Noninertial relativity groups consistent with Heisenberg commutation relations

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Introduction: Consistency of relativity group with Heisenberg commutation relations

Question: What is the condition on a relativity group for the Heisenberg commutation relations to hold for all states related by the group (on a flat manifold)

The Weyl-Heisenberg group $\mathcal{H}(m) \simeq \mathcal{A}(m) \otimes_s \mathcal{A}(m+1)$, $\mathcal{A}(m) \simeq (\mathbb{R}^m, +)$ is a matrix group with manifold $\mathbb{R}^{2m+1}$

Unitary representations ϱ on Hilbert space $\mathbf{H} = \mathbf{L}^2(\mathbb{R}^m, \mathbb{C})$ given by Stone - von Neuman or generally Mackey theorems

The Heisenberg commutation relations are the Hermitian representation of the Lie algebra $\hat{Z} = \varrho'(Z)$.

Any commuting subset of the operators $\{\hat{Z}_\alpha\} = \{\hat{T}, \hat{Q}_i, \hat{E}, \hat{P}_i, \hat{I}\}$ may be diagonalized. Diagonalize $\{\hat{T}, \hat{Q}_i\}$:

$$\langle q, t| \hat{Q}_i |\psi\rangle = q_i \psi(q, t), \quad \langle q, t| \hat{T} |\psi\rangle = t\,\psi(q, t), \quad \langle q, t| \hat{I} |\psi\rangle = \hbar\,\psi(q, t),$$

$$\langle q, t| \hat{P}_i |\psi\rangle = i\hbar \frac{\partial}{\partial \tilde{q}^i} \psi(q, t), \quad \langle \tilde{q}, \tilde{t}| \hat{E} |\psi\rangle = -i\hbar \frac{\partial}{\partial t} \psi(q, t), \quad \psi \in \mathbf{L}^2(\mathbb{R}^m, \mathbb{C}), \ \{q_i, t\} \in \mathbb{R}^m$$

Equally well diagonalize $\{\hat{T}, \hat{P}_i\}$ with wave function $\psi(p, t)$, $\{\hat{Q}_i, \hat{E}\}$ with $\psi(q, e)$or $\{\hat{E}, \hat{P}_i\}$ with $\psi(p, e)$.

The Weyl-Heisenberg group has no spacetime *bias* - all of these diagonalizations are equally valid.

We will show this requires that the central extension $\check{\mathcal{G}}$ is a subgroup of the automorphism group of $\mathcal{H}(m)$

Introduction to Problem: Relativity group for noninertial states

Second Question: How are the clocks for noninertial states related where the noninertial states are due to forces other than gravity and therefore the manifold is flat.

Special relativity answers this for inertial states. Clocks related by Minkowski metric

$$d\tau^2 = \eta_{a,b}\, d x^a\, d x^b = d t^2 - \frac{1}{c^2}\, d q^2 = d t^2 (1 - v^2 / c^2), \quad a, b = 0, 1 .. n = 3$$

If the noninertial states are due to gravity, then general relativity answers the question.

The equivalence principle of general relativity states that these states equivalent to locally inertial state in a curved manifold.

Particles only under the influence of gravity follow geodesics that are the *locally inertial* trajectories on the curved manifold. Neighboring local inertial frames are related by the connection.

Clocks related by the Riemannian line element $d\tau^2 = g_{a,b}(x)\, d x^a\, d x^b$

In a purely gravitating system, there are no noninertial states.

Question remains: What is the relativity group for noninertial states for a flat manifold where noninertial motion is caused by a force other than gravity?

Introduction: Noninertial relativity

We will investigate noninertial relativity groups consistent with the Heisenberg algebra with proper time for noninertial states defined by the Born-Green metric on $\mathbb{P} \simeq \mathbb{R}^{2m}$ and its limits, the Minkowski and Newtonian time line elements

Newtonian time $d t^2$:

> Hamilton mechanics relativity that has an absolute inertial rest frame. Velocity and force are additive and unbounded. Time $\{t\}$ is an invariant subspace in $\mathbb{P}$.

Minkowski proper time: $d\tau^2 = d t^2 - \frac{1}{c^2} d q^2$:

> Relativity with absolute inertial frame but no absolute rest frame. Velocity is relative to states and bounded by c, force f are unbounded. Spacetime $\{t, q^i\}$ is an invariant subspace in $\mathbb{P}$. Time is relative.

Born-Green proper time: $d s^2 = d t^2 - \frac{1}{c^2} d q^2 + \frac{1}{b^2}\left(\frac{1}{c^2} d e^2 - d p^2\right)$, $\quad b = \alpha_b c^4 / G$

> Reciprocal relativity with Born-Green time has neither an absolute inertial frame nor an absolute rest frame. Velocity is relative to states and bounded by c, force f is relative to states and bounded by b. Spacetime $\{t, q^i\}$ is not an invariant subspace in $\mathbb{P}$, it is relative to noninertial frame.

We call this *reciprocal relativity* after Max Born's reciprocity (Edinburgh 1939-1949)

Consistency of relativity with Heisenberg commutators

What is the largest relativity group consistent with Heisenberg commutation relations where the manifold is flat

Physical states in quantum mechanics are rays Ψ in a Hilbert space $\mathbf{H}$, $\Psi = \{e^{i\omega} |\psi\rangle \mid \omega \in \mathbb{R}\}$, $|\psi\rangle \in \mathbf{H}$

Relativity group $\mathcal{G}$ acts through projective representations on these states and observables.

This is equivalent to the unitary representations ϱ of the central extension $\check{\mathcal{G}}$.

