Difference valued fields viewed as valued modules over the Ore ring of Frobenius polynomials.

Joint work with Luc Bélair, june 29, 2007.

Let K be a field of charateristic $p > 0$.

- [T. Rohwer] (PhD Thesis, 2003) If $[K : K^p] < \infty$, then the theory of $K((x))$ as valued module over the Ore ring of Frobenius polynomials is model-complete, whenever $Th(K)$ is.

- [T. Pheidas, K. Zahidi] (JSL 2004) If K is perfect, then the theory of $(K[x], +, u \rightarrow u^p, \cdot x)$ is decidable/ model-complete, whenever $Th(K)$ is.

- [Z. Chatzidakis, E. Hrushovski] (2002) Let (F_p, v) be a non-trivially valued field of characteristic p, let $q = p^n$ and $\sigma_q : u \rightarrow u^q$. They describe in certain cases the asymptotic theory (namely the theory of the class of non-principal ultraproducts over the set of prime numbers of valued difference fields (F_p, v, σ_q), when the residue field is a model of $ACFA$.

Let A be a complete discrete valuation ring with perfect residue field $A/\pi.A \cong \bar{K}$ (*).

• If A and $\bar{K}$ have the same characteristic, then $A \cong \bar{K}[[x]]$.

• If A and $\bar{K}$ have different characteristic namely $ch(A) = 0$ and $ch(\bar{K}) = p \neq 0$ and if $v(p) = 1$,
then $A \cong W[\bar{K}]$, where $W[K]$ is the ring of Witt vectors of K.

In both cases, if $\bar{\sigma}$ is an automorphism of $\bar{K}$, then it lifts in an automorphism σ of A such that

$$v(a) = v(\sigma(a)).$$

We will say that σ is an isometry. (Conversely, if σ is an isometry, it induces an automorphism on $\bar{K}$.)

We will consider valued fields endowed with an isometry as valued modules over a ring of non commuting polynomials. This kind of skew polynomial ring has been studied by O. Ore; $R := K[t;\sigma]$ where the non-commuting rule is the following: for all $c \in K$,

$$c.t = t.c^{\sigma}.$$

Let $(R, \oplus, \otimes, 0, 1)$ be a commutative ring of characteristic p and R^m the product of m copies of R. Denote by $W_m[R]$ the ring: $(R^m, +, ., 0, 1)$, where $0 := (0, \cdots, 0)$ and $1 := (1, 0, \cdots, 0)$.

The ring of Witt vectors $W[R]$ on R is defined as the inverse limit of the rings $W_m[R]$, $m \in \omega$.

If R is a field of characteristic p, then $W[R]$ is a domain of characteristic 0 and we denote its field of fractions $W(R)$.

Set $\sigma(r) := (r_0^p, r_1^p, \cdots, r_{m-1}^p, \cdots)$ on $W[R]$, since R has characteristic p, it is an endomorphism called the Witt Frobenius. Note that the valuation of the image of an element by the Witt Frobenius is equal to the valuation of this element, namely

$$v(\sigma(r)) = v(r).$$

If R is a perfect, then σ is an automorphism of $W[R]$, which extends to $W(R)$.

Let $v : W[R] - \{0\} \to \mathbb{N}$ the map sending $(r_0, r_1, \cdots)$ to the smallest natural number n such that $r_n \neq 0$.

The model theory of these difference valued fields has been studied by T. Scanlon (Fields Institute, 2003); Luc Bélair, Angus Macintyre and Thomas Scanlon (see Amer.J. of Math., 2007).

In particular, they showed model-completeness of $W(\tilde{\mathbb{F}}_p)$, where $\tilde{\mathbb{F}}_p$ is the algebraic closure of $\mathbb{F}_p$ and its decidability

Note that as a byproduct, they obtained other (than $ACFA$) theories of existentially closed difference fields since v is algebraically definable.

A field F of characteristic p is p-closed if F has no finite algebraic extension of order divisible by p. Set $D := \sigma - 1$.

[B,M,S] Let $(K := W(k), v, D)$, where k is p-closed. Then, $Th(K)$ is axiomatized by:

1. K is a D-henselian field of characteristic 0,
2. its residue field is a model of $Th(k)$,
3. its value group is a $\mathbb{Z}$-group with smallest element $v(p)$ and $D(x) \equiv x^p - x \pmod{p}$ for $x \in \mathcal{O}_K := \{x \in K : v(x) \geq 0\}$.

