A no-pure-boost uncertainty principle from space-time noncommutativity

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The subject of this talk

In collaboration with Giovanni Amelino-Camelia, Giulia Gubitosi, Antonino Marcianó and Pierre Martinetti,

- F.M., Amelino-Camelia, Gubitosi, Marcianó, Martinetti, arXiv:0707.1863
- F.M., Amelino-Camelia, Gubitosi, Marcianó, Martinetti, Pranzetti, Altair Tacchi, Prog. Theor. Phys. Suppl. **171** (2007), 65-78

Field theory as a tool for studying the symmetries of non-commutative space-times
Amelino-Camelia *et al.* - first Noether analysis for translations:

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Our aim: extend the above analysis to the Lorentz sector.

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The κ-Minkowski noncommutative spacetime

κ-Minkowski coordinates commutation rules

$$[\hat{x}_j, \hat{x}_k] = 0 \ , \qquad [\hat{x}_j, \hat{x}_0] = i\lambda \hat{x}_j \ ,$$

First introduced in:

- Majid, Ruegg (1994)

as the homogeneous space associated to the κ-Poincaré algebra:

- Lukierski, Nowicki, Ruegg (1992)

the bicrossproduct basis was discovered in:

- Majid, Ruegg (1994)

and we will use it in the following

Algebraic sector

$$[P_\mu, P_\nu] = 0 \ , \qquad [R_j, P_0] = 0 \ , \qquad [R_j, P_k] = i\varepsilon_{jkl} P_l \ ,$$

$$[N_j, P_0] = iP_j \ , \qquad [N_j, P_k] = i\delta_{jk} \left(\frac{1 - e^{-2\lambda P_0}}{2\lambda} + \frac{\lambda}{2} |\vec{P}|^2 \right) \ ,$$

$$[R_j, R_k] = i\varepsilon_{jkl} R_l \ , \qquad [R_j, N_k] = i\varepsilon_{jkl} N_l \ , \qquad [N_j, N_k] = -i\varepsilon_{jkl} R_l \ .$$

Coalgebraic sector

$$\Delta P_j = P_j \otimes \mathbb{1} + e^{-\lambda P_0} \otimes P_j \ , \qquad \Delta P_j = P_j \otimes \mathbb{1} + \mathbb{1} \otimes P_j \ ,$$

$$\Delta R_j = R_j \otimes \mathbb{1} + \mathbb{1} \otimes R_j \ ,$$

$$\Delta N_k = N_k \otimes \mathbb{1} + e^{-\lambda P_0} \otimes N_k + i\lambda \varepsilon_{klm} P_l \otimes R_m \ ,$$

Antipodes

$$S(P_0) = -P_0 \ , \qquad S(P_j) = -e^{\lambda P_0} P_j \ ,$$

$$S(R_j) = -R_j \ , \qquad S(N_j) = -e^{\lambda P_0} N_j + \lambda \varepsilon_{jkl} e^{\lambda P_0} P_k R_l$$

The algebra of functions over κ-Minkowski

Functions over the κ-Minkowski spacetime = elements of the universal enveloping algebra generated by κ-Minkowski coordinates $\hat{x}_\mu$,

The Weyl maps

Label the κ-Minkowski functions with commutative functions, by giving an ordering prescription:

$$\Omega_R\left(x_0 x_1\right) = \hat{x}_1 \hat{x}_0 \qquad \Omega_S\left(x_0 x_1\right) = 1/2\left(\hat{x}_1 \hat{x}_0 + \hat{x}_0 \hat{x}_1\right)$$

The Weyl maps allow to define a Fourier transform and an integral:

$$f(\hat{x}) = \int d^4k\ \tilde{f}_R(k) \Omega_R\left(e^{ik\cdot x}\right) , \qquad \int d^4\hat{x}\ f(\hat{x}) = \tilde{f}_R(0) \equiv \tilde{f}_S(0)$$