Consider observables of time, position, energy, momentum $\{\hat{Z}_\alpha\} = \left\{\hat{X}_a, \hat{P}_a\right\} = \left\{\hat{T}, \hat{Q}_i, \hat{E}, \hat{P}_i\right\}$ that are Hermitian operators on $\mathbf{H}$. ($i,\ j = 1,\ ..\, m-1,\ a,\ b = 0,\ 1 \ldots m-1,\ \alpha,\ \beta = 1,\ \ldots 2\,m$)

These observables are the Hermitian representations of the algebra of the unitary representation ϱ of the Weyl-Heisenberg group $\mathcal{H}(m)$: $\hat{Z}_\alpha = T_e\,\varrho(Z_\alpha) = \varrho'(Z_\alpha)$, $\hat{g} = \varrho(g)$

$$[Z_\alpha, Z_\beta] = \zeta_{\alpha,\beta}\, I, \qquad \zeta = \begin{pmatrix} 0 & \eta \\ -\eta & 0 \end{pmatrix}$$

and Hermitian representation

$$\left[\hat{Z}_\alpha, \hat{Z}_\beta\right] = i\,\zeta_{\alpha,\beta}\,\hat{I}$$

$\mathcal{H}(m)$ is a one parameter central extension of $\mathcal{A}(2\,m) \simeq (\mathbb{R}^{2\,m}, +)$, the abelian *translation* group. Unitary representation of $\mathcal{H}(m)$ is a particular projective representation of $\mathcal{A}(2\,m)$.

Consistency of relativity with Heisenberg commutators

The requirement that the relativity group $\mathcal{G}$ be consistent with the Heisenberg commutation relations is that if $\hat{\tilde{Z}}_\alpha = \hat{g}\,\hat{Z}_\alpha\,\hat{g}^{-1}$ with $\hat{g} = \varrho(g)$, $\hat{Z}_\alpha = \varrho'(Z_\alpha)$

$$\left[\hat{Z}_\alpha, \hat{Z}_\beta\right] = i\,\zeta_{\alpha,\beta}\,\hat{I} \quad \Leftrightarrow \quad \left[\hat{\tilde{Z}}_\alpha, \hat{\tilde{Z}}_\beta\right] = i\,\zeta_{\alpha,\beta}\,\hat{I}$$

for all $g \in \check{\mathcal{G}}$, $\hat{Z}_\alpha \in \mathbf{a}(\check{\mathcal{G}})$

Then note that

$$\hat{\tilde{Z}}_\alpha = \varrho(g)\,\varrho'(Z_\alpha)\,\varrho(g)^{-1} = \varrho'(g\,Z_\alpha\,g^{-1})$$

and so as the representation is faithful

$$\{Z_\alpha, I\} \in \mathbf{a}(\mathcal{H}(m)) \quad \Leftrightarrow \quad \{\tilde{Z}_\alpha, I\} \in \mathbf{a}(\mathcal{H}(m))$$

where

$$\tilde{Z}_\alpha = g\,Z_\alpha\,g^{-1}$$

and so $g \in \check{\mathcal{G}} \subset \mathcal{A}ut_{\mathcal{H}(m)}$

A relativity group on a flat manifold consistent with Heisenberg commutation relations has a central extension that is a subgroup of the automorphism group $\mathcal{A}ut_{\mathcal{H}}$ of the Weyl-Heisenberg algebra

Weyl-Heisenberg group

The Weyl-Heisenberg group is a real matrix group $\mathcal{H}(m) \subset \mathcal{GL}(2\,m+2)$ (Major)

$$\mathrm{H}(z,\,\iota) \simeq \begin{pmatrix} I_{2\,m} & 0 & z \\ -{}^{t}z\cdot\zeta & 1 & 2\,\iota \\ 0 & 0 & 1 \end{pmatrix} \quad \zeta = \begin{pmatrix} 0 & \eta \\ -\eta & 0 \end{pmatrix}, \quad z \in \mathbb{R}^{2\,m}, \;\; \iota \in \mathbb{R}$$

Group operations of the Weyl-Heisenberg subgroup $\mathrm{H}(z,\,\iota)$ are

$$\mathrm{H}(z'',\,\iota'')\cdot\mathrm{H}(z',\,\iota') = \mathrm{H}(z''+z',\,\iota''+\iota' - \tfrac{1}{2}\,{}^{t}z''\,\zeta\,z'), \quad \mathrm{H}^{-1}(z,\,\iota) = \mathrm{H}(-z,\,-\iota)$$

Expand notation with $z = (x,\; p)$

$$\mathrm{H}(x,\;p,\;\iota) \simeq \begin{pmatrix} I_m & 0 & 0 & x^a \\ 0 & I_m & 0 & p^a \\ p_a & -x_a & 1 & 2\,\iota \\ 0 & 0 & 0 & 1 \end{pmatrix} \; x,\;p \in \mathbb{R}^m, \;\; \iota \in \mathbb{R}$$

$$\mathrm{H}(x'',\;p'',\,\iota'')\cdot\mathrm{H}(x',\;p',\,\iota') = \mathrm{H}(x''+x',\;p''+p',\,\iota''+\iota' - \tfrac{1}{2}\,(x''\,p' - p''\,x')),$$

Automorphism group of the Weyl-Heisenberg group

$\mathcal{A}ut_{\mathcal{H}}$ has been determined by Folland to be

$$\mathcal{A}ut_{\mathcal{H}} \simeq \overline{\mathcal{O}\mathcal{A}ut_{\mathcal{H}}} \otimes_s \mathcal{H}(m), \ \mathcal{O}\mathcal{A}ut_{\mathcal{H}} = \mathbb{Z}_2 \otimes \mathcal{D} \otimes \mathcal{S}p(2\,m)$$

where $\mathbb{Z}_2$ is discrete 2 element group, $\mathcal{D} \simeq (\mathbb{R}, \times)$ and $\mathcal{S}p(2\,m)$ is the symplectic group. $\mathcal{H}(m)$ are the inner automorphisms. The outer automorphisms $\mathcal{O}\mathcal{A}ut_{\mathcal{H}}$ that is the homogeneous group is directly computed from the condition on $\Omega \in \mathcal{G}\mathcal{L}(2\,m+2)$

$$\Omega\,\mathrm{H}(z, \iota)\,\Omega^{-1} = \mathrm{H}(z', \iota')$$

Matrix realization is

$$\Omega(\epsilon, \delta, \Sigma, z, \iota) = \begin{pmatrix} \delta\,\Sigma & 0 & z \\ -{}^t z\,\zeta\,\Sigma & \epsilon\,\delta^2 & 2\,\iota \\ 0 & 0 & \epsilon \end{pmatrix}$$

