

Edinburgh, July 2008

κ -DEFORMED OSCILLATORS AND
 κ -DEFORMED QUANTUM FIELDS

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Based on:

-hep-th/0703200 (Mod. Phys. Lett. A 23, 9 (2008))

-arXiv 0708.1571 (Phys. Rev. D 77, 105007 (2008))

PLAN:

1. Introduction

2. κ -deformed framework

a) κ -Poincare algebra, κ -Minkowski space

b) κ -deformed free fields

3. κ -deformed statistics and κ -oscillators

4. κ -deformed Fock space, states with bosonic statistics and clustering of multiparticle states

5. κ -deformed fields with standard star product and on-shell κ -deformed oscillators

6. κ -deformed fields with nonstandard star product and off-shell κ -deformed oscillators

7. Quantization rules for κ -deformed quantum fields

8. Different algebraic models of κ -statistics

9. Final Remarks

1. INTRODUCTION

Two possible ways of deforming classical free fields

$$\varphi(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^4\mathbf{p} \mathbf{A}(\mathbf{p}) e^{i\mathbf{p}\mathbf{x}}$$

α) deformation of space-time

$$\begin{array}{ccc} \mathbf{x}_\mu & \longrightarrow & \hat{\mathbf{x}}_\mu \\ \text{classical} & & \text{quantum} \\ [\mathbf{x}_\mu, \mathbf{x}_\nu] = \mathbf{0} & \rightsquigarrow & [\hat{\mathbf{x}}_\mu, \hat{\mathbf{x}}_\nu] = \theta_{\mu\nu} \end{array}$$

One can expand:

$$\frac{1}{\kappa^2} \theta_{\mu\nu}(\kappa \hat{\mathbf{x}}) = \frac{1}{\kappa^2} \theta_{\mu\nu}^{(0)} + \frac{1}{\kappa} \theta_{\mu\nu}^{(1)\rho} \hat{\mathbf{x}}_\rho + \theta_{\mu\nu}^{(2)\rho\sigma} \hat{\mathbf{x}}_\rho \hat{\mathbf{x}}_\sigma$$

\nearrow \uparrow \nwarrow
 canonical Lie-algebraic quadratic
 (κ -deformation)

β) deformation of field quanta

$$\begin{array}{ccc} \mathbf{A}(\mathbf{p}) & \longrightarrow & \hat{\mathbf{A}}(\mathbf{p}) \\ \text{classical} & & \text{braided} \\ [\mathbf{A}(\mathbf{p}), \mathbf{A}(\mathbf{p}')] = \mathbf{0} & \rightsquigarrow & [\hat{\mathbf{A}}(\mathbf{p}), \hat{\mathbf{A}}(\mathbf{p}')] \neq \mathbf{0} \end{array}$$

One can introduce braided commutator:

$$[\mathbf{A}(\mathbf{p}), \mathbf{A}(\mathbf{p}')] = \mathbf{0} \quad \rightsquigarrow \quad [\hat{\mathbf{A}}(\mathbf{p}), \hat{\mathbf{A}}(\mathbf{p}')]_{\text{Br}} = \mathbf{0}$$

where

$$[\hat{\mathbf{A}}(\mathbf{p}), \hat{\mathbf{A}}(\mathbf{p}')]_{\text{Br}} = \begin{cases} \hat{\mathbf{A}}(\mathbf{p}) \circ \hat{\mathbf{A}}(\mathbf{p}') - \hat{\mathbf{A}}(\mathbf{p}') \circ \hat{\mathbf{A}}(\mathbf{p}) \\ \hat{\mathbf{A}}(\mathbf{p})\hat{\mathbf{A}}(\mathbf{p}') - \hat{\tau} \left(\hat{\mathbf{A}}(\mathbf{p})\hat{\mathbf{A}}(\mathbf{p}') \right) \end{cases}$$

$\hat{\tau}$ -deformed flip operator τ_0

$$\tau_0 (\mathbf{A}(\mathbf{p})\mathbf{A}(\mathbf{p}')) = \mathbf{A}(\mathbf{p}')\mathbf{A}(\mathbf{p})$$

γ) quantization

	<i>undeformed</i>	$\xrightarrow{\theta}$	<i>deformed</i>
<i>class.</i>	$[\mathbf{A}(\mathbf{p}), \mathbf{A}(\mathbf{p}')] = \mathbf{0}$		$[\mathbf{A}(\mathbf{p}), \mathbf{A}(\mathbf{p}')]_{\text{Br}} = \mathbf{0}$
	$\downarrow \hbar$		$\downarrow \hbar$
<i>quant.</i>	$[\hat{\mathbf{A}}(\mathbf{p}), \hat{\mathbf{A}}(\mathbf{p}')] \sim \hbar$	$\xrightarrow{\theta}$	$[\hat{\mathbf{A}}(\mathbf{p}), \hat{\mathbf{A}}(\mathbf{p}')]_{\text{Br}} \sim \hbar$

quantum standard
field theory

→

quantum braided
field theory

For canonical deformation

$$[\hat{\mathbf{x}}_\mu, \hat{\mathbf{x}}_\nu] = \frac{\mathbf{i}}{\kappa^2} \theta_{\mu\nu} \quad \theta_{\mu\nu} = \text{const}$$

Homomorphic star product realization

$$\varphi(\hat{\mathbf{x}}) \cdot \chi(\hat{\mathbf{x}}) \quad \longleftrightarrow \quad \varphi(\mathbf{x}) \star_\theta \chi(\mathbf{x}) \quad \text{Moyal – Weyl} \\ \text{star-product}$$

where

$$\begin{aligned} \varphi(\mathbf{x}) \star_\theta \chi(\mathbf{x}) &= \varphi(\mathbf{x}) \exp \left[\frac{\mathbf{i}}{\kappa^2} \overleftarrow{\partial}_\mu \theta^{\mu\nu} \overrightarrow{\partial}_\nu \right] \chi(\mathbf{x}) \\ &= \frac{1}{(2\pi)^3} \int d^4\mathbf{p} d^4\mathbf{q} \mathbf{A}(\mathbf{p}) \cdot \mathbf{A}(\mathbf{q}) \cdot e^{i\mathbf{p}\theta\mathbf{q}} \cdot e^{i(\mathbf{p}+\mathbf{q})\mathbf{x}} \end{aligned}$$

If we define

$$\mathbf{A}(\mathbf{p}) \circ_\theta \mathbf{A}(\mathbf{q}) = \mathbf{A}(\mathbf{p}) \cdot \mathbf{A}(\mathbf{q}) \cdot e^{i\mathbf{p}\theta\mathbf{q}}$$

$$[\mathbf{A}(\mathbf{p}), \mathbf{A}(\mathbf{p}')] \rightarrow [\mathbf{A}(\mathbf{p}), \mathbf{A}(\mathbf{p}')]_{\circ_\theta} - \text{new statistics}$$

we get two equivalent descriptions $\alpha) \Leftrightarrow \beta)$

$$\varphi(\mathbf{x}) \star_\theta \chi(\mathbf{x}) = \varphi(\mathbf{x}) \circ_\theta \chi(\mathbf{x})$$

We look for similar equality for κ -deformation

$$\varphi(\mathbf{x}) \star_\kappa \chi(\mathbf{x}) = \varphi(\mathbf{x}) \circ_\kappa \chi(\mathbf{x})$$

and

$$\varphi(\mathbf{x}) \star_\kappa \chi(\mathbf{y}) = \varphi(\mathbf{x}) \circ_\kappa \chi(\mathbf{y})$$