From now on we will consider at the same time, valued fields with an isometry (K, σ, v), and valued fields with a continuous σ-derivation (K, ∂, v), namely $v(\partial(a)) \geq v(a)$, satisfying a σ-Leibnitz rule:

$$\partial(a.b) = \partial(a).b^{\sigma} + a.\partial(b).$$

Note that since K is commutative, any continuous σ-derivative $\sigma \neq 1$ is of the form

$$c.(\sigma - 1),$$

where $c \in K$.

Theory of modules

Let R be the skew polynomial ring $K[t;\sigma,\partial]$, where

$$c.t = t.c^{\sigma} + \partial(c),$$

$c \in K$. This is an integral domain, right and left Euclidian.

Let $q(t) \in R - \{0\}$, $q(t) = \sum_{i=0}^{n} t^{n}.a_i$, where $a_i \in K$. Define $v(q(t)) := min\{v(a_i) :\ 1 \leq i \leq n\}$.
Since $v(a^{\sigma}) = v(a)$ and $v(\partial(a)) \geq a$, for all $a \in K$, it is a valuation on R (see A. Duval, 1983 and al.).

Let $\mathcal{O}_K$ be the valuation ring of K. Let $\mathcal{I} := \{q(t) : v(q(t)) = 0\}$. We consider the subring $R_0 := \mathcal{O}_K[t;\sigma,\partial]$ of R. It is an Ore domain i.e.

$$\forall a \neq 0 \forall b \neq 0 \exists a_1 \neq 0 \exists b_1 \neq 0 \ a.a_1 = b.b_1.$$

Let $L_R := \{+, -, 0, \cdot r; r \in R\}$ be the R-module language and let T_R be the L_R-theory of right R-modules.

Let $T_{d,\sigma}$ (respectively T_d) be the theory T_R plus:

1. $\forall m \, \exists n \, (m = n.t)$, & $\forall m \, (m.t = 0 \rightarrow m = 0)$,

2. $\forall m \, \exists n \, (n.q(t) = m)$, where $q(t)$ varies over the irreducible polynomials of R_0 such that $q(0) \neq 0$.

Recall that a valued field (F, v) is p-closed iff F satisfies the *"condition résiduelle de Kaplansky"*:

$$\forall a_0 \forall a_1 \cdots \forall a_n \forall b \ \exists x \ (\sum_{i=0}^{n} a_i . x^{p^i} + b = 0),$$

where $v(a_0) = \cdots = v(a_n) = 0$. [F. Delon, Thesis, 1983.]

Let F be a p-closed field of charateristic p.

- Let $K := W(F)$, σ the Witt Frobenius and $R := K[t; \sigma]$. Then, K is a model of $T_{d,\sigma}$. If F is algebraically closed, then its theory as an R-module is axiomatized by $T_{Ore} := T_{d,\sigma} \cup \{\exists u \neq 0 \ u.q(t) = 0;\ q(t)$ irreducible polynomial with $q(0) \neq 0\}$.

- Let $K := F((x))$, $\sigma : \sum_i f_i . x^i \to \sum_i f_i^p . x^i$ and $R := K[t; \sigma]$. Then, K is a model of $T_{d,\sigma}$. If F is algebraically closed, then its theory as an R-module is axiomatized by $T_{Ore} := T_{d,\sigma} \cup \{\exists u \neq 0 \ u.q(t) = 0; \ \ q(t)$ irreducible polynomial with $q(0) \neq 0\}$.

Besides the fact that F is p-closed, we use that $W(F)$ and $F((x))$ are complete.

Denote that $\bar{\sigma}$ and $\bar{\partial}$ the induced actions of σ and ∂ on $\bar{K}$.

Let (K, v, σ, ∂) be either a difference valued field or a valued field with a continuous σ-derivative operator ∂.
K has the *linear Hensel property* if for any separable polynomial $q[X] \in \mathcal{O}_K[X]$ with $\bar{q}[X] \neq 0$, for the corresponding linear difference (respectively σ-derivative) operators $q(\sigma)$ or $q(\partial)$ the following holds.