Note the special cases, $\epsilon = \pm 1$, $\Delta(\epsilon, \delta) = \Omega(\epsilon, \delta, I_{2\,n+2}, 0, 0) \in \mathbb{Z}_2 \otimes \mathcal{D}$, $\Sigma \simeq \Omega(1, 1, \Sigma, 0, 0) \in \mathcal{S}p(2\,m)$ and $\mathrm{H}(z, \iota) = \Omega(1, 1, I_{2\,n+2}, z, \iota) \in \mathcal{H}(m)$

$$\Delta(\epsilon, \delta) = \begin{pmatrix} \delta\,I_{2\,m} & 0 & 0 \\ 0 & \epsilon\,\delta^2 & 0 \\ 0 & 0 & \epsilon \end{pmatrix}, \quad \Sigma \simeq \begin{pmatrix} \Sigma & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathrm{H}(z, \iota) = \begin{pmatrix} I_{2\,m} & 0 & z \\ -{}^t z\,\zeta & 1 & 2\,\iota \\ 0 & 0 & 1 \end{pmatrix}$$

and that $\Omega(\epsilon, \delta, \Sigma, \tilde{z}, \tilde{\iota}) = \Delta(\epsilon, \delta)\,\Sigma\mathrm{H}(z, \iota)$ where $\tilde{z} = \delta\,\Sigma\,z$ and $\tilde{\iota} = \epsilon\,\delta^2\,\iota$

Automorphism group of the Weyl-Heisenberg group

$\mathcal{A}ut_{\mathcal{H}}$ is the central extension of $\mathbb{Z}_2 \otimes \mathcal{D} \otimes \mathcal{I}\mathcal{S}p(2\,m)$ where the inhomogeneous symplectic group is

$$\mathcal{I}\mathcal{S}p(m) \simeq \mathcal{S}p(2\,m) \otimes_s \mathcal{A}(2\,m),$$

The central extension of the inhomogeneous symplectic group is

$$\mathcal{I}\check{\mathcal{S}}p(m) \simeq \overline{\mathcal{H}\mathcal{S}p}(2\,m) \simeq \overline{\mathcal{S}p}(2\,m) \otimes_s \mathcal{H}(m)$$

Thus the relativity group $\mathcal{G}$ can always be written in the form

$$\mathcal{G} \simeq \mathcal{K} \otimes_s \mathcal{N}$$

where the homogeneous relativity group

$$\mathcal{K} \subset \mathcal{O}\mathcal{A}ut_{\mathcal{H}} \simeq \mathbb{Z}_2 \otimes \mathcal{D} \otimes \mathcal{S}p(2\,m)$$

and the normal subgroup $\mathcal{N} \subset \mathcal{A}(2\,m)$ such that the central extension $\check{\mathcal{G}} \subset \mathcal{A}ut_{\mathcal{H}(m)}$

Example: Inhomogeneous Lorentz group $\mathcal{O}(1,\, m-1) \otimes_s \mathcal{A}(m)$.

$$O(1,\, m-1) \subset \mathcal{S}p(2\,m)\,, \quad \mathcal{A}(m) \subset \mathcal{A}(2\,m)$$

Poincaré group that is the central extension (it has no algebraic extension so this is the cover)

$$\mathcal{P}(1,\, m-1) \simeq \overline{O}(1,\, m-1) \otimes_s \mathcal{A}(m) \subset \mathcal{A}ut_{\mathcal{H}(m)}$$

Another example is the Galilei group that is the central extension of the inhomogeneous Euclidean group

Summary: Relativity group consistent with Heisenberg algebra

The assertion that the Weyl-Heisenberg commutation relations must hold in any frame related by a relativity group $\mathcal{G}$ implies that:

The relativity group has the form $\mathcal{G} \simeq \mathcal{K} \otimes_s \mathcal{N}$ where the homogeneous group is a subgroup of the outer automorphisms

$$\mathcal{K} \subset \mathcal{O}\mathcal{A}ut_{\mathcal{H}} \simeq \mathbb{Z}_2 \otimes \mathcal{D} \otimes \mathcal{S}p(2\,m)$$

and the normal subgroup $\mathcal{N} \subset \mathcal{A}(2\,m)$ such that the central extension $\check{\mathcal{G}} \subset \mathcal{A}ut_{\mathcal{H}(m)}$

Examples are the Poincaré group and Galilei group.

There is a Weyl-Cartan symplectic metric invariant up to a scale:

$$\zeta = \zeta_{\alpha,\beta}\, d\, z^{\alpha}\, d\, z^{\beta} = -d\, e \wedge d\, t + \delta_{i,j}\, d\, p^{i} \wedge d\, q^{j}, \qquad \zeta \to \delta^{2}\, \zeta$$

Noninertial homogeneous relativity groups

The homogeneous relativity groups for the orthogonal line that are a subgroup of $\mathcal{OAut}_{\mathcal{H}}$ may be determined simply by the conditions. $(n = m - 1)$ $\Gamma \in \mathcal{GL}(2\,n + 2, \mathbb{R})$

$$^{t}\Gamma \cdot \zeta \cdot \Gamma = \delta^2\, \zeta \quad ,$$

where $\zeta = \zeta_{\alpha,\beta}\, d\, z^{\alpha}\, \mathrm{d}z^{\beta} = -d\, e \wedge d\, t + \delta_{i,j}\, d\, p^{i} \wedge d\, q^{j}$ with metric components $(2\,n+2)\times(2\,n+2)$ matrices

$$\zeta = \begin{pmatrix} 0 & \eta \\ -\eta & 0 \end{pmatrix},$$

and orthogonal line element

$$^{t}\Gamma \cdot \tilde{\eta} \cdot \Gamma = \tilde{\eta},$$

were $d\, s^2 = \tilde{\eta}_{\alpha,\beta}\, d\, z^{\alpha}\, d\, z^{\beta}$ with where the homogeneous relativity group $\mathcal{K}$ are for the cases

$$d\, t^2: \qquad \tilde{\eta} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \mathcal{H}\hat{\mathcal{S}}p(2\,n) \simeq \mathbb{Z}_2 \otimes_s \mathcal{S}p(2\,n) \otimes_s \mathcal{H}(n)$$

$$d\, t^2 - \tfrac{1}{c^2}\, d\, q^2: \qquad \tilde{\eta} = \begin{pmatrix} \eta & 0 \\ 0 & 0 \end{pmatrix}, \qquad \mathcal{U}b(1,\, n) \simeq \mathcal{O}(1,\, n) \otimes_s \mathcal{A}(k),\;\; k = \tfrac{(n+1)\,(n+2)}{2}$$

$$d\, t^2 - \tfrac{1}{c^2}\, d\, q^2 - \tfrac{1}{c^2}\, d\, q^2 + \tfrac{1}{b^2\, c^2}\, d\, e^2: \; \tilde{\eta} = \begin{pmatrix} \eta & 0 \\ 0 & \eta \end{pmatrix} \qquad \mathcal{U}(1,\, n)$$