2. κ -DEFORMED FRAMEWORK

a) κ -Poincare algebra, κ -deformed Minkowski space

κ -deformed Poincare-Hopf algebra

We choose κ -deformed Poincare algebra in the modified Majid-Ruegg basis

$$\begin{aligned}
 [\mathbf{M}_{\mu\nu}, \mathbf{M}_{\lambda\sigma}] &= \mathbf{i} (\eta_{\mu\sigma} \mathbf{M}_{\nu\lambda} - \eta_{\nu\sigma} \mathbf{M}_{\mu\lambda} + \eta_{\nu\lambda} \mathbf{M}_{\mu\sigma} - \eta_{\mu\lambda} \mathbf{M}_{\nu\sigma}) \\
 [\mathbf{M}_i, \mathbf{P}_j] &= \mathbf{i} \epsilon_{ijk} \mathbf{P}_k \\
 [\mathbf{N}_i, \mathbf{P}_j] &= \mathbf{i} \delta_{ij} e^{\frac{\mathbf{P}_0}{2\kappa}} \left[\frac{\kappa}{2} \left(1 - e^{-\frac{2\mathbf{P}_0}{\kappa}} \right) + \frac{1}{2\kappa} e^{-\frac{\mathbf{P}_0}{\kappa}} \tilde{\mathbf{P}}^2 \right] - \frac{\mathbf{i}}{2\kappa} e^{-\frac{\mathbf{P}_0}{2\kappa}} \mathbf{P}_i \mathbf{P}_j \\
 [\mathbf{M}_i, \mathbf{P}_0] &= 0 \quad , \quad [\mathbf{N}_i, \mathbf{P}_0] = \mathbf{i} e^{-\frac{\mathbf{P}_0}{2\kappa}} \mathbf{P}_i \\
 [\mathbf{P}_\mu, \mathbf{P}_\nu] &= 0
 \end{aligned}$$

Relation with standard Majid-Ruegg basis

$$\mathbf{P}_i = e^{\frac{\mathbf{P}_0}{2\kappa}} \mathbf{P}_i^{\text{MR}}$$

κ -deformed mass Casimir

$$\begin{aligned}
 \mathbf{C}_\kappa^2(\mathbf{P}_\mu) &= \mathbf{C}_\kappa^2(\tilde{\mathbf{P}}, \mathbf{P}_0) = \left(2\kappa \sinh \left(\frac{\mathbf{P}_0}{2\kappa} \right) \right)^2 - \tilde{\mathbf{P}}^2 \\
 \Rightarrow \mathbf{p}_0 &\equiv \omega_\kappa = 2\kappa \operatorname{arcsinh} \frac{(\tilde{\mathbf{p}}^2 + \mathbf{m}^2)^{\frac{1}{2}}}{2\kappa}
 \end{aligned}$$

κ -deformed Minkowski space

$$[\hat{x}_0, \hat{x}_i] = \frac{i}{\kappa} \hat{x}_i \quad , \quad [\hat{x}_i, \hat{x}_j] = 0$$

Extension to the pair of κ -deformed Minkowski spaces
 \hat{x}_μ, \hat{x}_ν

$$[\hat{x}_0, \hat{y}_0] = [\hat{y}_0, \hat{y}_j] = 0$$

$$[\hat{x}_0, \hat{y}_i] = \frac{i}{\kappa} \hat{y}_i \quad , \quad [\hat{y}_0, \hat{x}_i] = \frac{i}{\kappa} \hat{x}_i$$

b) κ -deformed free fields

We choose the order in noncommutative exponential

$$:e^{ip_\mu \hat{x}^\mu} : = e^{-\frac{i}{2} p_0 \hat{x}_0} e^{ip_i \hat{x}_i} e^{-\frac{i}{2} p_0 \hat{x}_0}$$

The Weyl homomorphism leads to the star product

$$:e^{ip_\mu \hat{x}^\mu} : \cdot :e^{iq_\mu \hat{x}^\mu} : \leftrightarrow e^{ip_\mu x^\mu} \star e^{iq_\mu x^\mu} = e^{-i(p_0+q_0)x_0+i\Delta_i(\tilde{p},\tilde{q})x_i}$$

where

$$\Delta_i(\tilde{p}, \tilde{q}) = p_i e^{\frac{q_0}{2\kappa}} + e^{-\frac{p_0}{2\kappa}} q_i \quad (\text{coproduct formula})$$

The homomorphism extends to bilocal products if

$$:e^{ip_\mu \hat{x}^\mu} : \cdot :e^{iq_\mu \hat{y}^\mu} : \leftrightarrow e^{ip_\mu x^\mu} \star e^{iq_\mu y^\mu} = e^{-i(p_0 x_0 + q_0 y_0) + ip_i e^{\frac{q_0}{2\kappa} x_i} + q_i e^{-\frac{p_0}{2\kappa} y_i}}$$

Single κ -deformed free field

$$\phi(\hat{\mathbf{x}}) = \frac{1}{(2\pi)^{3/2}} \int d^4\mathbf{p} \mathbf{A}(\mathbf{p}_0, \tilde{\mathbf{p}}) \delta(\mathbf{C}_\kappa^2(\tilde{\mathbf{p}}, \mathbf{p}_0) - \mathbf{M}^2) :e^{ip\hat{\mathbf{x}}}:$$

Resolution of mass-shell condition

$$\mathbf{p}_0^\pm = \pm \omega_\kappa(\tilde{\mathbf{p}}) = \pm 2\kappa \operatorname{arcsinh} \left(\frac{\sqrt{\tilde{\mathbf{p}}^2 + \mathbf{M}^2}}{2\kappa} \right)$$

We have

$$\delta(\mathbf{C}_\kappa^2(\tilde{\mathbf{p}}, \mathbf{p}_0) - \mathbf{M}^2) = \frac{1}{\Omega_\kappa(\tilde{\mathbf{p}})} \delta(\mathbf{p}_0 - \mathbf{p}_0^+) + \frac{1}{\Omega_\kappa(\tilde{\mathbf{p}})} \delta(\mathbf{p}_0 - \mathbf{p}_0^-)$$

where

$$\Omega_\kappa(\tilde{\mathbf{p}}) = 2\kappa \sinh \left(\frac{\omega_\kappa(\tilde{\mathbf{p}})}{\kappa} \right)$$

If

$$\phi(\hat{\mathbf{x}}) = \phi_+(\hat{\mathbf{x}}) + \phi_-(\hat{\mathbf{x}})$$

and $\mathbf{A}(\mathbf{p}_0, \tilde{\mathbf{p}}) = \sqrt{\Omega_\kappa} \cdot \mathbf{a}(\mathbf{p}_0, \tilde{\mathbf{p}})$ one gets

$$\phi(\hat{\mathbf{x}}) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3\tilde{\mathbf{p}}}{\Omega_\kappa(\tilde{\mathbf{p}})} \left(\mathbf{a}(\mathbf{p}_0, \tilde{\mathbf{p}}) :e^{i\mathbf{p}\hat{\mathbf{x}}}: + \mathbf{a}^\dagger(\mathbf{p}_0, \tilde{\mathbf{p}}) :e^{-i\mathbf{p}\hat{\mathbf{x}}}: \right) \Big|_{\mathbf{p}_0=\omega_\kappa(\tilde{\mathbf{p}})}$$

The creation/annihilation operators describe excitations with definite fourmomenta

$$\mathbf{P}_\mu \triangleright \mathbf{a}_\kappa(\mathbf{p}_0, \tilde{\mathbf{p}}) = \mathbf{p}_\mu \mathbf{a}_\kappa(\mathbf{p}_0, \tilde{\mathbf{p}})$$

$$\mathbf{P}_\mu \triangleright \mathbf{a}_\kappa(\mathbf{p}_0, \tilde{\mathbf{p}}) \equiv \mathbf{m}(\Delta^{(1)}(\mathbf{P}_\mu) \mathbf{a}_\kappa \mathbf{S}(\Delta^{(2)}(\mathbf{P}_\mu)))$$

$$\mathbf{P}_\mu \triangleright (\mathbf{a}_\kappa(\mathbf{p}_0, \tilde{\mathbf{p}}) \mathbf{a}_\kappa(\mathbf{q}_0, \tilde{\mathbf{q}})) =$$

$$\left(\Delta^{(1)}(\mathbf{P}_\mu) \triangleright \mathbf{a}_\kappa(\mathbf{p}_0, \tilde{\mathbf{p}}) \right) \cdot \left(\Delta^{(2)}(\mathbf{P}_\mu) \triangleright \mathbf{a}_\kappa(\mathbf{q}_0, \tilde{\mathbf{q}}) \right)$$