κ-Poincaré as the algebra of symmetries of κ-Minkowski

Action of κ-Poincaré translations on functions over κ-Minkowski :

$$P_\mu \triangleright \hat{x}_\nu = -i\eta_{\mu\nu} \qquad \text{(duality)}$$

then, to extend it to polynomials, one uses the coproduct:

$$P_\mu \triangleright f(\hat{x}) \cdot g(\hat{x}) = \mu \left[\Delta P_\mu \triangleright f(\hat{x}) \otimes g(\hat{x}) \right] \qquad \mu[f \otimes g] = f \cdot g$$

for example, in the bicrossproduct basis:

$$P_j \triangleright [f(\hat{x}) \cdot g(\hat{x})] = [P_j \triangleright f(\hat{x})] \cdot g(\hat{x}) + [e^{-\lambda P_0} \triangleright f(\hat{x})] \cdot [P_j \triangleright g(\hat{x})]$$

other (translation) bases have different coproducts: if $P_j^S = e^{\frac{\lambda}{2} P_0} P_j$

$$P_j^S \triangleright [f(\hat{x}) \cdot g(\hat{x})] = [P_j^S \triangleright f(\hat{x})] \cdot \left[e^{\frac{\lambda}{2} P_0} \triangleright g(\hat{x}) \right] + \left[e^{-\frac{\lambda}{2} P_0} \triangleright f(\hat{x}) \right] \cdot \left[P_j^S \triangleright g(\hat{x}) \right]$$

One can write κ-Poincaré elements as linear operators acting "inside" the Weyl map:

$$P_\mu \triangleright \Omega_R(f(x)) = -i\Omega_R\left(\partial_\mu f(x)\right) \quad \text{(bicrossproduct basis)}$$

Choice of Weyl map $\rightarrow$ choice of basis for κ-Poincaré translations:

$$P^S_\mu \triangleright \Omega_S(f(x)) = -i\Omega_S\left(\partial_\mu f(x)\right)$$

Bicrossproduct Lorentz generators are written in the same way under every Weyl map (their definitions are Weyl map independent):

$$R_j\Omega(f(x)) = -i\varepsilon_{jkl}\Omega(\partial_l f(x) x_k) \ ,$$

$$N_j\Omega(f(x)) = \Omega\left[\left(\frac{1-e^{2i\lambda\partial_0}}{2\lambda} - \frac{\lambda}{2}|\vec{\nabla}|^2\right)x_j f(x) - x_0\partial_j f(x)\right] \ ,$$

- Amelino-Camelia, Arzano (2002)
- Agostini, Amelino-Camelia, D'Andrea (2003)

Scalar field in κ-Minkowski

Quadratic casimir in κ-Poincaré :

$\Box_\lambda = \tilde{P}_\mu \tilde{P}^\mu$, where $\tilde{P}_0 = \left(\frac{2}{\lambda}\right) \sinh\left(\frac{\lambda}{2} P_0\right)$, $\tilde{P}_j = e^{\frac{\lambda}{2} P_0} P_j$

Massless scalar field:

$$\Box_\lambda \Phi(\hat{x}) = 0 \ ,$$

EOM generated by this action:

$$S = \int d^4\hat{x} \ \mathscr{L}[\Phi] = \frac{1}{2} \int d^4\hat{x} \ \tilde{P}_\mu \Phi(\hat{x}) \tilde{P}^\mu \Phi(\hat{x}) \ ,$$

Naïve Noether analysis - (I)

"Infinitesimal" translation:

$$\delta \hat{x}_\mu = \hat{x}'_\mu - \hat{x}_\mu = i\varepsilon^\nu P_\nu \hat{x}_\mu \ ,$$