Noninertial relativity groups

The inhomgeneous relativity group $\mathcal{G} = \mathcal{I}\mathcal{K}$ that is the maximal group has a central extension $\check{\mathcal{G}} \subset \mathcal{A}ut_{\mathcal{H}(m)}$ that is the maximal group with the above invariant line elements, Born-Green, Minkowski, Newton are

	Inhomogeneous Group	Central Extension
Newton:	$\mathcal{HS}p(2\,n) \otimes_s \mathcal{A}(2\,n+2)$	$\mathcal{I}\mathcal{H}\check{\mathcal{S}p}(2\,n) \simeq \mathbb{Z}_2 \otimes_s \overline{\mathcal{HS}p}(2\,n) \otimes_s \mathcal{H}(n+1)$
Minkowski:	$\mathcal{U}b(1,\,n) \otimes_s \mathcal{A}(2\,n+2)$	$\overline{\mathcal{Q}b}(1,\,n) \simeq \overline{\mathcal{U}b}(1,\,n) \otimes_s \mathcal{H}(n+1)$
Born-Green:	$\mathcal{U}(1,\,n) \otimes_s \mathcal{A}(2\,n+2)$	$\overline{\mathcal{Q}}(1,\,n) \simeq \overline{\mathcal{U}}(1,\,n) \otimes_s \mathcal{H}(n+1)$

Related by Inőnű-Wigner contraction in limits $b \to \infty$ and $c \to \infty$

Wigner approach is special relativistic quantum mechanics is the projective representations of the inhomogeneous Lorentz group (unitary representations of the central extension, the Poincaré group): inertial quantum states

Relativity group relates physical states. States are rays in a Hilbert space, therefore related through projective represenation of group

The idea does not use the condition that it is a relativity group be for *inertial* states

Noninertial quantum states given by the projective representations of the above inhomogeneous groups

But.... we are getting ahead of ourselves, what is the physically meaning of these noninertial relativity groups

Hamilton group ($d\,t^2$)

Let's start with the simplest physical case, the classical (small velocity, nonquantum) physical meaning of the homogeneous group for the Newtonian time line element $d\,t^2$.

Elements Γ of the homogeneous relativity group $\mathcal{K} = \mathcal{HSp}(2\,m) \simeq \mathbb{Z}_2 \otimes_s \mathcal{Sp}(2\,n) \otimes_s \mathcal{H}(n)$ may be written as

$$\Gamma(\delta, \Sigma, w, r) \simeq \begin{pmatrix} \delta\Sigma & 0 & w \\ -\,{}^t w \cdot \zeta^\circ & \delta & r \\ 0 & 0 & \delta \end{pmatrix}, \quad \delta = \pm 1,\; w \in \mathbb{R}^{2\,n}, r \in \mathbb{R}\,,\; \Sigma \in \mathcal{Sp}(2\,n), \zeta^\circ = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

Setting $\Sigma = \begin{pmatrix} I_n & 0 \\ 0 & I_n \end{pmatrix}$, δ=1, $w = (f, v)$ with $f, v \in \mathbb{R}^n$ we can write the Weyl-Heisenberg subgroup

$$\mathrm{H}(f, v, r) \simeq \begin{pmatrix} I_n & 0 & 0 & f \\ 0 & I_n & 0 & v \\ v & -f & 1 & r \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The group has an invariant line element and symplectic metric

$$d\,\tau^2 = d\,t^2, \qquad \zeta = -d\,e \wedge d\,t + \delta_{i,j}\, d\,p^i \wedge d\,q^j$$

Hamilton's equations

Consider the diffeomorphisms $\varphi : \mathbb{P} \to \mathbb{P} : z \mapsto \tilde{z} = \varphi(z)$ or

$$\tilde{t} = \varphi_t(t, q, p, e), \ \ \tilde{q}^i = \varphi_q^i(t, q, p, e), \ \ \tilde{e} = \varphi_e(t, q, p, e), \ \tilde{p}^i = \varphi_p^i(t, q, p, e),$$

Then $\left[\frac{\partial\varphi(z)}{\partial z}\right] \in \mathcal{HSp}(2\,n)$ and in particular $\left[\frac{\partial\varphi(z)}{\partial z}\right] \in \mathcal{H}(n)$. Expanding this

$$\left[\frac{\partial\varphi(z)}{\partial z}\right] = \begin{pmatrix} \frac{\partial\varphi_t}{\partial p} & \frac{\partial\varphi_t}{\partial q} & \frac{\partial\varphi_t}{\partial e} & \frac{\partial\varphi_t}{\partial t} \\ \frac{\partial\varphi_q}{\partial p} & \frac{\partial\varphi_q}{\partial q} & \frac{\partial\varphi_q}{\partial e} & \frac{\partial\varphi_q}{\partial t} \\ \frac{\partial\varphi_e}{\partial p} & \frac{\partial\varphi_e}{\partial q} & \frac{\partial\varphi_e}{\partial e} & \frac{\partial\varphi_e}{\partial t} \\ \frac{\partial\varphi_p}{\partial p} & \frac{\partial\varphi_p}{\partial q} & \frac{\partial\varphi_p}{\partial e} & \frac{\partial\varphi_p}{\partial t} \end{pmatrix} = \mathrm{H} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ v & I_n & 0 & 0 \\ r & -f & 1 & v \\ f & 0 & 0 & I_n \end{pmatrix}$$

with $i, \ j = 1, \ ..n$. Zero partials reduce functional dependence of transformations to

$$\tilde{t} = \varphi_t(p, q, e, t) = \varphi_t(t) = t, \qquad \tilde{q}^i = \varphi_q^i(p, q, e, t) = \varphi_q^i(q, t) = q^i + \varphi_q^i(t),$$
$$\tilde{e} = \varphi_e(p, q, e, t) = e + H(p, q, t), \qquad \tilde{p}^i = \varphi_p^i(p, q, e, t) = \varphi^i(p, t) = p^i + \varphi_p^i(t).$$

resulting in Hamilton's equations (invariant under $\mathcal{Sp}(2\,n)$ for general $\mathcal{HSp}(2\,n)$)

$$\frac{d\,\varphi_q^i(t)}{d\,t} = v^i = \frac{\partial H(p,q,t)}{\partial p^i}, \ \ \frac{d\,\varphi_p^i(t)}{d\,t} = f^i = -\frac{\partial H(p,q,t)}{\partial q^i}, \ \ \frac{\partial H(p,q,t)}{\partial t} = r$$