Explicitly

$$\mathbf{P}_0 \triangleright (\mathbf{a}_\kappa(\mathbf{p}_0, \tilde{\mathbf{p}}) \mathbf{a}_\kappa(\mathbf{q}_0, \tilde{\mathbf{q}})) = (\mathbf{p}_0 + \mathbf{q}_0) (\mathbf{a}_\kappa(\mathbf{p}_0, \tilde{\mathbf{p}}) \mathbf{a}_\kappa(\mathbf{q}_0, \tilde{\mathbf{q}}))$$

$$\mathbf{P}_i \triangleright (\mathbf{a}_\kappa(\mathbf{p}_0, \tilde{\mathbf{p}}) \mathbf{a}_\kappa(\mathbf{q}_0, \tilde{\mathbf{q}})) = (\mathbf{p}_i e^{\frac{\mathbf{q}_0}{2\kappa}} + e^{-\frac{\mathbf{p}_0}{2\kappa}} \mathbf{q}_i) (\mathbf{a}_\kappa(\mathbf{p}_0, \tilde{\mathbf{p}}) \mathbf{a}_\kappa(\mathbf{q}_0, \tilde{\mathbf{q}}))$$

$$\begin{array}{cc} \uparrow & \uparrow \end{array}$$

nonsymmetric (nonabelian) addition law

Problem

Classical bosonic statistics

$$\mathbf{a}_\kappa(\mathbf{p}_0, \tilde{\mathbf{p}}) \cdot \mathbf{a}_\kappa(\mathbf{q}_0, \tilde{\mathbf{q}}) = \mathbf{a}_\kappa(\mathbf{q}_0, \tilde{\mathbf{q}}) \cdot \mathbf{a}_\kappa(\mathbf{p}_0, \tilde{\mathbf{p}})$$

nonconsistency with nonsymmetric coproduct !

We should introduce κ -statistics to remove this inconsistency. We start with the following general deformation

$$\begin{aligned} \mathbf{a}_\kappa(\mathbf{f}_0(\mathbf{p}, \mathbf{q}), \mathbf{f}_i(\mathbf{p}, \mathbf{q})) \cdot \mathbf{a}_\kappa(\mathbf{g}_0(\mathbf{p}, \mathbf{q}), \mathbf{g}_i(\mathbf{p}, \mathbf{q})) &= \\ &= \mathbf{a}_\kappa(\mathbf{h}_0(\mathbf{p}, \mathbf{q}), \mathbf{h}_i(\mathbf{p}, \mathbf{q})) \cdot \mathbf{a}_\kappa(\mathbf{k}_0(\mathbf{p}, \mathbf{q}), \mathbf{k}_i(\mathbf{p}, \mathbf{q})) \end{aligned}$$

We choose

$$\mathbf{f}_0 = \mathbf{p}_0 \quad , \quad \mathbf{g}_0 = \mathbf{q}_0 \quad , \quad \mathbf{h}_0 = \mathbf{q}_0 \quad , \quad \mathbf{k}_0 = \mathbf{p}_0$$

$$\mathbf{f}_i = \mathbf{p}_i e^{-\frac{\mathbf{q}_0}{2\kappa}} \quad , \quad \mathbf{g}_i = \mathbf{q}_i e^{\frac{\mathbf{p}_0}{2\kappa}} \quad , \quad \mathbf{h}_i = \mathbf{q}_i e^{-\frac{\mathbf{p}_0}{2\kappa}} \quad , \quad \mathbf{k}_i = \mathbf{p}_i e^{\frac{\mathbf{q}_0}{2\kappa}}$$

One gets

$$\begin{aligned} \mathbf{P}_\mu \triangleright \left(\mathbf{a}_\kappa \left(\mathbf{p}_0, e^{-\frac{\mathbf{q}_0}{2\kappa}} \tilde{\mathbf{p}} \right) \mathbf{a}_\kappa \left(\mathbf{q}_0, e^{\frac{\mathbf{p}_0}{2\kappa}} \tilde{\mathbf{q}} \right) \right) = \\ = (\mathbf{p}_\mu + \mathbf{q}_\mu) \left(\mathbf{a}_\kappa \left(\mathbf{p}_0, e^{-\frac{\mathbf{q}_0}{2\kappa}} \tilde{\mathbf{p}} \right) \mathbf{a}_\kappa \left(\mathbf{q}_0, e^{\frac{\mathbf{p}_0}{2\kappa}} \tilde{\mathbf{q}} \right) \right) \end{aligned}$$

$$\begin{aligned} \mathbf{P}_\mu \triangleright \left(\mathbf{a}_\kappa \left(\mathbf{q}_0, e^{-\frac{\mathbf{p}_0}{2\kappa}} \tilde{\mathbf{q}} \right) \mathbf{a}_\kappa \left(\mathbf{p}_0, e^{\frac{\mathbf{q}_0}{2\kappa}} \tilde{\mathbf{p}} \right) \right) = \\ = (\mathbf{p}_\mu + \mathbf{q}_\mu) \left(\mathbf{a}_\kappa \left(\mathbf{q}_0, e^{-\frac{\mathbf{p}_0}{2\kappa}} \tilde{\mathbf{q}} \right) \mathbf{a}_\kappa \left(\mathbf{p}_0, e^{\frac{\mathbf{q}_0}{2\kappa}} \tilde{\mathbf{p}} \right) \right) \end{aligned}$$

We get consequently standard Abelian addition law

$$\mathbf{p}_0^{(1+2)} = \mathbf{p}_0 + \mathbf{q}_0 \quad , \quad \tilde{\mathbf{p}}^{(1+2)} = \tilde{\mathbf{p}} + \tilde{\mathbf{q}}$$

In our approach the energy and momenta of individual particles which we add in classical (Abelian) way are put on κ -deformed mass-shells:

$$\mathbf{p}_0^2 = \omega_\kappa(\tilde{\mathbf{p}}^2) \quad , \quad \mathbf{q}_0^2 = \omega_\kappa(\tilde{\mathbf{q}}^2)$$

This is consistent with the energy conservation

$$\omega_\kappa(\tilde{\mathbf{p}}^2) + \omega_\kappa(\tilde{\mathbf{q}}^2) = \omega_\kappa(\tilde{\mathbf{q}}^2) + \omega_\kappa(\tilde{\mathbf{p}}^2)$$

But the oscillators $\mathbf{a}(\mathbf{p}_0, e^{-\frac{\mathbf{q}_0}{2\kappa}} \tilde{\mathbf{p}})$, ... are off-shell!

Two options with the mass-shell condition

Our option: new coupled mass-shell conditions \Leftrightarrow going off-shell in particular way:

$$\mathbf{a}_\kappa(\mathbf{p}_0, \tilde{\mathbf{p}})\mathbf{a}_\kappa(\mathbf{q}_0, \tilde{\mathbf{q}}) \rightarrow \mathbf{a}_\kappa(\mathbf{p}_0, \tilde{\mathcal{P}} = \tilde{\mathbf{p}}e^{-\mathbf{q}_0/2\kappa})\mathbf{a}_\kappa(\mathbf{q}_0, \tilde{\mathcal{Q}} = \tilde{\mathbf{q}}e^{\mathbf{p}_0/2\kappa})$$

$$\mathbf{p}_0^2 = \omega_\kappa^2(\tilde{\mathbf{p}}) \quad \Leftrightarrow \quad \mathbf{p}_0^2 = \omega_\kappa^2(\tilde{\mathcal{P}}e^{\mathbf{q}_0/2\kappa})$$

$$\mathbf{q}_0^2 = \omega_\kappa^2(\tilde{\mathbf{q}}) \quad \Leftrightarrow \quad \mathbf{q}_0^2 = \omega_\kappa^2(\tilde{\mathcal{Q}}e^{-\mathbf{p}_0/2\kappa})$$

↑

coupled mass-shells

Such entangled dispersion relations are consistent with addition law of energy

Other option: standard mass-shell for the fourmomenta $\tilde{\mathcal{P}}, \tilde{\mathcal{Q}}$ carried by the κ -oscillators:

$$\mathbf{p}_0^2 = \omega_\kappa^2(\tilde{\mathbf{p}}) \quad \rightarrow \quad \mathbf{p}_0^2 = \omega_\kappa^2(\tilde{\mathcal{P}}) = \omega_\kappa^2(\tilde{\mathbf{p}}e^{-\mathbf{q}_0/2\kappa})$$

$$\mathbf{q}_0^2 = \omega_\kappa^2(\tilde{\mathbf{q}}) \quad \rightarrow \quad \mathbf{q}_0^2 = \omega_\kappa^2(\tilde{\mathcal{Q}}) = \omega_\kappa^2(\tilde{\mathbf{q}}e^{\mathbf{p}_0/2\kappa})$$