Variation of the Φ field under a κ-Poincaré transformation:

$$\begin{aligned} \delta_T \Phi(\hat{x}) &= \Phi'(\hat{x}') - \Phi(\hat{x}) = \Phi'(\hat{x}') - \Phi(\hat{x}') + \Phi(\hat{x}') - \Phi(\hat{x}) \\ &\simeq \delta\Phi(\hat{x}) + d\Phi(\hat{x}) \ , \end{aligned}$$

Imposing the scalarity of the field:

$$\delta_T \Phi(\hat{x}) = 0 \qquad \Rightarrow \qquad \delta\Phi(\hat{x}) = -d\Phi(\hat{x}) \ ,$$

where, in the case of translations:

$$d\Phi(\hat{x}) = \Phi(\hat{x}') - \Phi(\hat{x}) = i\varepsilon^\mu P_\mu \Phi(\hat{x}) \ ,$$

Naïve Noether analysis - (II)

Variation of the action:

$$\delta S = \int d^4\hat{x}\ \delta_T \mathscr{L}[\Phi] = \int d^4\hat{x} \{\mathscr{L}[\Phi + \delta\Phi] - \mathscr{L}[\Phi] + d\mathscr{L}[\Phi]\} =$$
$$= \frac{1}{2}\int d^4\hat{x} \left\{ \tilde{P}_\mu \delta\Phi \tilde{P}^\mu \Phi + \tilde{P}_\mu \Phi \tilde{P}^\mu \delta\Phi - \delta\left(\tilde{P}_\mu \Phi \tilde{P}^\mu \Phi\right)\right\}$$

The action can be invariant:

$$\delta S = 0$$

only if $\delta = -i\varepsilon^\mu P_\mu$ satisfies Leibniz rule.

P_μ does not comply with Leibniz rule $\Rightarrow$ ε_μ isn't commutative

- Agostini, Amelino-Camelia, Arzano, Marcianò, Tacchi (2007)

Noncommutative transformation parameters

Imposing Leibniz rule:

$$\varepsilon^\mu P_\mu (fg) = \left(\varepsilon^\mu P_\mu f\right) g + f \left(\varepsilon^\mu P_\mu g\right)$$

one obtains for ε_j (ε_0 is trivial):

$$\varepsilon_j [P_j f] g + \varepsilon_j \left[e^{-\lambda P_0} f\right] [P_j g] = [\varepsilon_j P_j f] g + f \varepsilon_j [P_j g]$$

Commutation rules for ε_j

$$f(\hat{x})\varepsilon_j = \varepsilon_j e^{-\lambda P_0} f(\hat{x}) \quad \Leftrightarrow \quad [\varepsilon_j, \hat{x}_0] = i\lambda \varepsilon_j$$

Successful Noether analysis

Agostini,Amelino-Camelia,Arzano,Marcianò,Tacchi (2007):
First successful Noether analysis for the translation symmetries of a scalar field theory in κ-Minkowski :

$$\delta S = \frac{i}{2}\int d^4\hat{x}\,\varepsilon_\nu P_\mu T^{\mu\nu} = 0$$

$$T^{\mu\nu} = \left[e^{\lambda\delta^0_\nu P_0}\tilde{P}^\mu\Phi(\hat{x})\right]\left[e^{\frac{\lambda}{2}P_0}P^\nu\Phi(\hat{x})\right] - \left[e^{\lambda(\delta^0_\nu-\frac{1}{2})P_0}\Phi(\hat{x})\right]\left[\tilde{P}^\mu P^\nu\Phi(\hat{x})\right]$$

$$\Downarrow$$

$$Q^P_\mu = -\frac{1}{2}\int d^4q\,\frac{\tilde{q}_0}{|\tilde{q}_0|}e^{\lambda q_0\delta^0_\mu}\delta(\tilde{q}^2)\tilde{\Phi}(-q_0,-e^{\lambda q_0}\vec{q})\tilde{\Phi}(q)q_\mu = \text{Conserved}\ .$$

Bicovariant differential calculus over an Hopf algebra $\mathscr{A}$

A *bimodule* Γ over $\mathscr{A}$ + a map $d : \mathscr{A} \to \Gamma$ which satisfies Leibniz rule.