Hamilton relativity group of noninertial states

If we also assume orthonormal frames such that length $d\,q^2$ and $d\,p^2$ is invariant, then the symplectic group reduces to $\Sigma \simeq \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}$, $R \in \mathcal{O}(n)$ and the Hamilton relativity group with element Γ is

$$\hat{\mathcal{H}a}(n) \simeq \mathbb{Z}_2 \otimes_s \mathcal{O}(n) \otimes_s \mathcal{H}(n),$$

$$\Gamma(\delta, R, f, v, r) = \begin{pmatrix} \delta & 0 & 0 & 0 \\ v & \delta R & 0 & 0 \\ r & -f & \delta & v \\ f & 0 & 0 & \delta R \end{pmatrix}, \delta = \pm 1, \; v, \; f \in \mathbb{R}^n, \; r \in \mathbb{R}, \; R \in \mathcal{O}(n)$$

where v, f, r, are velocity, force and power and R are rotations. Action on basis is $d\,\tilde{z} = \Gamma\, d\,z$

$$d\,\tilde{t} = d\,t$$
$$d\,\tilde{p} = R\,d\,p + f\,d\,t$$
$$d\,\tilde{q} = R\,d\,q + v\,d\,t$$
$$d\,\tilde{e} = d\,e + v \cdot d\,p - f \cdot d\,q + r\,d\,t$$

From group composition $(R = I_n)$, $\mathrm{H}(f, v, r) = \mathrm{H}(f'', v'', r'') \cdot \mathrm{H}(f', v', r')$

$$f = f'' + f', \quad v = v'' + v', \quad r = r'' + r' - f'' \cdot v' + v'' \cdot f'$$

These are the expected transformations to noninertial states in Hamilton's mechanics.

Inertial special case $\Gamma(1, R, 0, v, 0) \in \mathcal{E}(n) \subset \mathcal{H}a(n)$

Hamilton summary

Started with two assumptions:

1) Relativity group is consistent with Heisenberg commutation relations

2) Newtonian time line element $d\,t^2$ (Classical approximation to Born-Green line element)

Homogeneous relativity group is

$$\mathcal{K} \simeq \mathcal{H}\hat{\mathcal{S}}p(2\,n) \simeq \mathbb{Z}_2 \otimes_s \mathcal{S}p(n) \otimes \mathcal{H}(n)$$

With the orthonormal frame assumption, homogeneous group is $\hat{\mathcal{H}}\,a(n) \simeq \mathbb{Z}_2 \otimes_s \mathcal{O}(n) \otimes \mathcal{H}(n)$.

Euclidean inertial group (that is the homogeneous subgroup of the Galilei group) is the inertial special case of the Hamilton group: $\hat{\mathcal{E}}(n) \subset \hat{\mathcal{H}}\,a(n)$

Results in Hamilton's equations describing general noninertial trajectory of classical point particle.

Results in 'expected relativity group' between all physical noninertial states in Hamilton mechanics. Momentum, energy dimensions are as *real* as position and time dimensions to define a particle state.

This homogeneous noninertial relativity group is as physically fundamental as the homogeneous inertial relativity group.

Time is invariant: all observers agree on the time subspace spanned by $d\,t$

Velocity and force are simply additive and unbounded.

There is an absolute inertial rest frame that all observers agree on

Relativity group with invariant Born-Green metric

Consider a non-degenerate Born-Green orthogonal metric on $\mathbb{P}$

$$d s^2 = \tilde{\eta}_{\alpha,\beta}\, d z^\alpha\, d z^\beta = \eta_{a,b}\, d x^a\, d x^b + \eta_{a,b}\, d p^a\, d p^b$$

or with dimensional constants explicit

$$d s^2 = d t^2 - \frac{1}{c^2} d q^2 - \frac{1}{b^2} d p^2 + \frac{1}{b^2 c^2} d e^2 \,,$$

This is the only new physical assumption in the talk. It was considered by Born in his reciprocity investigations in Edinburgh 1939-1949. $\{t, e\} \to \{e, -t\}$, $\{q, p\} \to \{p, -q\}$

b is a dimensional physical constant (dimensions of force - Newtons) that we use for the third dimensional constant $\{c, b, \hbar\}$ rather than usual $\{c, G, \hbar\}$. Note that $G = \alpha_b \frac{c^4}{b}$, $b \approx \alpha_b\, 10^{44}$ Newtons.

$$\lambda_t = \sqrt{\frac{\hbar}{b\,c}},\ \lambda_q = \sqrt{\frac{\hbar\,c}{b}},\ \lambda_p = \sqrt{\frac{\hbar\,b}{c}},\ \lambda_e = \sqrt{\hbar\,b\,c}\,.$$

Scaling set by $z = (t/\lambda_t,\ q/\lambda_q,\ e/\lambda_e,\ p/\lambda_p)$ and $(v/c,\ f/b,\ r/b\,c)$

If $\alpha_b = 1$ these are numerically the usual Planck scales.