If $\exists b \in \mathcal{O}_K \ \bar{q}(\bar{\sigma})(\bar{b}) = \bar{a}$, where $a \in \mathcal{O}_K$ (respectively $\bar{q}(\bar{\partial})(\bar{b}) = \bar{a}$) and if for any $\lambda \in \mathcal{O}_K - \{0\}$ there exists $u \in \mathcal{O}_K$ such that $1 + \overline{q^{\lambda}}(\bar{\sigma})(\bar{u}) = \bar{0}$ (respectively $1 + \overline{q^{\lambda}}(\bar{\partial})(\bar{u}) = \bar{0}$), then there is an element c in $\mathcal{O}_K$ with $q(\sigma)(c) = a$ (respectively $q(\partial)(c) = a$) with $v(b - c) > 0$.

Any discrete complete valued field K endowed either with an isometry or a continuous σ-derivative operator has the linear Hensel property.

The field $\bar{K}$ is linearly $\bar{\sigma}$-closed (respectively $\bar{\partial}$-closed) if for any separable element $q(t)$ of $\mathcal{I}$, for any element $\bar{b} \in \bar{K}$ there is an element in $\bar{u}$ in $\bar{K}$ such that $\bar{q}(\bar{\sigma})(\bar{u}) = \bar{b}$ (respectively $\bar{q}(\bar{\partial})(\bar{u}) = \bar{b}$).

If σ induces the Frobenius on $\bar{K}$, then $\bar{K}$ is $\bar{\sigma}$-closed if it is p-closed.

Let K be a difference valued field or a valued field with a continuous σ-derivative operator ∂. Assume that $\bar{K}$ is either linearly $\bar{\sigma}$-closed or $\bar{\partial}$-closed and suppose that K has the linear Hensel property. Then, K, viewed as an $A_0 := \mathcal{O}_K[t; \sigma, \partial]$-module, is $\mathcal{I}$-divisible.

Let (K, σ, ∂, v) be a difference (respectively endowed with a σ-derivative operator) valued field.
Assume that K has the linear Hensel property and that its residue field is linearly $\bar{\sigma}$-closed (respectively $\bar{\partial}$-closed).
In the case where K is a valued field with a σ-derivative, assume in addition that for any a in K there exists b such that $a = \partial b$.
Then K is a $K[t; \sigma, \partial]$-module satisfying T_d.

Let F be a linearly ∂ closed field of characteristic 0, where for any $a \in F$ there exists b such that $a = \partial b$.
Let $K := F((x^{-1}))$ and set $v(x^{-1}) = 1$, and $R := K[t; \partial]$.
Then, K is a model of T_d.

T_d admits positive quantifier elimination (namely that every positive primitive (p.p.) formula is equivalent to a conjunction of atomic formulas).

The positive q.e. result follows from classical result in the model theory of modules over a right and left Euclidean ring and in particular from the fact that any p.p. formula is equivalent to divisibility or annihilators conditions on the parameters.

The completions of T_d are obtained, in the case where $Fix(\sigma) \cap K_\partial$ is infinite by specifying for which irreducible polynomials $q(t)$ with $q(0) \neq 0$ one has $ann(q(t)) \neq \{0\}$. Each of the completions admits q.e.

Let K be a difference valued field ($\partial = 0$). Assume that $\bar{\sigma}$ and $\bar{\sigma} - \bar{\mu}$ with $\bar{\mu} \in \bar{K}$, are surjective on $\bar{K}$ and that K has the linear Hensel property. Then, $Fix(\sigma)$ is infinite, whenever the valuation is non trivial on K.

When $Fix(\sigma) \cap K_\partial$ is finite, we have to specify the cardinality of the annihilators $ann(q(t))$ where $q(t)$ varies over the irreducible polynomials.

Decidability issues.

It depends on the fact that the ring R is (countable) recursively presented and part of its existential theory is decidable. Then one has to decide $|ann(r(t))|$ for each irreducible element $r(t)$ of R.

?Ax-Kochen-Ershov implies that if $Th_{\exists}(F)$ decidable then $Th_{\exists}(W(F))$ decidable.