Oscillators are on-shell, but problems with addition law of energies, because

$$\omega_\kappa(\tilde{\mathbf{p}}e^{-\mathbf{q}_0/2\kappa}) + \omega_\kappa(\tilde{\mathbf{q}}e^{\mathbf{p}_0/2\kappa}) \neq \omega_\kappa(\tilde{\mathbf{q}}e^{-\mathbf{p}_0/2\kappa}) + \omega_\kappa(\tilde{\mathbf{p}}e^{\mathbf{q}_0/2\kappa}) !$$

Our next step which we perform:

Deformation of n -product of oscillators which provide the classical value of the total fourmomentum

$$\begin{array}{ccc} \Delta^{(n)}(\mathbf{p}_1, \dots, \mathbf{p}_n) & \longrightarrow & \mathbf{p}_1 + \dots + \mathbf{p}_n \\ \text{standard product} & & \text{deformed product} \\ \text{of oscillators} & & \text{of oscillators} \end{array}$$

3. κ -DEFORMED STATISTICS AND κ -OSCILLATORS

a) κ -deformed oscillator algebra

i) Binary relations

We define new product

$$\mathbf{a}_\kappa(\tilde{\mathbf{p}}) \circ \mathbf{a}_\kappa(\tilde{\mathbf{q}}) = \mathbf{a}_\kappa\left(\mathbf{p}_0, e^{-\frac{q_0}{2\kappa}} \tilde{\mathbf{p}}\right) \mathbf{a}_\kappa\left(\mathbf{q}_0, e^{\frac{p_0}{2\kappa}} \tilde{\mathbf{q}}\right)$$

$$\mathbf{a}_\kappa^\dagger(\tilde{\mathbf{p}}) \circ \mathbf{a}_\kappa^\dagger(\tilde{\mathbf{q}}) = \mathbf{a}_\kappa^\dagger\left(\mathbf{p}_0, e^{\frac{q_0}{2\kappa}} \tilde{\mathbf{p}}\right) \mathbf{a}_\kappa^\dagger\left(\mathbf{q}_0, e^{-\frac{p_0}{2\kappa}} \tilde{\mathbf{q}}\right)$$

$$\mathbf{a}_\kappa^\dagger(\tilde{\mathbf{p}}) \circ \mathbf{a}_\kappa(\tilde{\mathbf{q}}) = \mathbf{a}_\kappa^\dagger\left(\mathbf{p}_0, e^{-\frac{q_0}{2\kappa}} \tilde{\mathbf{p}}\right) \mathbf{a}_\kappa\left(\mathbf{q}_0, e^{-\frac{p_0}{2\kappa}} \tilde{\mathbf{q}}\right)$$

$$\mathbf{a}_\kappa(\tilde{\mathbf{p}}) \circ \mathbf{a}_\kappa^\dagger(\tilde{\mathbf{q}}) = \mathbf{a}_\kappa\left(\mathbf{p}_0, e^{\frac{q_0}{2\kappa}} \tilde{\mathbf{p}}\right) \mathbf{a}_\kappa^\dagger\left(\mathbf{q}_0, e^{\frac{p_0}{2\kappa}} \tilde{\mathbf{q}}\right)$$

We define κ -deformed oscillators as follows ($[\mathbf{A}, \mathbf{B}]_\circ := \mathbf{A} \circ \mathbf{B} - \mathbf{B} \circ \mathbf{A}$)

$$[\mathbf{a}_\kappa(\tilde{\mathbf{p}}), \mathbf{a}_\kappa(\tilde{\mathbf{q}})]_\circ = [\mathbf{a}_\kappa^\dagger(\tilde{\mathbf{p}}), \mathbf{a}_\kappa^\dagger(\tilde{\mathbf{q}})]_\circ = 0$$

$$[\mathbf{a}_\kappa^\dagger(\tilde{\mathbf{p}}), \mathbf{a}_\kappa(\tilde{\mathbf{q}})]_\circ = \delta^{(3)}(\tilde{\mathbf{p}} - \tilde{\mathbf{q}})$$

The algebra is standard, multiplication prescription can be extended in such a way that

$$(\mathbf{a}_\kappa(\tilde{\mathbf{p}}) \circ \mathbf{a}_\kappa(\tilde{\mathbf{q}})) \circ \mathbf{a}_\kappa(\tilde{\mathbf{r}}) = \mathbf{a}_\kappa(\tilde{\mathbf{p}}) \circ (\mathbf{a}_\kappa(\tilde{\mathbf{q}}) \circ \mathbf{a}_\kappa(\tilde{\mathbf{r}}))$$

ii) Arbitrary monomial

The κ -deformed product of n oscillators looks as follows

$$\begin{aligned}
& \mathbf{a}_\kappa(\tilde{\mathbf{p}}^{(1)}) \circ \dots \circ \mathbf{a}_\kappa(\tilde{\mathbf{p}}^{(k)}) \circ \dots \circ \mathbf{a}_\kappa(\tilde{\mathbf{p}}^{(n)}) = \\
& \mathbf{a}_\kappa \left(\mathbf{p}_0^{(1)}, \exp \left(- \sum_{i=2}^n \frac{\mathbf{p}_0^{(i)}}{2\kappa} \right) \tilde{\mathbf{p}}^{(1)} \right) \dots \\
& \dots \mathbf{a}_\kappa \left(\mathbf{p}_0^{(k)}, \exp \left(- \sum_{i=k+1}^n \frac{\mathbf{p}_0^{(i)}}{2\kappa} + \sum_{i=1}^{k-1} \frac{\mathbf{p}_0^{(i)}}{2\kappa} \right) \tilde{\mathbf{p}}^{(k)} \right) \dots \\
& \dots \mathbf{a}_\kappa \left(\mathbf{p}_0^{(n)}, \exp \left(\sum_{i=1}^{n-1} \frac{\mathbf{p}_0^{(i)}}{2\kappa} \right) \tilde{\mathbf{p}}^{(n)} \right)
\end{aligned}$$

One can extend the above formula to general product of $a_\kappa(p)$ and $a_\kappa^\dagger(p)$. One can also introduce relativistic extension of these formula

$$\begin{aligned}
& \mathbf{a}_\kappa(\tilde{\mathbf{p}}^{(1)}) \circ_\kappa \dots \circ_\kappa \mathbf{a}_\kappa(\tilde{\mathbf{p}}^{(k)}) \circ_\kappa \dots \circ_\kappa \mathbf{a}_\kappa(\tilde{\mathbf{p}}^{(n)}) = \\
& = \mathbf{F}_\kappa^{(n)}(\mathbf{p}_0^{(1)}, \dots, \mathbf{p}_0^{(n)}) \cdot \mathbf{a}_\kappa(\tilde{\mathbf{p}}^{(1)}) \circ \dots \circ \mathbf{a}_\kappa(\tilde{\mathbf{p}}^{(k)}) \circ \dots \circ \mathbf{a}_\kappa(\tilde{\mathbf{p}}^{(n)})
\end{aligned}$$

where $\mathbf{F}_\kappa^{(n)}(\mathbf{p}_0^{(1)}, \dots, \mathbf{p}_0^{(n)}) = e^{\frac{3}{2\kappa} \sum_{k=1}^n (n+1-2k)\mathbf{p}_0^{(k)}}$ \Leftarrow normalization factor

One can check associativity !!!