Reciprocal relativity of noninertial states: Unitary group

Require that it is a subgroup of the outer automorphisms to be consistent with the Weyl-Heisenberg algebra

$$\mathcal{O}(2, 2n) \bigcap \mathcal{O}\mathcal{A}ut_{\mathcal{H}} \simeq \mathcal{U}(1, n)$$

Unitary group may be factored: $\mathcal{U}(1, n) \simeq \mathcal{U}(1) \otimes_s \mathcal{S}\mathcal{U}(1, n)$

The groups satisfy the Inőnű-Wigner contraction criteria

$$\mathcal{U}(1, n) \underset{b\to\infty}{\to} \mathcal{U}b(1, n) \underset{c\to\infty}{\to} \hat{\mathcal{H}}a(n)$$

With the corresponding contraction of the orthogonal line element

$$d\,t^2 - \frac{1}{c^2}\,d\,q^2 - \frac{1}{b^2}\,d\,p^2 + \frac{1}{b^2\,c^2}\,d\,e^2 \underset{b\to\infty}{\to} d\,t^2 - \frac{1}{c^2}\,d\,q^2 \underset{c\to\infty}{\to} d\,t^2$$

Reciprocal relativity $\mathcal{SU}(1,1)$ matrix group

Consider the one dimensional case and consider first the $\Gamma(v, f, r) \in \mathcal{SU}(1,1)$ group realized by matrix with basis ordering $dz = (dt/\lambda_t, dq/\lambda_q, de/\lambda_e, dp/\lambda_p)$.

$$\Gamma(v, f, r) = \begin{pmatrix} \Lambda & -\mathrm{M} \\ \mathrm{M} & \Lambda \end{pmatrix} = \gamma \begin{pmatrix} 1 & \frac{v}{c} & -\frac{r}{b\,c} & \frac{f}{b} \\ \frac{v}{c} & 1 & -\frac{f}{b} & \frac{r}{b\,c} \\ \frac{r}{b\,c} & -\frac{f}{b} & 1 & \frac{v}{c} \\ \frac{f}{b} & -\frac{r}{b\,c} & \frac{v}{c} & 1 \end{pmatrix}, \quad \Gamma(v, 0, 0) = \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix}$$

with $\gamma = (1 - v^2/c^2 - f^2/b^2 + r^2/b^2 c^2)^{-1/2}$ and $\gamma^\circ = (1 - v^2/c^2)^{-1/2}$

Transformation equations are $d\tilde{z} = \Gamma \cdot dz$.

$$\frac{d\tilde{t}}{\lambda_t} = \gamma\Big(\frac{dt}{\lambda_t} + \frac{v}{c}\frac{dq}{\lambda_q} + \frac{f}{b}\frac{dp}{\lambda_p} - \frac{r}{c\,b}\frac{de}{\lambda_e}\Big),$$

$$\frac{d\tilde{q}}{\lambda_q} = \gamma\Big(\frac{dq}{\lambda_q} + \frac{v}{c}\frac{dt}{\lambda_t} + \frac{r}{c\,b}\frac{dp}{\lambda_p} - \frac{f}{b}\frac{de}{\lambda_e}\Big),$$

$$\frac{d\tilde{p}}{\lambda_p} = \gamma\Big(\frac{dp}{\lambda_p} + \frac{f}{b}\frac{dt}{\lambda_t} - \frac{r}{c\,b}\frac{dq}{\lambda_q} + \frac{v}{c}\frac{de}{\lambda_e}\Big),$$

$$\frac{d\tilde{e}}{\lambda_e} = \gamma\Big(\frac{de}{\lambda_e} + \frac{v}{c}\frac{dp}{\lambda_p} - \frac{f}{b}\frac{dq}{\lambda_q} + \frac{r}{c\,b}\frac{dt}{\lambda_t}\Big).$$

Transformation equations

Transformation equations are $d\,\tilde{z} = \Gamma \cdot d\,z$. *Spacetime* itself is relative to noninertial states.

$$d\,\tilde{t} = \gamma\left(d\,t + \tfrac{v}{c^2}\,d\,q + \tfrac{f}{b^2}\,d\,p - \tfrac{r}{b^2\,c^2}\,d\,e\right),$$
$$d\,\tilde{q} = \gamma\left(d\,q + v\,d\,t + \tfrac{r}{b^2}\,d\,p - \tfrac{f}{b^2}\,d\,e\right),$$
$$d\,\tilde{p} = \gamma(d\,p + f\,d\,t - \tfrac{r}{c^2}\,d\,q + \tfrac{v}{c^2}\,d\,e),$$
$$d\,\tilde{e} = \gamma(d\,e + v\,d\,p - f\,d\,q + r\,d\,t).$$

$$\gamma = (1 - v^2/c^2 - f^2/b^2 + r^2/b^2\,c^2)^{-1/2}$$

Limit of $b \to \infty$,

$$d\,\tilde{t} = \gamma^\circ(d\,t + \tfrac{v}{c^2}\,d\,q),$$
$$d\,\tilde{q} = \gamma^\circ(d\,q + v\,d\,t),$$
$$d\,\tilde{p} = \gamma^\circ(d\,p + f\,d\,t - \tfrac{r}{c^2}\,d\,q + \tfrac{v}{c^2}\,d\,e),$$
$$d\,\tilde{e} = \gamma^\circ(d\,e + v\,d\,p - f\,d\,q + r\,d\,t).$$

$$\gamma^\circ = (1 - v^2/c^2)^{-1/2}$$

Limit of $b, c \to \infty$

$$d\,\tilde{t} = d\,t,$$
$$d\,\tilde{q} = d\,q + v\,d\,t,$$
$$d\,\tilde{p} = d\,p + f\,d\,t,$$
$$d\,\tilde{e} = d\,e + v\,d\,p - f\,d\,q + r\,d\,t.$$

Matrix groups (with dimensions)

Dimensioned basis $(d\,t,\,d\,q,\,d\,e,\,d\,p)$

$$\Gamma(v,\ f,\ r) = \begin{pmatrix} \Lambda & -\mathrm{M} \\ \mathrm{M} & \Lambda \end{pmatrix} = \gamma \begin{pmatrix} 1 & \frac{v}{c^2} & -\frac{r}{b^2\,c^2} & \frac{f}{b^2} \\ v & 1 & -\frac{f}{b^2} & \frac{r}{b^2} \\ r & -f & 1 & v \\ f & -\frac{r}{c^2} & \frac{v}{c^2} & 1 \end{pmatrix}, \quad \Gamma(v,\,0,\,0) = \gamma^\circ \begin{pmatrix} 1 & \frac{v}{c^2} & 0 & 0 \\ v & 1 & 0 & 0 \\ 0 & 0 & 1 & v \\ 0 & 0 & \frac{v}{c^2} & 1 \end{pmatrix}$$

with $\gamma = (1 - v^2/c^2 - f^2/b^2 + r^2/b^2\,c^2)^{-1/2}$ and $\gamma^\circ = (1 - v^2/c^2)^{-1/2}$