This is exactly what we have shown above, if we consider κ-Minkowski as an Hopf algebra:

$$\Delta \hat{x}_\mu = \hat{x}_\mu \otimes \mathbb{1} + \mathbb{1} \otimes \hat{x}_\mu \ ,$$

with ε^μ = basis of Γ, and $\delta = -i\varepsilon^\mu P_\mu$ = differential map.

but bicovariant differential calculi are far from unique...

One can choose one on the basis of some notion of Lorentz covariance after specifying some action of Lorentz generators over one-forms

5D-Bicovariant differential calculus

A natural action is the one that preserves the coproduct:

$$M_{\mu\nu} \triangleright f(\hat{x}) dg(\hat{x}) = \left[M^{(1)}_{\mu\nu} \triangleright f(\hat{x}) \right] \; d \left[M^{(2)}_{\mu\nu} g(\hat{x}) \right]$$

and it leads to a 5D-differential calculus.

To render the 4D calculus ε_μ Lorentz covariant one has to assume an action incompatible with the coproducts.

Also using the 5D calculus the Noether analysis can be accomplished, leading to 5 conserved charges [Freidel, Kowalski-Glikman, Nowak (2007)] [Amelino-Camelia, Marcianò, Pranzetti (2007)]
The question remains open: How many "translation" symmetries are there in κ-Minkowski ?

Extension to the Lorentz sector of the above results

$$\Delta P_j = P_j \otimes \mathbb{1} + e^{-\lambda P_0} \otimes P_j \ ,$$

$$\downarrow ?$$

$$\Delta N_k = N_k \otimes \mathbb{1} + e^{-\lambda P_0} \otimes N_k + i\lambda \varepsilon_{klm} P_l \otimes R_m$$

Rotations

Rotations already satisfy Leibniz rule, so that in $\delta = -i\sigma_j R_j$ we can take σ_j commutative.

Variation of the action:

$$\delta S = \frac{i}{2} \int d^4\hat{x}\, \sigma_k P_\mu J^\mu{}_k \ ,$$

$$J^\mu{}_k = \left[e^{\frac{\lambda}{2}P_0} \Phi(\hat{x}) \right] \left[\tilde{P}^\mu R_j \Phi(\hat{x}) \right] - \left[e^{\lambda P_0} \tilde{P}^\mu \Phi(\hat{x}) \right] \left[e^{\frac{\lambda}{2}P_0} R_j \Phi(\hat{x}) \right] \ ,$$

The associated charge results conserved:

$$Q^R_j = -\frac{i}{2} \int d^4 q \ \frac{\tilde{q}_0}{|\tilde{q}_0|} \delta(\tilde{q}^2) e^{\lambda q_0} \tilde{\Phi}(-q_0, -e^{\lambda q_0} \vec{q}) \varepsilon_{jkl} q_l \frac{\partial \tilde{\Phi}(q)}{\partial q_k} \ .$$

Boosts - (I)

Coproduct of N_j:

$$N_k[f(\hat{x})g(\hat{x})] = [N_k f(\hat{x})]g(\hat{x}) + \left[e^{-\lambda P_0} f(\hat{x})\right][N_k g(\hat{x})] + \\ + i\lambda\varepsilon_{klm}[P_l f(\hat{x})][R_m g(\hat{x})] \ ,$$

due to the red term, $\delta = -i\tau_j N_j$ cannot satisfy Leibniz rule by means of simple algebraic properties of τ_j:

$$\tau_k\left[e^{-\lambda P_0} f\right][N_k g] + i\lambda\varepsilon_{klm}\tau_k[P_l f][R_m g] = f\tau_k[N_k g] \ .$$

Boosts - the 1+1-dimensional case

simplified case, in which the κ-Lorentz algebra reduces to only one generator:

$$\Delta N = N \otimes \mathbb{1} + e^{-\lambda P_0} \otimes N \ ,$$

coproduct identical to that of P_j, so:

$$[\tau, \hat{x}_0] = i\lambda\tau \ , \qquad [\tau, \hat{x}_j] = 0 \ ,$$

renders $\delta = -i\tau N$ Leibniz-compatible.

