Signatures of Spin-Statistics Violations in Noncommutative QFTs

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Introduction

1. Motivations for Spacetime Noncommutativity
2. Problems with Traditional Quantization
3. Formalism of Twisted Quantization
   - Commutative and Noncommutative Algebras
   - Implementing Poincaré Symmetry
   - Hopf Algebras and Drinfel'd Twist
   - Quantum Mechanics and Field Theory
4. Gauge Fields on Moyal Space
   - Covariant Derivatives and Field Strength
   - Noncommutative Gauge Theories
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Why study noncommutative spacetime?

Heisenberg (letter to Peierls, sometime in the 30's):

Noncommutativity may help to cut-off ultraviolet divergences in QFT's.

Quantum Gravity near Planck Scale

Properties of spacetime are almost certainly expected to change at short distances. Spacetime points replaced by some other structure.

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End-points of an open string in a B-field background "live" on in a noncommutative space.

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Arguments from uncertainty principle

- In order to probe physics at the Planck length scale l_P, the Compton wavelength $\hbar/Mc$ of the probe must satisfy

$$\frac{\hbar}{Mc} \leq l_P \implies M \geq \frac{\hbar}{l_P c}$$

- Such a large mass concentrated in so small a volume (l_P^3) will lead to the formation of black holes and horizons.
- This suggests a fundamental length limiting spatial localization.
- Similar arguments can also be made about time localization.

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Noncommutative Spacetime (Moyal Algebra)

A concrete model that realizes these uncertainties is the algebra generated by operators $\hat{x}_\mu$:

(Doplicher-Fredenhagen-Roberts, 1995)

$[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu}, \quad \theta_{\mu\nu} =$ (fixed) constant antisymmetric matrix.

- Analogy with Standard Quantum Mechanics:
- Classical phase space (a commutative manifold) is replaced in QM by a “noncommutative” manifold $[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}$.
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Oddities of Conventional Quantization - Kinematics

Conventional Quantization

Free quantum fields on Moyal space are identical to those in ordinary space.

- The fundamental commutation relation $[x_\mu, x_\nu] = i\theta_{\mu\nu}$ seems to break Lorentz invariance explicitly, making it difficult to use standard tensor analysis to define scalars, spinors, vector fields and so on.
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Oddities of Standard Quantization - Dynamics

- Field Theories (even of the simplest kind) that are quantized conventionally exhibit weird behaviour – no decoupling between low- and high-energy degrees of freedom. So scattering amplitudes at, say, LHC will be exceedingly sensitive to details of Planck scale physics.
- If $\theta_{0i} \neq 0$, then QFT's are apparently non-unitary, even if the Hamiltonian is hermitian.
- Despite these difficulties, an energetic researcher may soldier on and try to construct gauge theories, only to discover that only $U(N)$ gauge theories can be consistently constructed. So it is very difficult to make models that have interesting phenomenology.

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One can do better!

Twisted Quantization

There is an alternative quantization that preserves Poincaré invariance.

Unitarity

The QFT is unitary (if the Hamiltonian is hermitian) for a general $\theta_{\mu\nu}$.

Immaculate High Energy Behaviour

Decoupling of high- and low-energy degrees of freedom.

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We can construct gauge theories with any gauge group.

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The Moyal algebra

- Elements of the noncommutative algebra are expressions like $a_0 + a_1^\mu \hat{x}_\mu + a_2^{\mu\nu} \hat{x}_\mu \hat{x}_\nu + \cdots$ (modulo the basic commutation relations).
- A simple realization in terms of ordinary functions: $\hat{x}_\mu$'s don't commute $\equiv$ change the multiplication rule.