Valued Modules (see L. van den Dries, 1981, B.S.M.B.).

Let (K, σ, ∂, v) be a difference valued field with valuation v, isometry σ and continuous σ-derivative ∂. The value group $(\Gamma, +, \leq, 0, 1)$ of K is a totally ordered abelian group.
Let M be an R-module, where $R := K[t; \sigma, \partial]$.
Let $w : M \to \Delta \cup \{+\infty\}$, where $(\Delta, \leq)$ is a totally ordered and $+\infty$ a new element strictly bigger than Δ with the following properties:
$\forall m_1 \in M \forall m_2 \in M \ w(m_1 + m_2) \geq min\{w(m_1), w(m_2)\}$,
$w(0) = +\infty$,
$\forall m \in M \ w(m.t) \geq w(m)$,
$\forall m \in M \ w(m.\lambda) = w(m) + v(\lambda)$, for all $\lambda \in K - \{0\}$, where $+$ denotes the action of the group Γ on Δ.
(So, $\Gamma \subseteq Aut(\Delta, \leq)$.)

The structure $((\Delta, \leq), (\Gamma, +, -, 0, \leq), +)$ satisfies the following axioms T_{Δ}:

- For all $\delta \in \Delta$ and $\gamma, \gamma_1, \gamma_2 \in \Gamma$, then $\delta + \gamma \in \Delta$ and $(\delta + \gamma_1) + \gamma_2 = \delta + (\gamma_1 + \gamma_2)$.
- Moreover this action respects the order: for all $\delta_1, \delta_2 \in \Delta$ and $\gamma_1, \gamma_2 \in \Gamma$, if $\Delta \models \delta_1 \leq \delta_2$ and $\Gamma \models \gamma_1 \leq \gamma_2$, then $\delta_1 + \gamma_1 \leq \delta_2 + \gamma_2$.

Abelian Structures.

Let M be a valued R-module. We endow M with a chain of subgroups, indexed by the elements of $\Gamma = v(R)$.

$$V_\gamma = \{m \in M \;\; w(m) \geq \gamma\}.$$

Every subgroup V_γ is invariant by multiplication by $\mathcal{O}_K$; and so every V_γ is an $R_0 = \mathcal{O}_K[t;\sigma]$-module.

Let $\mathcal{L}_V := \mathcal{L}_R \cup \{V_\gamma; \gamma \in \Gamma\}$, let

$$\mathcal{M} := (M, +, 0, .r; r \in R, V_\gamma; \gamma \in \Gamma).$$

Let T_V be the theory T_R with the axioms (1) up to (6).
Let T_V^* be the theory T_V with axiom scheme (7).
Recall that $\mathcal{I} := \{q(t) \in \mathcal{O}_K[t;\sigma] : \; v(q(t)) = 0\}$.

1. $\forall m \, \exists n \; (m = n.t)$, [& $\forall m \; (m.t = 0 \rightarrow m = 0)$],

2. $\forall m \, \exists n \, (n.q(t) = m)$, where $q(t)$ varies over irreducible polynomials of R such that $q(0) \neq 0$,

3. $\forall m \; (V_\delta(m) \to V_\gamma(m)$, for all $\delta > \gamma$,

4. $\forall m \; (V_\gamma(m) \to V_\gamma(m.t))$,

5. $\forall m_1 \, \forall m_2 \; (V_\gamma(m_1) \,\&\, V_\gamma(m_2) \to V_\gamma(m_1 + m_2))$,

6. $\forall m \; (V_\gamma(m) \to V_{\gamma+v(\lambda)}(m.\lambda))$, for any $\lambda \in K$,

7. $\forall m \; (V_\gamma(m) \to V_{\gamma+v(q(t))}(m.q(t)))$, where $q(t) \in K[t, \sigma]$,

8. $\forall m \in V_\gamma \, \exists n \in V_\gamma \; n.q(t) = m$ for all $q(t) \in \mathcal{I}$.

A model of T_V^* is for instance $W(F)$, where F is p-closed, endowed with the chain of subgroups
$\{m \in W(F) : v(m) \geq \gamma\}$, $\gamma \in v(K)$.