The simplification here is the lack of rotation generators R_j

Boosts - (II)

3+1-D case: add not only two other boost generators, but also three rotations:

$$\tau N \nrightarrow \tau_j N_j \qquad \Rightarrow \qquad \tau N \rightarrow \sigma_j R_j + \tau_k N_k \ .$$

Idea: work with boosts and rotations together:

Requirement: the whole $\delta = -i\tau_j N_j - i\sigma_k R_k$ satisfies Leibniz rule

This is feasible:

The commutation rules

$$[\tau_j, \hat{x}_0] = i\lambda\tau_j \ , \quad [\tau_j, \hat{x}_k] = 0 \ ,$$

$$[\sigma_j, \hat{x}_k] = i\lambda\varepsilon_{jlk}\tau_l \ , \quad [\sigma_j, \hat{x}_0] = 0 \ .$$

The fate of pure rotations

Problem: we've changed the properties of σ_j: is the Noether analysis for rotations still possible?

$$[\sigma_j, \hat{x}_k] = i\lambda\varepsilon_{jlk}\tau_l$$

From the commutation rules above, one observes that

The σ_j are noncommutative only if $\tau_k \neq 0$

So:

PURE ROTATIONS REMAIN COMMUTATIVE/CLASSICAL

Noether charges for boosts

Variation of the action:

$$\delta S = \frac{i}{2} \int d^4\hat{x} \, \{\sigma_k P_\mu J^\mu{}_k + \tau_j P_\mu K^\mu{}_j\} \ ,$$

$J^\mu{}_k$ = the same rotation current as before in accordance with the "pure rotations" remark,
$K^\mu{}_j$ leads to the following conserved charges:

$$Q_j^N = -\frac{i}{2} \int d^4q \, \frac{\tilde{q}}{|\tilde{q}_0|} \delta(\tilde{q}^2) e^{\lambda q_0} \tilde{\Phi}(-q_0, -e^{\lambda q_0}\vec{q}) \cdot$$
$$\cdot \left\{ q_j \frac{\partial \tilde{\Phi}(q)}{\partial q_0} + \frac{\partial}{\partial q_j} \left[\left(\frac{1 - e^{-2\lambda q_0}}{2\lambda} - \frac{\lambda}{2} |\vec{q}|^2 \right) \tilde{\Phi}(q) \right] - \lambda q_j \tilde{\Phi}(q) \right\} \ .$$

A "no-pure-boost uncertainty principle"

Let's have another look at this commutation rule:

$$[\sigma_j, \hat{x}_k] = i\lambda\varepsilon_{jlk}\tau_l \ ,$$

corresponding uncertainty relation:

$$\Delta(\sigma_j)\Delta(\hat{x}_k) \geq \frac{\lambda}{2}|\langle\tau_l\rangle|$$

"No pure boost - uncertainty principle"

$$\tau_j \neq 0 \qquad \Rightarrow \qquad \Delta(\sigma_k) \neq 0$$

Conclusions

- Hopf algebras seems to describe well the symmetries of noncommutative spacetimes
- First-order transformations needs noncommuting parameters
- Noether analysis for the whole κ-Poincaré algebra concluded
- No pure boosts are allowed by the theory

Some open issues

- Over all: we still lack *observables* and a well-defined *measurement process*
- Quantisation of the theory
- Finite transformations $T =^{?} exp(\varepsilon^{\mu} P_{\mu})$
- Interacting theories
- Covariance of the conserved charges
- ...