4. κ -DEFORMED FOCK SPACE, STATES WITH BOSONIC STATISTICS AND CLUSTERING OF MULTIPARTICLE STATES

We introduce

$$\mathbf{a}_\kappa^\dagger(\mathbf{p}_0, \tilde{\mathbf{p}})|\mathbf{0}\rangle = \mathbf{0} \quad , \quad \langle \mathbf{0}|\mathbf{0}\rangle = 1$$

$$|\tilde{\mathbf{p}}\rangle = \mathbf{a}_\kappa(\mathbf{p}_0, \tilde{\mathbf{p}})|\mathbf{0}\rangle \quad , \quad \mathbf{p}_0 = \omega_\kappa(\tilde{\mathbf{p}})$$

Two-particle state:

$$|\tilde{\mathbf{p}}, \tilde{\mathbf{q}}\rangle = \mathbf{a}_\kappa(\mathbf{p}_0, \tilde{\mathbf{p}}) \circ \mathbf{a}_\kappa(\mathbf{q}_0, \tilde{\mathbf{q}})|\mathbf{0}\rangle$$

$$|\tilde{\mathbf{q}}, \tilde{\mathbf{p}}\rangle = \mathbf{a}_\kappa(\mathbf{q}_0, \tilde{\mathbf{q}}) \circ \mathbf{a}_\kappa(\mathbf{p}_0, \tilde{\mathbf{p}})|\mathbf{0}\rangle$$

$$\mathbf{P}_\mu |\tilde{\mathbf{p}}, \tilde{\mathbf{q}}\rangle = (\mathbf{p}_\mu + \mathbf{q}_\mu) |\tilde{\mathbf{p}}, \tilde{\mathbf{q}}\rangle = (\mathbf{q}_\mu + \mathbf{p}_\mu) |\tilde{\mathbf{q}}, \tilde{\mathbf{p}}\rangle$$

We get classical bosonic symmetry of 2-partical states

$$|\tilde{\mathbf{p}}, \tilde{\mathbf{q}}\rangle = |\tilde{\mathbf{q}}, \tilde{\mathbf{p}}\rangle$$

n-particle state:

$$|\tilde{\mathbf{p}}^{(1)}, \dots, \tilde{\mathbf{p}}^{(k)}, \dots, \tilde{\mathbf{p}}^{(n)}\rangle = \mathbf{a}_\kappa(\mathbf{p}^{(1)}) \circ \dots \circ \mathbf{a}_\kappa(\mathbf{p}^{(k)}) \circ \dots \circ \mathbf{a}_\kappa(\mathbf{p}^{(n)})|\mathbf{0}\rangle$$

We get

$$\begin{aligned}
\mathbf{P}_\mu |\tilde{\mathbf{p}}^{(1)}, \dots, \tilde{\mathbf{p}}^{(k)}, \dots, \tilde{\mathbf{p}}^{(n)}\rangle &= \\
&= \left[\Delta^{(n)}(\mathbf{P}_\mu) \triangleright \left(\mathbf{a}_\kappa(\mathbf{p}^{(1)}) \circ \dots \circ \mathbf{a}_\kappa(\mathbf{p}^{(n)}) \right) \right] |\mathbf{0}\rangle = \\
&= \sum_{i=1}^n \mathbf{p}_\mu^{(i)} |\tilde{\mathbf{p}}^{(1)}, \dots, \tilde{\mathbf{p}}^{(k)}, \dots, \tilde{\mathbf{p}}^{(n)}\rangle
\end{aligned}$$

We also get classical bosonic symmetry

$$\begin{aligned}
|\tilde{\mathbf{p}}^{(1)}, \dots, \tilde{\mathbf{p}}^{(i)}, \dots, \tilde{\mathbf{p}}^{(j)}, \dots, \tilde{\mathbf{p}}^{(n)}\rangle &= \\
&= |\tilde{\mathbf{p}}^{(1)}, \dots, \tilde{\mathbf{p}}^{(j)}, \dots, \tilde{\mathbf{p}}^{(i)}, \dots, \tilde{\mathbf{p}}^{(n)}\rangle
\end{aligned}$$

κ -deformed scalar product:

$$\langle \tilde{\mathbf{k}}^{(1)}, \dots, \tilde{\mathbf{k}}^{(n)} | = \langle \mathbf{0} | \mathbf{a}_\kappa^\dagger(\mathbf{k}_0^{(1)}, \tilde{\mathbf{k}}^{(1)}) \circ \dots \circ \mathbf{a}_\kappa^\dagger(\mathbf{k}_0^{(n)}, \tilde{\mathbf{k}}^{(n)})$$

$$\begin{aligned}
& \langle \tilde{\mathbf{k}}^{(1)}, \dots, \tilde{\mathbf{k}}^{(m)} | \tilde{\mathbf{p}}^{(1)}, \dots, \tilde{\mathbf{p}}^{(n)} \rangle_{\kappa} := \\
& \quad = \langle \tilde{\mathbf{k}}^{(1)}, \dots, \tilde{\mathbf{k}}^{(m)} | \circ | \tilde{\mathbf{p}}^{(1)}, \dots, \tilde{\mathbf{p}}^{(n)} \rangle = \\
& \quad = \langle \mathbf{0} | \mathbf{a}_{\kappa}^{\dagger}(\mathbf{k}_0^{(1)}, \tilde{\mathbf{k}}^{(1)}) \circ \dots \circ \mathbf{a}_{\kappa}^{\dagger}(\mathbf{k}_0^{(m)}, \tilde{\mathbf{k}}^{(m)}) \circ \\
& \quad \quad \mathbf{a}_{\kappa}(\mathbf{p}_0^{(1)}, \tilde{\mathbf{p}}^{(1)}) \circ \dots \circ \mathbf{a}_{\kappa}(\mathbf{p}_0^{(n)}, \tilde{\mathbf{p}}^{(n)}) | \mathbf{0} \rangle
\end{aligned}$$

We get metric in space of states described by \circ . One can calculate that

$$\begin{aligned}
& \langle \tilde{\mathbf{k}}^{(1)}, \dots, \tilde{\mathbf{k}}^{(m)} | \tilde{\mathbf{p}}^{(1)}, \dots, \tilde{\mathbf{p}}^{(n)} \rangle_{\kappa} = \\
& \quad = \delta_{\mathbf{nm}} \sum_{\text{perm}(\mathbf{i}_1, \dots, \mathbf{i}_n)} \delta^{(3)}(\tilde{\mathbf{p}}^{(1)} - \tilde{\mathbf{k}}^{(\mathbf{i}_1)}) \dots \delta^{(3)}(\tilde{\mathbf{p}}^{(n)} - \tilde{\mathbf{k}}^{(\mathbf{i}_n)})
\end{aligned}$$

→ operator-valued metric in space of states

→ one gets properties of **STANDARD BOSONIC FOCK SPACE**

κ -deformed clustered multiparticle:

$$|\tilde{\mathbf{p}}, \tilde{\mathbf{q}}\rangle = \mathbf{a}_\kappa(\mathbf{p}_0, \tilde{\mathbf{p}}) \circ \mathbf{a}_\kappa(\mathbf{q}_0, \tilde{\mathbf{q}}) |0\rangle \equiv \mathbf{a}_\kappa(\mathbf{p}_0, \underbrace{e^{-\frac{q_0}{2\kappa}} \tilde{\mathbf{p}}}_{\tilde{\mathcal{P}}}) \mathbf{a}_\kappa(\mathbf{q}_0, \underbrace{e^{\frac{p_0}{2\kappa}} \tilde{\mathbf{q}}}_{\tilde{\mathcal{Q}}}) |0\rangle$$

where

$$\mathbf{p}_0^2 = \omega_\kappa^2(\tilde{\mathcal{P}} e^{\frac{q_0}{2\kappa}}) \quad , \quad \mathbf{q}_0^2 = \omega_\kappa^2(\tilde{\mathcal{Q}} e^{-\frac{p_0}{2\kappa}})$$

coupled mass-shells of two particles

For n-particle states ($i = 1, 2, \dots, n$)

$$\mathbf{p}_0^{(i)2} = \omega_\kappa^2 \left(\tilde{\mathcal{P}}^{(i)} \exp \left\{ - \sum_{i=1}^{k-1} \frac{\mathbf{p}_0^{(i)}}{2\kappa} + \sum_{i=k+1}^n \frac{\mathbf{p}_0^{(i)}}{2\kappa} \right\} \right)$$

coupled set of nonlinear equations

We see that there is a kinematic coupling between particles - changing energies of remaining n-1 particles

$$\mathbf{p}_0^{(1)}, \mathbf{p}_0^{(2)}, \dots, \mathbf{p}_0^{(i-1)}, \mathbf{p}_0^{(i+1)}, \dots, \mathbf{p}_0^{(n)}$$