The index sentences are of the form:

$$|(V_{\gamma_1} \cap ann(q_1(t)))/(V_{\gamma_2} \cap ann(q_2(t)))| \geq n,$$

$\gamma_1, \gamma_2 \in \Delta$, $q_1(t), q_2(t) \in \mathcal{I}$.

[Bélair, P] T_V^* admits q.e. modulo index sentences.

Back to two-sorted structures.

(M, w, Δ); $(\Delta, +, \Gamma)$, where $v(K) = \Gamma$ et $w : M \to \Delta$.

Assume that
either $(\Delta, \leq)$ is densely ordered,
or $(\Gamma, +, -, \leq, 0, 1)$ is an abelian discretely ordered group with 1 as the smallest strictly positive element and
Δ satisfies: $(\star)$

$$\forall \delta_1 \exists \delta_2 \forall \delta_3 \, (\delta_2 > \delta_1 \ \& \ (\delta_3 > \delta_1 \to (\delta_2 \leq \delta_3 \ \& \ \delta_2 = \delta_1 + 1)).$$

Let $T_{\Delta,dense}$ be the theory T_Δ plus les axioms expressing that $(\Delta, \leq)$ is densely ordered and let $T_{\Delta,discrete}$ the theory T_Δ together with:
Γ discretely ordered, Δ satisfies the axiom $(\star)$ above.

In both cases, the structure $< (\Delta, \leq), (\Gamma, +, -, \leq, 0), + >$ admits quantifier elimination in sort Δ.

Let T_w be the following theory of valued R-modules considered as two-sorted structures $(M, \Delta \cup \{\infty\}, w)$ where M is an R-module, $w : M - \{0\} \rightarrow \Delta$, $w(0) = +\infty$ such that:

1. $M \models T_R$,

2. $((\Delta, \leq), (\Gamma, +, -, 0, \leq), +) \models T_\Delta$, where $\Gamma = v(R)$

3. $\forall m_1 \in M \forall m_2 \in M \; w(m_1 + m_2) \geq min\{w(m_1), w(m_2)\}$, $w(0) = +\infty$,

4. $\forall m \in M \; w(m.t) \geq w(m)$,

5. $\forall m \in M \; w(m.\lambda) = w(m) + v(\lambda)$, for all $\lambda \in K - \{0\}$,

Let T (respectively T_{dense} or $T_{discrete}$) the theory of valued R-modules in the two-sorted language $\mathcal{L}_w$.

1. T_w,

2. T_Δ, (respectively $T_{\Delta,dense}$ or $T_{\Delta,discrete}$)

3. Divisibility axioms (DG): let $q(t) \in \mathcal{I}$:
   $\forall u \in M - \{0\}\ \exists u_1\ (u = u_1.q(t)\ \&\ w(u) = w(u_1)$.

4. Axioms (IR): Given $p_1(t), \cdots, p_m(t) \in \mathcal{I}$ and $\gamma \in \Delta$:
   $\forall u_1, \cdots, \forall u_m \exists u \in M\ \ (\bigwedge_{i=1}^m w(u_i) \geq \gamma) \rightarrow$

$$\bigwedge_{i=1}^{m} w(u.p_i(t) + u_i) = \gamma).$$

An index residual formula $\chi(\delta)$ is a formula of the form: $r(t) \in R_0$, $V_{\delta+} := \{m \in V_\delta : w(m) > \delta\}$, $n \in \omega$,

$$\exists u_1 \cdots \exists u_n \bigwedge_{1 \leq i < j \leq n} u_i \not\equiv_{V_{\delta+}} u_j \ \& \ u_i.r(t) \equiv_{V_\delta} 0.$$

Equivalently, $|ann(r(t)) \cap V_\delta / V_\delta^+| \geq n$.
A residual sentence is of the form: $\exists \delta \in \Delta \quad \chi(\delta)$, where $\chi(.)$ is an index residual sentence.

A valued index formula is of the form $|ann(r(t)) \cap V_\delta| \geq n$.

[Bélair-P.]In T, every existential $\mathcal{L}_w$-formula in sort M is equivalent to a quantifier-free formula in sort M, and to an existential formula in sort Δ together with index residual formula.