Moyal (or star) product in terms of commutative product

$$(f * g)(x) = m_\theta(f \otimes g)(x) = m_0(e^{\frac{i}{2}\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu} f \otimes g)(x),$$

$$= m_0(\mathcal{F} f \otimes g)(x) = (f \cdot g)(x) + \frac{i}{2}\theta^{\mu\nu}(\partial_\mu f \cdot \partial_\nu g)(x) + \cdots$$

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(Naive) Lorentz Transformations

- Under a Lorentz transformation Λ, functions f and g transform as

$$f(x) \rightarrow f^{\Lambda}(x) = f(\Lambda^{-1}x), \quad g(x) \rightarrow g^{\Lambda}(x) = g(\Lambda^{-1}x)$$

$$\begin{aligned} (f \cdot g)^{\Lambda}(x) &= (f^{\Lambda} \cdot g^{\Lambda})(x), \quad \text{BUT} \\ (f * g)^{\Lambda}(x) &\neq (f^{\Lambda} * g^{\Lambda})(x)!! \end{aligned}$$

- This is the reason why we can(not) do Lorentz invariant quantum theories on (non)commutative $R^{1,3}$.
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A Closer Look at the Moyal Algebra

- Left- and right- multiplications are not the same:

$$\hat{x}^L_\mu f = x_\mu * f, \quad \hat{x}^R_\mu f = f * x_\mu.$$

- The left and right actions satisfy:

$$[\hat{x}^L_\mu, \hat{x}^L_\nu] = i\theta_{\mu\nu} = -[\hat{x}^R_\mu, \hat{x}^R_\nu], \quad [\hat{x}^L_\mu, \hat{x}^R_\nu] = 0.$$

- Define (a commuting) $\hat{x}^c_\mu$ in terms $\hat{x}^L_\mu, \hat{x}^R_\mu$ as

$$\hat{x}^c_\mu \equiv \frac{1}{2}\left(\hat{x}^L_\mu + \hat{x}^R_\mu\right), \quad [\hat{x}^c_\mu, \hat{x}^c_\nu] = 0,$$

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A Second Look at Lorentz Transformations

- Under Λ, $f(x) \to f^{\Lambda}(x) = f(\Lambda^{-1}x)$. This is an operation on a single function, and does not require the star (or any) product.
- Under an infinitesimal transformation $\Lambda \simeq \mathbf{1} + i\epsilon^{\mu\nu}M_{\mu\nu}$,

$$f^{\Lambda}(x) \simeq f(x) - i\epsilon^{\mu\nu}(x_\mu\partial_\nu - x_\nu\partial_\mu)f(x).$$

  Notice that in the above, there is no star!
- So $M_{\mu\nu} = \hat{x}^c_\mu \hat{p}_\nu - \hat{x}^c_\nu \hat{p}_\mu \qquad (\hat{p}_\mu = -i\partial_\mu)$
- Actually, this is how an arbitrary vector field also acts on noncommutative functions: $\hat{v}f = [v(\hat{x}^c_\mu)\partial_\mu f](x)$.
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Modified Leibnitz rule

- Although the $M_{\mu\nu}$ correctly generate the Lorentz algebra, their action on the star product of two functions is different:

$$
\begin{aligned}
& M_{\mu\nu}(\alpha * \beta) = (M_{\mu\nu}\alpha) * \beta + \alpha * (M_{\mu\nu}\beta) \\
- \quad & \frac{1}{2}\left[((\hat{p}\cdot\theta)_\mu\alpha) * (\hat{p}_\nu\beta) - (\hat{p}_\nu\alpha) * ((\hat{p}\cdot\theta)_\mu\beta) - \mu \leftrightarrow \nu\right], \\
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On Coproduct

- The usual coproduct $\Delta_0(\Lambda) = \Lambda \times \Lambda$ is compatible ordinary multiplication, but not with Moyal multiplication.
- But a twisted coproduct Δ_θ defined as
$$\Delta_\theta(\Lambda) = \mathcal{F}^{-1}\Delta_0(\Lambda)\mathcal{F}$$
is compatible with Moyal product!
- Indeed, $m_\theta[\Delta_\theta(\Lambda) f \otimes g] = \rho(\Lambda) m_\theta(f \otimes g)$.
- For infinitesimal Lorentz transformations, the twisted coproduct reproduces our earlier result:

$$\begin{aligned} \Delta_\theta(M_{\mu\nu}) &= M_{\mu\nu} \otimes \mathbf{1} + \mathbf{1} \otimes M_{\mu\nu} \\ &- \frac{1}{2}\left[(\hat{p}\cdot\theta)_\mu \otimes \hat{p}_\nu - \hat{p}_\nu \otimes (\hat{p}\cdot\theta)_\mu - (\mu \leftrightarrow \nu)\right] \end{aligned}$$