Limits are:

$$\Gamma^\circ(v,\ f,\ r) = \lim_{b\to\infty} \Gamma(v,\ f,\ r) \in \mathcal{U}b(1,1), \qquad \Phi(v,\ f,\ r) = \lim_{b,c\to\infty} \Gamma(v,\ f,\ r) \in \mathcal{H}a(1),$$

$$\Gamma^\circ(v) = \begin{pmatrix} \Lambda & 0 \\ \mathrm{M} & \Lambda \end{pmatrix} = \gamma^\circ \begin{pmatrix} 1 & \frac{v}{c^2} & 0 & 0 \\ v & 1 & 0 & 0 \\ r & -f & 1 & v \\ f & -\frac{r}{c^2} & \frac{v}{c^2} & 1 \end{pmatrix}, \qquad \Phi(v,\ f,\ r) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ v & 1 & 0 & 0 \\ r & -f & 1 & v \\ f & 0 & 0 & 1 \end{pmatrix},$$

Group composition and null surfaces

Group multiplication is

$$\Gamma(v'', f'', r'') \cdot \Gamma(v', f', r') = \Gamma(v, f, r)$$

where

$$v = \left(v'' + v' + \frac{1}{b^2}(r' f'' - f' r'')\right) \Big/ \left(1 + \frac{v' v''}{c^2} + \frac{f' f''}{b^2} - \frac{r' r''}{b^2 c^2}\right),$$

$$f = \left(f'' + f' + \frac{1}{c^2}(-r' v'' + v' r'')\right) \Big/ \left(1 + \frac{v' v''}{c^2} + \frac{f' f''}{b^2} - \frac{r' r''}{b^2 c^2}\right),$$

$$r = \left(r'' + r' - f' v'' + v' f''\right) \Big/ \left(1 + \frac{v' v''}{c^2} + \frac{f' f''}{b^2} - \frac{r' r''}{b^2 c^2}\right)$$

Null surfaces (b, c universal constants) from fixed point of group product

$$\frac{v^2}{c^2} + \frac{f^2}{b^2} - \frac{r^2}{c^2 b^2} = 1, \quad \text{inertial case: } \frac{v^2}{c^2} = 1$$

Using $d\mu^2 = \frac{1}{c^2} d e^2 - d p^2$, $\left(\frac{d\mu}{dt}\right)^2 = \frac{1}{c^2} r^2 - f^2$

$$\frac{v^2}{c^2} = 1 + \frac{c^2}{b^2}\left(\frac{d\mu}{dt}\right)^2 = 1 + \kappa^2$$

For a W meson decay, $dt \sim 10^{-25}\, s$, $d\mu \sim 100 \frac{\text{GeV}}{c^2}$, $\qquad \kappa = \frac{c}{b}\frac{d\mu}{dt} \sim \frac{1}{\alpha_b} 10^{-19}$

Value of α_b sets scale where effects show up, $\alpha_b \sim 10^{-19}$ at weak scale, $\alpha_b \sim 1$ at usual Planck scale. In very strongly interacting regime (early universe) $\kappa \sim 1$ or greater, probe null surface where $v > c$,

Second summary

New physical hypothesis is the Born-Green metric .

Reciprocal relativity of noninertial states given by $\mathcal{U}(1, n)$ transformations. Transform energy-momentum into position-time and vice versa.

All physical states (not just inertial states) are related by the relativity group as observers must be in a physical state and can only transform measurements from other physical states through the relativity group. E.g. If there is not a relativity group transform for a state, you could not *interpret* how that states clocks are ticking.

No absolute rest frame, no absolute inertial frame. *Spacetime* itself is relative to noninertial state.

Natural dimensional basis is $\{c, b, \hbar\}$.

Velocities are between particle states. Forces are between particle states. (Mach's principle)

Velocity and force are bounded. Null hypersurface elliptical hyperboloid.

In $b \to \infty$ limit *spacetime* is again invariant. An *apparent* global inertial frame emerges and it appears that forces are relative to it.

The groups satisfy the Inönü-Wigner contraction criteria $\mathcal{U}(1, n) \underset{b\to\infty}{\to} \mathcal{U}b(1, n) \underset{c\to\infty}{\to} \hat{\mathcal{H}}a(n)$

With the corresponding contraction of the orthogonal line element

$$d\,t^2 - \frac{1}{c^2}\,d\,q^2 - \frac{1}{b^2}\,d\,p^2 + \frac{1}{b^2\,c^2}\,d\,e^2 \underset{b\to\infty}{\to} d\,t^2 - \frac{1}{c^2}\,d\,q^2 \underset{c\to\infty}{\to} d\,t^2$$

Relativity group with invariant Minkowski line element

Consider the case where proper time is invariant that is given by the Minkowski line element on

$$d\tau^2 = \tilde{\eta}_{\alpha,\beta}\, d z^\alpha\, d z^\beta = \eta_{a,b}\, d x^a\, d x^b = d t^2 - \frac{1}{c^2}\, d q^2$$

where $z \in \mathbb{P} \simeq \mathbb{R}^{2m}$, $\{z^\alpha\} = \{x^a, p^a\}$, $\tilde{\eta} = \begin{pmatrix} \eta & 0 \\ 0 & 0 \end{pmatrix}$

Group is

$$\mathcal{U}b(1, n) \simeq \mathcal{O}(1, n) \otimes_s \mathcal{A}(k) \text{ , } k = (n+1)(n+2)/2$$

with elements $\Gamma \in \mathcal{U}b(1, n)$ given by

$$\Gamma(\Lambda, \mathrm{M}) = \begin{pmatrix} \Lambda & 0 \\ \mathrm{M} & \Lambda \end{pmatrix} \text{ with } \Gamma(\Lambda, 0) \in \mathcal{O}(1, n), \Gamma(I_m, \mathrm{M}) \in \mathcal{A}(k), {}^t\mathrm{M} = \eta\, \mathrm{M}\, \eta$$

Group product is $\Gamma(\Lambda', \mathrm{M}')\, \Gamma(\Lambda, \mathrm{M}) = \Gamma(\Lambda' \Lambda, \Lambda' \mathrm{M} + \mathrm{M}' \Lambda)$, $\Gamma(\Lambda, \mathrm{M})^{-1} = \Gamma(\Lambda^{-1}, -\Lambda^{-1} \mathrm{M} \Lambda^{-1})$