Let $\mathcal{M} := (M, w, \Delta_M)$, $\mathcal{N} := (N, w, \Delta_N)$ where $\mathcal{M} \subset \mathcal{N}$, are two valued R-modules satisfying T.
Assume that $\mathcal{M}$ et $\mathcal{N}$ satisfy the same index residual (respectively index) formulas and that $\Delta_M \subset_{ec} \Delta_N$. Then, $\mathcal{M} \subset_{ec} \mathcal{N}$.

If the action of Γ on Δ is transitive, then in the theory T_{dense} (respectively $T_{discrete}$), every $\mathcal{L}_w$-formula $\varphi(\bar{x})$ in sort M is equivalent to a q.e. formula $\theta(\bar{x}, \delta_0)$ of $\mathcal{L}_w$, where $\delta_0 \in \Delta$.

NIP. Let T be a complete theory, then T has the independance property (IP) if one of its formula has IP.
A formula $\varphi(x,\bar{y})$ has the IP if the following axioms I_n are consequences of T:

$$I_n : \exists x_0 \cdots \exists x_{n-1} \exists \bar{y}_{\emptyset} \cdots \exists \bar{y}_w \cdots \exists \bar{y}_{2^n} \bigwedge_{i \in w} \varphi(x_i, \bar{y}_w) \ \& \ \bigwedge_{i \notin w} \neg\varphi(x_i, \bar{y}_w).$$

We will abbreviate the property of "not having the independence property" by NIP.

• The theories $T_{\Delta,dense}$ and $T_{\Delta,discrete}$ of $(\Gamma, \Delta, +)$ have the NIP.

•• Using this result and the previous q.e. result, we show that $T_{discrete}$ (respectively T_{dense}) has the NIP.
Therefore the definable sets can be endowed with a VC-dimension.

Ultraproducts/ Torsion.

• Can we characterize the elements of $W(\tilde{\mathbb{F}}_p)$ which belong to the torsion submodule and also these which belong to $ann(q(t))$ for a specific $q(t)$?
We have that $Tor(W(\tilde{\mathbb{F}_p})$ is included in $\bigcup_n W((\mathbb{F}_{p^n}^{p^{-m}})_m))$, but there exists $a \in W(\mathbb{F}_p)^*$, $ann(t+a) \not\subset \bigcup_n W(\mathbb{F}_{p^n})$.

•• Let U be a non-principal ultrafilter on $\mathcal{P}$. Let F_p be a p-closed field of characteristic p of cardinality at most $\aleph_1$. Let σ_{pwf} be the Witt Frobenius on $W(F_p)$ and let σ_{pc} be the automorphism sending $\sum_{i\geq 0} x^i.a_i$ to $\sum_{i\geq 0} x^i.a_i^p$ in $F_p((x))$. We consider the difference valued fields corps $(W(F_p), \sigma_{pwf})$, and $(F_p((x)), \sigma_{pc})$. We denote by σ_{wf} and σ_c the automorphisms induced on the ultraproducts of these fields.

Let $A_1 := \prod_U W(\mathbb{F}_p)[t]$ the skew polynomial ring on $\prod_U \mathbb{Q}_p$. We consider $\prod_U W(F_p)$ and $\prod_U F_p((t))$ as A_1-modules with t acting as the Witt Frobenius σ_{wf} on $\prod_U W(F_p)$ and as σ_c on $\prod F_p((t))$.

As modules, they are elementarily equivalent whenever the annihilator of $\sum_i t^i.a_i$, where $a_i \in \prod_U \mathbb{Q}_p$ is non trivial in $\prod_U W(F_p)$ iff it is in $\prod_U F_p((x))$. Since these two structures satisfy Hensel's Lemma and since the residue fields are isomorphic to $\prod_U F_p$ and since σ_{wf} and σ_c act as the Frobenius on the residue fields, we get the result.

As valued modules, we show that they satisfy the schemes (DG) and (IR), then we calculate the cardinality of the annihilators in the quotient V_0/V_0^+ ou in the subgroup V_0; and so either in $\prod_U F_p$ or in the subgroup $\prod_U W[F_p]$ (respectively in $\prod_U F_p[[x]]$). We use the fact that $\prod_U F_p$ is infinite, and does not satisfy any identities; the axiom (DG) is also satisfied (Hensel's Lemma and F_p is p-closed).