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Implications for Quantum Statistics

- In usual quantum mechanics, the wavefunction of two identical particles is the (anti-)symmetrized tensor product of single particle wavefunctions:

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Implications for Quantum Statistics

- For example, for plane waves $e_p(x) = e^{-ip\cdot x}$ we get

$$\begin{aligned} e_p \otimes_{S_\theta, A_\theta} e_q &= \pm e^{-ip_\mu \theta^{\mu\nu} q_\nu} e_q \otimes_{S_\theta, A_\theta} e_p \,, \\ (e_p \otimes_{S_\theta, A_\theta} e_q)(x_1, x_2) &= \pm e^{-i \frac{\partial}{\partial x_1^\mu} \theta^{\mu\nu} \frac{\partial}{\partial x_2^\nu}} (e_p \otimes_{S_\theta, A_\theta} e_q)(x_2, x_1) \,. \end{aligned}$$

Twisted Quantum Fields

- Suppose $\Phi(x)$ is a second-quantized field, and $a_p^\dagger$ the creation operator with momentum p. As usual we require that

$$\begin{aligned} \langle 0|\Phi^{(-)}(x)a_p^\dagger|0\rangle &= e_p(x), \\ \langle 0|\Phi^{(-)}(x_1)\Phi^{(-)}(x_2)a_q^\dagger a_p^\dagger|0\rangle &= (\mathbf{1} \pm \tau_\theta)\,(e_p \otimes e_q)(x_1, x_2) \\ &\equiv (e_p \otimes_{S_\theta, A_\theta} e_q)(x_1, x_2) \end{aligned}$$

- This gives us the twisted commutation relations:

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- Interestingly, we can realize the twisted operators $a_p, a_p^\dagger$ in terms on usual Fock space operators $c_p, c_p^\dagger$:

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Simple Implications of Twisted Statistics

- Consider a two-fermion state

$$|\alpha, \beta\rangle = \int d\mu)p_1)d\mu(p_2)\alpha(p_1)\beta(p_2)a^\dagger(p_1)a^\dagger(p_2)|0\rangle$$

Notice that $|\alpha, \alpha\rangle$ does not vanish!

- This is an example of a "Pauli-forbidden" state.
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Gauge transformations

- Gauge fields A_λ transform as one-forms under diffeos generated by vector fields. They could be functions of $\hat{x}^c$ or $\hat{x}^L$.
- If $A_\lambda = A_\lambda(\hat{x}^c)$, then we can write gauge theories for arbitrary gauge groups. These theories are identical to the corresponding commutative ones.
- If $A_\lambda = A_\lambda(\hat{x}^L)$, then we can only construct $U(N)$ gauge theories.

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Gauge Covariant Derivatives

- Under a gauge transformation $g(\hat{x}^c)$, a charged matter field $\Phi(x)$ transforms as $\Phi(x) \rightarrow g(x)\Phi(x)$.
- The quantum covariant derivative D_μ must respect this *module* property of the gauge group:

$$D_\mu(g\Phi) = gD_\mu\Phi + (\partial_\mu g)\Phi$$

- D_μ must also respect (twisted) statistics, and Poincaré covariance.
- The only one which does this is

$$D_\mu\Phi = (D^c_\mu\Phi^c)e^{\frac{1}{2}\overleftarrow{\partial}_\mu\theta^{\mu\nu}\mathcal{P}_\nu}$$

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Matter-Gauge Interactions

- The interaction Hamiltonian is of the form

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H^I_\theta &= \int d^3x [\mathcal{H}^{MG}_\theta + \mathcal{H}^G_\theta], \\
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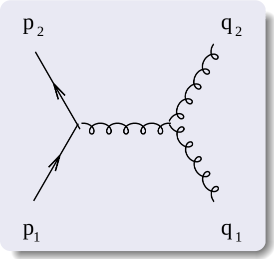
Non-Abelian Gauge Theories



- Processes like $qg \to qg$ violate Lorentz invariance (the propagator is “frame-dependent”).

