbb{T}^d \times \mathbb{T}^d)$ having the same marginals of $(e_s,e_t)_{\#}\boldsymbol{\eta}_a$.

Conversely, if (i), (ii) hold with $\bar{p}$ replaced by some function q with $Mq \in L^1_{\rm loc}\left((0,1); L^1(\mathbb{T}^d)\right)$, then $\boldsymbol{\eta}$ is optimal, and q is the pressure field.

The second condition becomes meaningful only when $(e_s)_{\#}\boldsymbol{\eta}_a$ are *not* Dirac masses: it corresponds to the case when $(e_t, \pi_a)_{\#}\mu_D$ is *not* induced by a map, a phenomenon that can't be ruled out.

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Necessary and sufficient optimality conditions

Theorem

Let $\boldsymbol{\eta} = \boldsymbol{\eta}_a \otimes \mu_{\mathbb{T}}$ be an optimal incompressible flow between $\eta = \eta_a \otimes \mu_{\mathbb{T}}$ and $\gamma = \gamma_a \otimes \mu_{\mathbb{T}}$. Then

(i) *$\boldsymbol{\eta}$ is concentrated on locally minimizing paths for $\mathcal{L}_{\bar{p}}$;*

(ii) *for all intervals $[s,t] \subset (0,T)$, for $\mu_{\mathbb{T}}$-a.e. a, the plan $(e_s, e_t)_{\#}\boldsymbol{\eta}_a$ is $c_{\bar{p}}^{s,t}$-optimal, i.e.*

$$\int_{\mathbb{T}^d \times \mathbb{T}^d} c_{\bar{p}}^{s,t}(x,y)\, d(e_s,e_t)_{\#}\boldsymbol{\eta}_a \le \int_{\mathbb{T}^d \times \mathbb{T}^d} c_{\bar{p}}^{s,t}(x,y)\, d\lambda$$

for any $\lambda \in \mathscr{P}(\mathbb{T}^d \times \mathbb{T}^d)$ having the same marginals of $(e_s,e_t)_{\#}\boldsymbol{\eta}_a$.

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SCUOLA NORMALE SUPERIORE

Conclusion

These result show a connection with the theory of action-minimizing measures, first investigated by Mather, and then by Bangert, Bernard-Buffoni and others; the main difference is that in our case the Lagrangian $\int_0^1 \frac{1}{2}|\dot{\omega}|^2 - \bar{p}(t,\omega)\,dt$ is possibly non-smooth and not given a priori, but *generated* by the variational problem itself.

Here we see a nice variation on a classical theme of Calculus of Variations: a field of (smooth, nonintersecting) *extremals* gives rise both to *minimizers* and to an incompressible flow *in phase space*. Here, instead, we have a field of (possibly nonsmooth, or intersecting) *minimizers* which has to produce an incompressible flow *in the state space*.

This structure seems to be rigid, and might lead to new regularity results for the pressure field.

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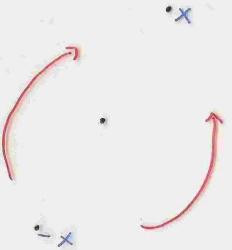
Non-deterministic geodesics

Back



$$\mathcal{L}(w) := \frac{1}{2}\int_0^{\pi} |\dot{w}|^2 - |w|^2 \, dt$$

$$w(t) = x\cos t + B\,\mathrm{sen}\,t, \qquad B \in \mathbb{R}^2$$

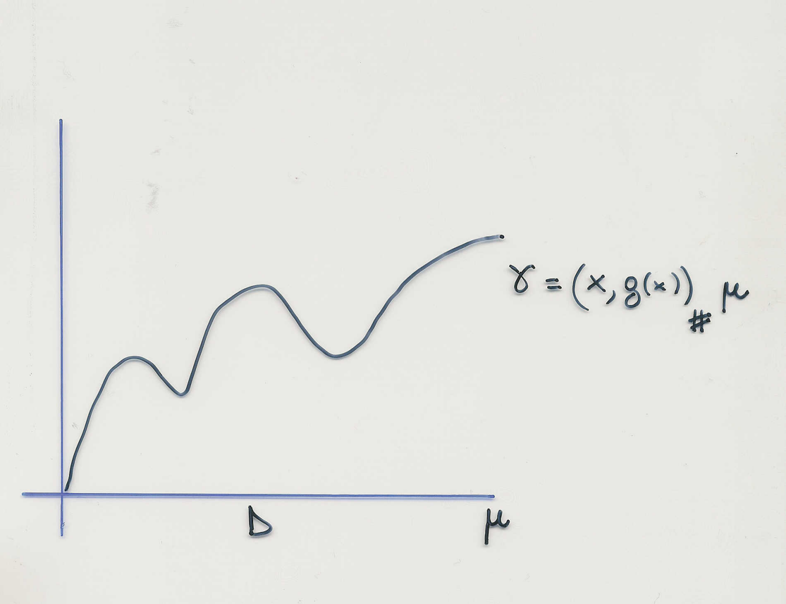
$$B = x^{\perp} \qquad P(t,x) = \frac{x^2}{2}$$

$$B = \sqrt{1-|x|^2}\,(\cos\theta, \mathrm{sen}\,\theta)$$

SCUOLA NORMALE SUPERIORE

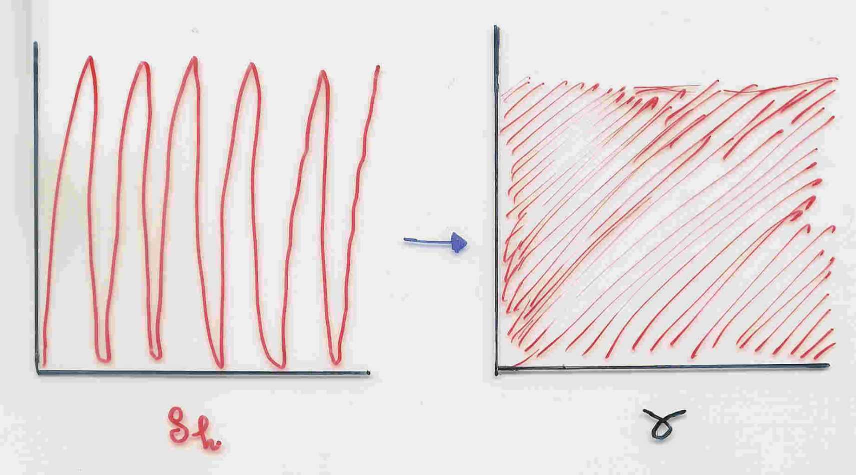
Transport maps and plans

Back



Density of maps among plans

Back



Branching

Back