changes the energies $\mathbf{p}_0^{(i)}$ of i-th particle

5. κ -DEFORMED FIELDS WITH STANDARD STAR PRODUCT AND ON-SHELL κ -DEFORMED OSCILLATORS

A) Standard \star_κ star product

$$\begin{aligned}
& \varphi(\hat{\mathbf{x}}) \cdot \varphi(\hat{\mathbf{y}}) \leftrightarrow \varphi(\mathbf{x}) \star_\kappa \varphi(\mathbf{y}) = \\
& = \frac{1}{(2\pi)^3} \int d^4\mathbf{p} \int d^4\mathbf{q} e^{(p_0x_0+q_0y_0)-i(p_i e^{\frac{q_0}{2\kappa}} x_i + q_i e^{-\frac{p_0}{2\kappa}} y_i)} \cdot \\
& \cdot \delta\left(C_\kappa^2(\tilde{\mathbf{p}}, p_0) - M^2\right) \delta\left(C_\kappa^2(\tilde{\mathbf{q}}, q_0) - M^2\right) \cdot \\
& \cdot \mathbf{A}(p_0, \tilde{\mathbf{p}}) \mathbf{A}(q_0, \tilde{\mathbf{q}})
\end{aligned}$$

After change of variables

$$\begin{aligned}
\mathcal{P}_0 &= p_0 \quad , \quad \mathcal{P}_i = p_i e^{\frac{q_0}{2\kappa}} \\
\mathcal{Q}_0 &= q_0 \quad , \quad \mathcal{Q}_i = q_i e^{-\frac{p_0}{2\kappa}}
\end{aligned}$$

one obtains

$$\begin{aligned}
\varphi(\mathbf{x}) \star_\kappa \varphi(\mathbf{y}) &= \frac{1}{(2\pi)^3} \int d^4\mathcal{P} \int d^4\mathcal{Q} e^{\frac{3(\mathcal{P}_0 - \mathcal{Q}_0)}{2\kappa}} \cdot \\
& \cdot e^{i(\mathcal{P}_\mu x^\mu + \mathcal{Q}_\mu y^\mu)} \mathbf{A}\left(\mathcal{P}_0, e^{-\frac{\mathcal{Q}_0}{2\kappa}} \mathcal{P}_i\right) \mathbf{A}\left(\mathcal{Q}_0, e^{\frac{\mathcal{P}_0}{2\kappa}} \mathcal{Q}_i\right) \cdot \\
& \cdot \delta\left(C_\kappa^2\left(e^{-\frac{\mathcal{Q}_0}{2\kappa}} \mathcal{P}_i, \mathcal{P}_0\right) - M^2\right) \delta\left(C_\kappa^2\left(e^{\frac{\mathcal{P}_0}{2\kappa}} \mathcal{Q}_i, \mathcal{Q}_0\right) - M^2\right)
\end{aligned}$$

In the field commutator

$$\begin{aligned}
[\varphi(\mathbf{x}), \varphi(\mathbf{y})]_{\star\kappa} &= \frac{1}{(2\pi)^3} \int d^4\mathcal{P} \int d^4\mathcal{Q} e^{i(\mathcal{P}_\mu \mathbf{x}^\mu + \mathcal{Q}_\mu \mathbf{y}^\mu)} \cdot \\
&\cdot \left[e^{\frac{3(\mathcal{P}_0 - \mathcal{Q}_0)}{2\kappa}} \mathbf{A} \left(\mathcal{P}_0, e^{-\frac{\mathcal{Q}_0}{2\kappa}} \mathcal{P}_i \right) \mathbf{A} \left(\mathcal{Q}_0, e^{\frac{\mathcal{P}_0}{2\kappa}} \mathcal{Q}_i \right) \cdot \right. \\
&\cdot \delta \left(\mathbf{C}_\kappa^2 \left(e^{\frac{\mathcal{P}_0}{2\kappa}} \mathcal{Q}_i, \mathcal{Q}_0 \right) - \mathbf{M}^2 \right) \delta \left(\mathbf{C}_\kappa^2 \left(e^{-\frac{\mathcal{Q}_0}{2\kappa}} \mathcal{P}_i, \mathcal{P}_0 \right) - \mathbf{M}^2 \right) + \\
&\quad - e^{\frac{-3(\mathcal{P}_0 - \mathcal{Q}_0)}{2\kappa}} \mathbf{A} \left(\mathcal{Q}_0, e^{-\frac{\mathcal{P}_0}{2\kappa}} \mathcal{Q}_i \right) \mathbf{A} \left(\mathcal{P}_0, e^{\frac{\mathcal{Q}_0}{2\kappa}} \mathcal{P}_i \right) \cdot \\
&\left. \cdot \delta \left(\mathbf{C}_\kappa^2 \left(e^{-\frac{\mathcal{P}_0}{2\kappa}} \mathcal{Q}_i, \mathcal{Q}_0 \right) - \mathbf{M}^2 \right) \delta \left(\mathbf{C}_\kappa^2 \left(e^{\frac{\mathcal{Q}_0}{2\kappa}} \mathcal{P}_i, \mathcal{P}_0 \right) - \mathbf{M}^2 \right) \right]
\end{aligned}$$

one can not factorize product of mass-shell deltas. We obtain the κ -deformed product of oscillators which lie on κ -deformed mass-shell. If we employ the κ -deformed oscillator algebra we get

$$[\varphi(\tilde{\mathbf{x}}), \varphi(\tilde{\mathbf{y}})]_{\star\kappa} = \int \mathbf{K}(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}; \mathbf{z}_1, \mathbf{z}_2) \varphi(\mathbf{z}_1) \varphi(\mathbf{z}_2) d\mathbf{z}_1 d\mathbf{z}_2$$

$\mathbf{K}(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}; \mathbf{z}_1, \mathbf{z}_2)$ - numerical kernel

commutator - q-number bilinear in fields φ

6. κ -DEFORMED FIELDS WITH NONSTANDARD STAR PRODUCT AND OFF-SHELL κ -DEFORMED OSCILLATORS

One can introduce modified multiplication of κ -deformed noncommutative fields on κ -Minkowski space

$$\varphi(\hat{\mathbf{x}}) \cdot \varphi(\hat{\mathbf{y}}) \rightarrow \varphi(\hat{\mathbf{x}}) \odot \varphi(\hat{\mathbf{y}})$$

with

$$\begin{aligned} \varphi(\hat{\mathbf{x}}) \odot \varphi(\hat{\mathbf{y}}) &= \frac{1}{(2\pi)^3} \int d^4\mathbf{p} \int d^4\mathbf{q} \mathbf{A}(\mathbf{p}_0, \tilde{\mathbf{p}}) \mathbf{A}(\mathbf{q}_0, \tilde{\mathbf{q}}) \cdot \\ &\cdot \delta \left(\mathbf{C}_{\kappa}^2(\tilde{\mathbf{p}} e^{-\frac{\mathbf{q}_0}{2\kappa}}, \mathbf{p}_0) - \mathbf{M}^2 \right) \delta \left(\mathbf{C}_{\kappa}^2(\tilde{\mathbf{q}} e^{\frac{\mathbf{p}_0}{2\kappa}}, \mathbf{q}_0) - \mathbf{M}^2 \right) \cdot \\ &\cdot \exp(i\mathbf{p}\hat{\mathbf{x}}) \cdot \exp(i\mathbf{q}\hat{\mathbf{y}}) \end{aligned}$$

where we deform the products of mass-shells

$$\begin{aligned} \delta \left(\mathbf{C}_{\kappa}^2(\tilde{\mathbf{p}}, \mathbf{p}_0) - \mathbf{M}^2 \right) \delta \left(\mathbf{C}_{\kappa}^2(\tilde{\mathbf{q}}, \mathbf{q}_0) - \mathbf{M}^2 \right) &\rightarrow \\ \rightarrow \delta \left(\mathbf{C}_{\kappa}^2(\tilde{\mathcal{P}}, \mathbf{p}_0) - \mathbf{M}^2 \right) \delta \left(\mathbf{C}_{\kappa}^2(\tilde{\mathcal{Q}}, \mathbf{q}_0) - \mathbf{M}^2 \right) \end{aligned}$$

with $\tilde{\mathcal{P}} = \tilde{\mathbf{p}} e^{-\frac{\mathbf{q}_0}{2\kappa}}$, $\tilde{\mathcal{Q}} = \tilde{\mathbf{q}} e^{\frac{\mathbf{p}_0}{2\kappa}}$

