Deformed particle dynamics from (quantum) gravity?[1]

Jerzy Kowalski-Glikman

Noncommutative deformations of special relativity workshop

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[1] With Artem; with thanks to Laurent, Florian, Catherine, & Bernd

The problem

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DSR cannot be fundamental

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Step 1. Find the formulation of (quantum) gravity, in which this limit can be naturally taken.

Step 2. Derive the effective deformed dynamics of particles.

Outline

1. 3D versus 4D
    - Results in 3D gravity
    - 4D gravity as a constrained BF theory
    - Coupling to particles
2. Perturbative quantization and holography
    - Perturbative quantum gravity
    - The holography
3. Chern–Simons and particles deformation
    - Alekseev–Malkin procedure
    - Deformed vs undeformed
    - The result

Results in 3D gravity

Gravity in 3D can be described by topological field theory (and thus no local degrees of freedom of gravitational field.) One can treat geometry quantum mechanically, and the theory is exactly soluble.

One can couple 3D gravity to matter, treated as **topological defects, carrying charges of an appropriate gauge group.**

Integrating the gravity degrees of freedom out, one gets an effective theory of matter degrees of freedom – a theory of **Doubly Special Relativity** type (with quantum deformed action and Hopf algebra of space-time symmetries.)

Could we do something similar in 4D? YES!

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BF theory as a theory of gravity

The starting point is the *BF*-theory for SO$(4,1)$ group.

The dynamical variables are:
A^{IJ} – so$(4,1)$-connection one-form;
$F^{IJ} = dA^{IJ} + A^{I}{}_{K} \wedge A^{KJ}$ – its curvature
B^{IJ} – so$(4,1)$-valued 2-form; $I, J = 0, \ldots, 4$.

If we add to such BF theory a term that breaks gauge symmetry down to $SO(3,1)$ we get action of General Relativity.

The action for gravity

$$S = \int \left(B^{IJ} \wedge F_{IJ} - \frac{\beta}{2} B^{IJ} \wedge B_{IJ} - \frac{\alpha}{4} B^{IJ} \wedge B^{KL} \epsilon_{IJKL4} \right)$$

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Equivalence with GR

Step 1. Decompose the SO(4, 1) connection

$$A_\mu{}^{a4} = \frac{1}{2\ell}\, e_\mu{}^a, \quad A_\mu{}^{ab} = \omega_\mu{}^{ab}$$

Step 2. Set the values of the parameters

$$\frac{1}{l^2} = \frac{\Lambda}{3}, \quad \alpha = \frac{G\Lambda}{3}\frac{1}{(1+\gamma^2)}, \quad \beta = \frac{G\Lambda}{3}\frac{\gamma}{(1+\gamma^2)}$$

Step 3. Solve for B

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$$S_{GR} = \frac{1}{2G} \int R^{ij}(\omega) \wedge e^k \wedge e^l \epsilon_{ijkl}$$

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- The α parameter is small and all terms are manifestly diff-invariant. Thus we can try perturbations in α, both classical and quantum. In particular quantum perturbation theory will be manifestly diff-invariant.
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Particles as Wilson lines

Matter can arise as breaking the gauge symmetry in a localized way. The gauge degrees of freedom are promoted to dynamical degree of freedom and reproduce the dynamics of a relativistic particle coupled to gravity. This realizes explicitly in 4D the idea that matter (relativistic particles) can arise as a charged (under $SO(4,1)$) topological gravitational defect.

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The simplest possible localized gauge breaking coupling to the gravitational field A is obtained by choosing a worldline P and a fixed element D of the so$(4,1)$ Lie algebra

$$S_P(\mathrm{A}) = -\int d\tau\, \mathrm{Tr}\, \left(\mathrm{D}\, \mathrm{A}_\tau(\tau)\right)$$

$\mathrm{A}_\tau(\tau) \equiv \mathrm{A}_\mu(z(\tau))\, \dot{z}^\mu$ and τ parameterizes the world line $z^\mu(\tau)$.