M transforms as an $(1, 1)$ tensor under the Lorentz transformations

$$\Gamma(\Lambda', \mathrm{M}')\, \Gamma(I_n, \mathrm{M})\, \Gamma(\Lambda', \mathrm{M}')^{-1} = \Gamma(I_n, \Lambda' \mathrm{M} \Lambda'^{-1})$$

Transformation equations $d\tilde{z} = \Gamma\, d z$ are

$$d\tilde{x}^a = \lambda^a_b\, d x^b \qquad \mu_{a,b} = \mu_{b,a}$$

$$d\tilde{p}^a = \lambda^a_b\, d p^b + \mu^a_b\, d x^b \qquad \mu^a_b = \frac{d}{d\tau} t^a_b \text{ , } t^a_b \text{ stress-energy-momentum}$$

Comments on quantum mechanics

Special relativistic quantum mechanics is given by the projective representations of the inhomogeneous Lorentz group. These are equivalent to the unitary representations of the Poincaré group that is its central extension (that is its cover as no algebraic extension) (Wigner, Mackey, Bargmann). For $n = 3$

$$\mathcal{P}(1,3) \simeq \mathbb{Z}_{2,2} \otimes_s \mathcal{SL}(2,\mathbb{C}) \otimes_s \mathcal{A}(4)$$

Invariants are the Casimir operators C_α in the enveloping algebra. Wave equations are the eigenvalue equation for the Hermitian representations of the Casimir operators

$$\varrho'(C_\alpha)\,|\psi\rangle = \nu_\alpha\,|\psi\rangle,\ \ \nu_\alpha \in \mathbb{R}$$

This gives the Klein-Gordon, Dirac, Maxwell, etc single inertial particle SRQM wave equations

Wigner takes the active view of relativity in that the unitary representations of the group *defines* the Hilbert space that is the space of physical states. (Mackey method for semidirect products.)

There is nothing in this definition that is inherently inertial.
Do the same thing for noninertial relativity groups.

Consider a noninertial homogeneous relativity group $\mathcal{K}$. Quantum mechanics is the projective representation of the inhomogeneous group $\mathcal{G} = \mathcal{IK}$. This is equivalent to the unitary representations of the central extension $\check{\mathcal{G}}$.

Wave equations given in terms of Casimir eigenvalue equations as above.

Comments on quantum mechanics

Born-Green case $\mathcal{K} = \mathcal{U}(1, n)$. $\overline{\mathcal{Q}}(1, n) = \mathcal{I}\check{\mathcal{U}}(1, n)$

$$\mathcal{Q}(1, n) \simeq \mathcal{U}(1, n) \otimes_s \mathcal{H}(n+1), \qquad \overline{\mathcal{Q}}(1, n) \simeq (\mathcal{D} \otimes \mathcal{SU}(1, n)) \otimes_s \mathcal{H}(n+1),$$

The Mackey theorems for nonabelian normal subgroups apply.

Again, wave equations given by

$$\varrho'(C_\alpha)\,|\psi\rangle = \nu_\alpha\,|\psi\rangle, \;\; \nu_\alpha \in \mathbb{R}$$

In general, the wave equations are second order equations of wave functions $\psi(q, t)$ that appear to be 'towers of spinning oscillators'. The 'scalar case' is the relativistic oscillator.

Minkowski case $\mathcal{K} = \mathcal{U}b(1, n)$. $\overline{\mathcal{Q}b}(1, n) = \mathcal{I}\check{\mathcal{U}}b(1, n)$

$$\overline{\mathcal{Q}b}(1, n) \simeq \overline{\mathcal{U}b}(1, n) \otimes_s \mathcal{H}(n+1),$$

That can be rewritten as

$$\overline{\mathcal{Q}b}(1, n) \simeq \mathcal{P}(1, n) \otimes_s (\mathcal{A}((n+1)\,(n+2)\,/\,3) \,\otimes \mathcal{A}(n+2))$$

An immediate consequence is that, as there is a homomorphism $\pi : \overline{\mathcal{Q}b}(1, n)\,(1, n) \to \mathcal{P}(1, n)$ with $\ker(\pi) \simeq (\mathcal{A}((n+1)\,(n+2)\,/\,2) \,\otimes \mathcal{A}(n+2))$, a degenerate representation is the standard Poincaré SRQM theory.

The Mackey theorems give this directly.

Final summary (60 min talk)

Question: How are the clocks for non inertial states related where the noninertial states are due to forces other than gravity and so manifold is flat.

Question: What is the most general relativity group consistent with quantum mechanics on a flat manifold

Newtonian time $d\,t^2$: $\quad \mathcal{H}\hat{\mathcal{S}p}(2\,n) \simeq \mathbb{Z}_2 \otimes_s \mathcal{S}p(2\,n) \otimes_s \mathcal{H}(n)$

> Hamilton relativity with has an absolute inertial rest frame. Velocity and force are additive and unbounded. Time $\{t\}$ is an invariant subspace in $\mathbb{P}$.

Minkowski proper time: $d\,\tau^2 = d\,t^2 - \frac{1}{c^2}\,d\,q^2$: $\quad \mathcal{U}b(1, n) \simeq \mathcal{O}(1, n) \otimes_s \mathcal{A}(k)$,

> Relativity with absolute inertial frame but no absolute rest frame. Velocity is relative to states and bounded by c, force f are unbounded. Spacetime $\{t, q^i\}$ is an invariant subspace in $\mathbb{P}$. Time is relative.

Born-Green proper time: $d\,s^2 = d\,t^2 - \frac{1}{c^2}\,d\,q^2 - \frac{1}{b^2}\,d\,p^2 + \frac{1}{b^2\,c^2}\,d\,e^2$: $\quad \mathcal{U}(1, \text{n})$

> Reciprocal relativity with Born-Green time has neither an absolute inertial frame nor an absolute rest frame. Velocity is relative to states and bounded by c, force f is relative to states and bounded by b (Mach's principle). Spacetime $\{t, q^i\}$ is not an invariant subspace in $\mathbb{P}$, it is relative to noninertial frame.

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