For relativistic scalar field case we should introduce modified κ -star product

$$\begin{aligned}
& \varphi(\hat{\mathbf{x}}) \odot \varphi(\hat{\mathbf{x}}) \leftrightarrow \varphi(\mathbf{x}) \hat{\star}_\kappa \varphi(\mathbf{x}) = \\
& = \frac{1}{(2\pi)^3} \int d^4\mathbf{p} \int d^4\mathbf{q} \mathbf{A}(\mathbf{p}_0, \tilde{\mathbf{p}}) \mathbf{A}(\mathbf{q}_0, \tilde{\mathbf{q}}) \\
& \cdot \delta \left(\mathbf{C}_\kappa^2(\tilde{\mathbf{p}} e^{-\frac{\mathbf{q}_0}{2\kappa}}, \mathbf{p}_0) - \mathbf{M}^2 \right) \delta \left(\mathbf{C}_\kappa^2(\tilde{\mathbf{q}} e^{\frac{\mathbf{p}_0}{2\kappa}}, \mathbf{q}_0) - \mathbf{M}^2 \right) \cdot \\
& \cdot \exp \left(-i(\mathbf{p}_0 + \mathbf{q}_0) \mathbf{x}^0 + i\Delta_i(\tilde{\mathbf{p}}, \tilde{\mathbf{q}}) \mathbf{x}^i \right)
\end{aligned}$$

After change of variables one gets

$$\begin{aligned}
& \varphi(\mathbf{x}) \hat{\star}_\kappa \varphi(\mathbf{y}) = \\
& = \frac{1}{(2\pi)^3} \int d^4\mathcal{P} \int d^4\mathcal{Q} e^{\frac{3(\mathcal{P}_0 - \mathcal{Q}_0)}{2\kappa}} \mathbf{A} \left(\mathcal{P}_0, e^{-\frac{\mathcal{Q}_0}{2\kappa}} \mathcal{P}_i \right) \mathbf{A} \left(\mathcal{Q}_0, e^{\frac{\mathcal{P}_0}{2\kappa}} \mathcal{Q}_i \right) \\
& \cdot \delta \left(\mathbf{C}_\kappa^2(\tilde{\mathcal{P}}, \mathcal{P}_0) - \mathbf{M}^2 \right) \cdot \delta \left(\mathbf{C}_\kappa^2(\tilde{\mathcal{Q}}, \mathcal{Q}_0) - \mathbf{M}^2 \right) \cdot e^{i(\mathcal{P}\mathbf{x} + \mathcal{Q}\mathbf{y})}
\end{aligned}$$

If we introduce $\mathbf{F}_\kappa^{(2)}(\mathcal{P}_0, \mathcal{Q}_0) = e^{\frac{3(\mathcal{P}_0 - \mathcal{Q}_0)}{2\kappa}}$ and relativistic multiplication formula

$$\mathbf{A}(\mathcal{P}) \circ_\kappa \mathbf{A}(\mathcal{Q}) = \mathbf{F}_\kappa^{(2)}(\mathcal{P}_0, \mathcal{Q}_0) \cdot \mathbf{A}\left(\mathcal{P}_0, e^{-\frac{\mathcal{Q}_0}{2\kappa}} \mathcal{P}_i\right) \mathbf{A}\left(\mathcal{Q}_0, e^{\frac{\mathcal{P}_0}{2\kappa}} \mathcal{Q}_i\right)$$

one can show that

$$\varphi(\mathbf{x}) \hat{\star}_\kappa \varphi(\mathbf{y}) = \varphi(\mathbf{x}) \circ_\kappa \varphi(\mathbf{y})$$

where the oscillators $\mathbf{A}(\mathcal{P}_0, e^{-\frac{\mathcal{Q}_0}{2\kappa}} \mathcal{P}_i)$ satisfy κ -deformed algebra but are off-shell because

$$\mathbf{C}_\kappa^2(\tilde{\mathcal{P}}, \mathcal{P}_0) - \mathbf{M}^2 = 0 \Leftrightarrow \mathcal{P}_0 = \pm \omega_\kappa(\tilde{\mathcal{P}})$$

We define creation and annihilation operators

$$\mathbf{a}(\tilde{\mathbf{p}}) = \mathbf{A}(\mathbf{p}_0 = \omega_\kappa(\tilde{\mathbf{p}}), \tilde{\mathbf{p}}) \rightarrow \text{positive energy}$$

$$\mathbf{a}^\dagger(\tilde{\mathbf{p}}) = \mathbf{A}(\mathbf{p}_0 = -\omega_\kappa(\tilde{\mathbf{p}}), \tilde{\mathbf{p}}) \rightarrow \text{negative energy}$$

The field decomposes into

$$\varphi(\mathbf{x}) = \varphi^{(+)}(\mathbf{x}) + \varphi^{(-)}(\mathbf{x})$$

Now for positive frequency parts

$$[\mathbf{a}_\kappa(\tilde{\mathbf{p}}), \mathbf{a}_\kappa(\tilde{\mathbf{q}})]_{\circ_\kappa} = \mathbf{0}$$

$$\Updownarrow$$

$$[\varphi^{(+)}(\mathbf{x}), \varphi^{(+)}(\mathbf{y})]_{\hat{\mathbf{x}}_\kappa} = [\varphi^{(+)}(\mathbf{x}), \varphi^{(+)}(\mathbf{y})]_{\circ_\kappa} = \mathbf{0}$$

Similarly for negative frequencies

$$[\mathbf{a}_\kappa^\dagger(\tilde{\mathbf{p}}), \mathbf{a}_\kappa^\dagger(\tilde{\mathbf{q}})]_{\circ_\kappa} = \mathbf{0}$$

$$\Updownarrow$$

$$[\varphi^{(-)}(\mathbf{x}), \varphi^{(-)}(\mathbf{y})]_{\hat{\mathbf{x}}_\kappa} = [\varphi^{(-)}(\mathbf{x}), \varphi^{(-)}(\mathbf{y})]_{\circ_\kappa} = \mathbf{0}$$

We postulate

$$[\mathbf{a}_\kappa^\dagger(\tilde{\mathbf{p}}), \mathbf{a}_\kappa(\tilde{\mathbf{q}})]_{\circ_\kappa} = 2\omega_\kappa(\tilde{\mathbf{p}})\delta^{(3)}(\tilde{\mathbf{p}} - \tilde{\mathbf{q}})$$

$$\Updownarrow$$

$$[\varphi(\mathbf{x}), \varphi(\mathbf{y})]_{\hat{\mathbf{x}}_\kappa} = [\varphi(\mathbf{x}), \varphi(\mathbf{y})]_{\circ_\kappa} =$$

$$= -\frac{\mathbf{i}}{(2\pi)^3} \int \frac{\mathbf{d}^3\mathbf{p}}{\Omega_\kappa(\tilde{\mathbf{p}})} \sin(\omega_\kappa(\tilde{\mathbf{p}})(\mathbf{x}_0 - \mathbf{y}_0)) \mathbf{e}^{\mathbf{i}\tilde{\mathbf{p}}(\tilde{\mathbf{x}} - \tilde{\mathbf{y}})} =$$

$$= \frac{1}{\mathbf{i}} \Delta_\kappa(\mathbf{x} - \mathbf{y}; \mathbf{M}^2)$$

$$\Uparrow$$

obtained by replacement $\delta(\mathbf{p}^2 - \mathbf{M}^2) \rightarrow \delta(\mathbf{C}_\kappa(\mathbf{p}_0, \tilde{\mathbf{p}}) - \mathbf{M}^2)$