QED from Spontaneously Broken $SU(2) \times U(1)$

- The gauge group for the Standard Model is non-Abelian, and will show similar effects.
- In particular, signatures of Lorentz (or spin-statistics) violation can be seen in QED.
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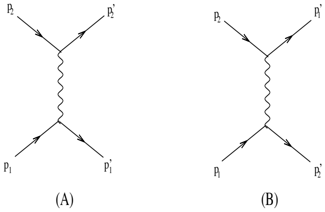
Möller scattering in QED

The interaction Hamiltonian is
$H_I = \frac{e}{2} \int d^3x [\bar{\psi}(x) * \left(\not{A}(\hat{x}^c)\psi(x)\right) + h.c.]$

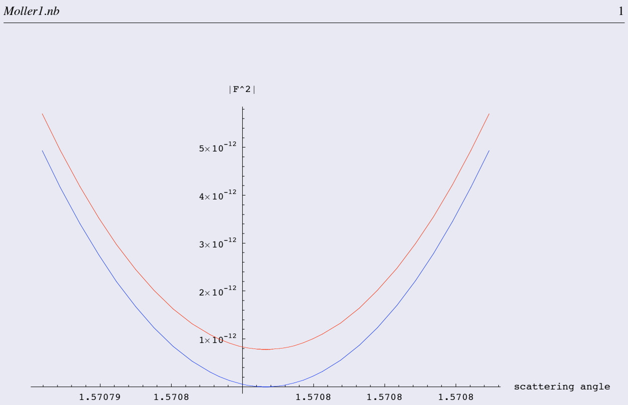


Möller scattering in QED

- We can calculate the scattering amplitude $\mathcal{T}_\theta$ in the centre-of-momentum frame, with the spins of the electrons aligned. It depends on scattering angle Θ_M, dimensionless c.m energy $x = E/m$, and $t = m^2\theta_{ij}\epsilon^{ijk}(\hat{p}_F \times \hat{p}_I)^k$.

Möller scattering in QED

Normalized scattering cross-section for $t = 10^{-5}$ and $x = 100$



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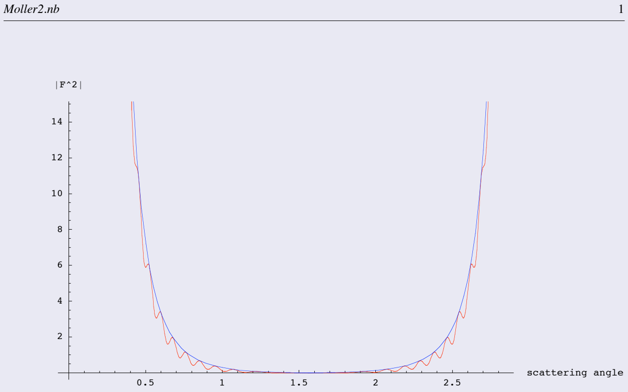
Normalized scattering cross-section for $t = 10^{-2}$ and $x = 100$



Summary

- By taking advantage of new algebraic structures (twists) from Hopf algebra theory, it is indeed possible to discuss Lorentz-invariant QFT's on Moyal space.
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Future Directions

- Outlook
  - Spontaneous Symmetry breaking can also be discussed in this framework (with A. P. Balachandran and T. R. Govindarajan). This will give us the noncommutative Standard Model, and phenomenological signatures.
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Collaborators I

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- T. R. Govindarajan, G. Mangano
- F. Lizzi and P. Vitale
- B. Chakraborty, F. Scholtz and J. Goevarts

References

- `arXiv:hep-th/0508002`
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- `arXiv:0707.3858 [hep-th]`, `arXiv:0708.0069 [hep-th]`, `arXiv:0708.1379 [hep-th]`, `arXiv:0709.3357 [hep-th]`