7. QUANTIZATION RULES κ -DEFORMED QUANTUM FIELDS

$$[\varphi(\mathbf{x}_0, \tilde{\mathbf{x}}), \varphi(\mathbf{x}_0, \tilde{\mathbf{y}})]_{\hat{\star}_\kappa} = \frac{1}{i} \Delta_\kappa(\mathbf{x} - \mathbf{y}; \mathbf{M}^2)|_{\mathbf{x}_0=\mathbf{y}_0} = \mathbf{0}$$

$$[\partial_0 \varphi(\mathbf{x}_0, \tilde{\mathbf{x}}), \varphi(\mathbf{x}_0, \tilde{\mathbf{y}})]_{\hat{\star}_\kappa} = \frac{1}{i} \partial_0 \Delta_\kappa(\mathbf{x} - \mathbf{y}; \mathbf{M}^2)|_{\mathbf{x}_0=\mathbf{y}_0} =$$

$$= \delta_\kappa^{(3)}(\tilde{\mathbf{x}} - \tilde{\mathbf{y}}) = \frac{1}{(2\pi)^3} \int d^3 \tilde{\mathbf{p}} f_\kappa(\tilde{\mathbf{p}}) e^{i\tilde{\mathbf{p}}(\tilde{\mathbf{x}}-\tilde{\mathbf{y}})}$$

$$f_\kappa(\tilde{\mathbf{p}}) = \frac{\omega_\kappa(\tilde{\mathbf{p}})}{\Omega_\kappa(\tilde{\mathbf{p}}^2)} = \frac{2 \operatorname{arcsinh} \left(\frac{\omega_\kappa(\tilde{\mathbf{p}})}{2\kappa} \right)}{\sinh \left(\frac{\omega_\kappa(\tilde{\mathbf{p}})}{\kappa} \right)}$$

We introduce κ -deformed time derivative

$$\partial_0^\kappa = \kappa \sin \left(\frac{\partial_0}{\kappa} \right) = \frac{\kappa}{2i} \left(e^{i\frac{\partial_0}{\kappa}} - e^{-i\frac{\partial_0}{\kappa}} \right)$$

One can check that

$$\partial_0^\kappa \Delta_\kappa(\mathbf{x} - \mathbf{y}; \mathbf{M}^2)|_{\mathbf{x}_0=\mathbf{y}_0} = \delta^{(3)}(\tilde{\mathbf{x}} - \tilde{\mathbf{y}})$$

$\partial_0^\kappa \varphi(\mathbf{x}) \leftarrow \kappa$ - deformed canonical momenta

$\partial_0^\kappa \varphi(\mathbf{x})$ - related with bicovariant time derivative

8. DIFFERENT ALGEBRAIC MODELS OF κ -STATISTICS

a) Class of κ -deformed multiplications

One can introduce

$$[\mathbf{a}_\kappa(\tilde{\mathbf{p}}), \mathbf{a}_\kappa(\tilde{\mathbf{q}})]_\odot = [\mathbf{a}_\kappa^\dagger(\tilde{\mathbf{p}}), \mathbf{a}_\kappa^\dagger(\tilde{\mathbf{q}})]_\odot = \mathbf{0}$$

$$[\mathbf{a}_\kappa^\dagger(\tilde{\mathbf{p}}), \mathbf{a}_\kappa(\tilde{\mathbf{q}})]_\odot = \delta^{(3)}(\tilde{\mathbf{p}} - \tilde{\mathbf{q}})$$

where

$$\mathbf{a}_\kappa(\tilde{\mathbf{p}}) \odot \mathbf{a}_\kappa(\tilde{\mathbf{q}}) = \mathbf{J} \left(\begin{matrix} \mathcal{P}, \mathcal{Q} \\ \mathbf{p}, \mathbf{q} \end{matrix} \right) \hat{\mathbf{A}}(\mathcal{P}_0, \tilde{\mathcal{P}}) \hat{\mathbf{A}}(\mathcal{Q}_0, \tilde{\mathcal{Q}})$$

where if we wish to obtain the c-number commutator function we should have

$$\tilde{\mathcal{P}}(\mathbf{p}, \mathbf{q}) = \tilde{\mathcal{Q}}(\mathbf{q}, \mathbf{p}) e^{-\frac{\mathcal{Q}_0}{\kappa}}$$

$$\mathcal{P}_0(\mathbf{p}, \mathbf{q}) = \mathcal{Q}_0(\mathbf{q}, \mathbf{p})$$

We obtain a freedom of choice of three functions

$$\tilde{\mathcal{P}}(\mathbf{p}, \mathbf{q}) = \tilde{\mathbf{p}}\mathbf{f}(\mathbf{p}, \mathbf{q}) + \tilde{\mathbf{q}}\mathbf{g}(\mathbf{p}, \mathbf{q}) \quad , \quad \mathcal{P}_0(\mathbf{p}, \mathbf{q})$$

b) Deformation via κ -deformed flip operators

$$\tau_0 (\mathbf{a}(\tilde{\mathbf{p}}) \mathbf{a}(\tilde{\mathbf{q}})) = \mathbf{a}(\tilde{\mathbf{q}}) \mathbf{a}(\tilde{\mathbf{p}})$$

↓

↓

$$\tau_\kappa (\hat{\mathbf{A}}(\mathbf{p}_0, \tilde{\mathbf{p}}) \hat{\mathbf{A}}(\mathbf{q}_0, \tilde{\mathbf{q}})) = \tilde{\mathbf{F}}^{(2)}(\mathbf{p}_0, \mathbf{q}_0) \hat{\mathbf{A}}(\mathbf{q}_0, \tilde{\mathbf{q}} e^{-\frac{\mathbf{p}_0}{\kappa}}) \hat{\mathbf{A}}(\mathbf{p}_0, \tilde{\mathbf{p}} e^{\frac{\mathbf{q}_0}{\kappa}})$$

where

$$\tilde{\mathbf{F}}^{(2)}(\mathbf{p}_0, \mathbf{q}_0) = (\mathbf{F}^{(2)}(\mathbf{p}_0, \mathbf{q}_0))^2 = \exp \frac{3(\mathbf{p}_0 - \mathbf{q}_0)}{\kappa}$$

If we wish to obtain the c-number field commutator the choice of τ_κ is unique. It satisfies

$$\tau_\kappa^2 = 1$$

More general κ -deformed flip operators used by Young and Zegers (arXiv: 0711.2206) even for off-shell oscillators do not lead to the c-number commutator function

c) The most general algebra of κ -deformed off-shell oscillators which leads to c-number field commutator depends on arbitrary five functions

9. FINAL REMARKS

We have the following basic alternative:

i) The theory of κ -deformed noncommutative "free" fields is described by the action

$$\int d^4 \hat{\mathbf{x}} \hat{\varphi}(\hat{\mathbf{x}}) \left(\hat{\square}_{\hat{\mathbf{x}}} - \mathbf{M}^2 \right) \hat{\varphi}(\hat{\mathbf{x}}) \leftrightarrow$$

$$\leftrightarrow \int d^4 \mathbf{x} \hat{\varphi}(\mathbf{x}) \star_{\kappa} \left[\mathbf{C}_{\kappa}^2 \left(\frac{1}{\mathbf{i}} \tilde{\nabla}, \frac{1}{\mathbf{i}} \frac{\partial}{\partial \mathbf{x}_0} \right) - \mathbf{M}^2 \right] \hat{\varphi}(\mathbf{x})$$

κ -deformed statistics \Rightarrow q-number field commutator

ii) The theory of κ -deformed noncommutative "free" fields is described by modified action

$$\int d^4 \hat{\mathbf{x}} \hat{\varphi}(\hat{\mathbf{x}}) \odot \left(\hat{\square}_{\hat{\mathbf{x}}} - \mathbf{M}^2 \right) \hat{\varphi}(\hat{\mathbf{x}}) \leftrightarrow$$

$$\leftrightarrow \int d^4 \mathbf{x} \hat{\varphi}(\mathbf{x}) \hat{\star}_{\kappa} \left[\mathbf{C}_{\kappa}^2 \left(\frac{1}{\mathbf{i}} \tilde{\nabla}, \frac{1}{\mathbf{i}} \frac{\partial}{\partial \mathbf{x}_0} \right) - \mathbf{M}^2 \right] \hat{\varphi}(\mathbf{x})$$

κ -deformed statistics \Rightarrow c-number field commutator

Only in framework ii) one can introduce κ -deformed theory with κ -deformed Feymann diagrams !