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de Sitter group acts by conjugation on its algebra; the orbits are labeled by two numbers (m, s): the mass and spin of the particle. For each orbit we choose a fixed representative element of so$(4,1)$ ($T^{IJ} = \gamma^{IJ} \equiv \frac{1}{2}[\gamma^I, \gamma^J]$)

$$\mathrm{D} \equiv m\ell\gamma^0\gamma^4/2 + s\gamma^2\gamma^3/4$$

Lagrangian of a particle in gravitational field is given by an embedding of its worldline $z(\tau)$ and a function $\mathrm{h}(\tau)$ in the Lorentz subgroup, representing Lorentz transformation from the rest frame, in which the Poincaré charges of the particle are described by D to an actual frame, in which the particle has momentum p_a and spin s_{ab}

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In terms of $\mathsf{A}^{\mathsf{h}} = \mathsf{h}^{-1}\mathsf{A}\mathsf{h} + \mathsf{h}^{-1}d\mathsf{h}$, Lagrangian takes the simple form

$$L(z,\mathsf{h};\mathsf{A}) = -\mathrm{Tr}\,\left(\mathsf{D}\mathsf{A}^{\mathsf{h}}_{\tau}(\tau)\right) \qquad S = \int d\tau\, L(z,\mathsf{h};\mathsf{A})$$

or

$$L(z,\mathsf{h};\mathsf{A}) = -\mathrm{Tr}\,(\mathsf{C}\mathsf{A}_{\tau}) + L_1(z,\mathsf{h})$$

with

$$\mathsf{C} \equiv \mathsf{h}\,\mathsf{D}\,\mathsf{h}^{-1} = \frac{\ell}{2} p_a\, \gamma^a \gamma + \frac{1}{4} s_{ab}\, \gamma^{ab}$$

and

$$L_1(z,\mathsf{h}) = -\mathrm{Tr}\,(\mathsf{h}^{-1}\dot{\mathsf{h}}\mathsf{D})$$

Equations of motion of particle(s) in gravitational field reproduce the spin precession equation and Mathisson–Papapetrou equation (in the presence of torsion.)

Gravitational field equations for gravity–particles system result in Einstein–Cartan equations with momentum being a source for curvature, and spin – for torsion.

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The partition function for gravity coupled to particles is formally given by

$$Z = \int \mathcal{D}A\,\mathcal{D}B\,\exp\left(S_{BF} + \frac{\alpha}{4}\,S_{pert}\right) =$$

$$= \int \mathcal{D}A\,\mathcal{D}B\,\left(\sum_{n=0}^{\infty} \frac{1}{n!}\,\left(\frac{\alpha}{4}\right)^n\,S^n_{pert}\right)\exp\left(S_{BF}\right)$$

where
S_{BF} is the BF and particles actions
S_{pert} is the gauge breaking term, treated as perturbation.

This expression provides quantum gravity perturbative theory around topological vacuum (but one must fully specify what the measure $\mathcal{D}A\,\mathcal{D}B$ is.)

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Since we are interested only in the "no gravity limit" we take just the zeroth order of this expansion, which after integrating B field out reads

$$Z^{(0)} = \int \mathcal{D}\mathrm{A}\, \mathcal{D}B \; \exp\left(S_{BF}\right) =$$

$$= \int \mathcal{D}A \; \exp\left(\frac{1}{\beta}\int F^{IJ}\wedge F_{IJ} + \int_P \mathrm{Tr}\,(\mathrm{DA}^h)\right)$$

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The quantum theory defined by $Z^{(0)}$ is holographic in the sense that only the connections A on the boundary and only the particles' paths there, $P \in \partial M$, contribute to path integral.

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To see why $Z^{(0)}$ describes a boundary theory let us write the classical action explicitly, assuming that the boundary has the form of the product of two-sphere S^2_n (with n punctures corresponding to particles) and the real time line.

$$S = \int_{\mathbb{R}} dx^0 \int_{S^2_n} \frac{k}{4\pi} \langle \partial_0 \mathrm{A}_S \wedge \mathrm{A}_S \rangle - \int_{\mathbb{R}} dx^0 \sum_{i=1}^{n} \langle \mathrm{D}_i \, , \, h_i^{-1} \partial_0 h_i \rangle$$

$$+ \int_{\mathcal{M}} dx^0 \left\langle \mathrm{A}_0 \, , \, \frac{k}{2\pi} \mathrm{F}_S \delta(\partial\mathcal{M}) - \sum_{i=1}^{n} \mathrm{D}_i \delta^{(3)}(x - x_{(i)}) d^3x \right\rangle$$

A decomposes into $\mathrm{A} = \mathrm{A}_0 \, dx^0 + \mathrm{A}_S$; F_S is the curvature of A_S; $\delta(\partial\mathcal{M})$ means that we formally enforce the term to belong to $\partial\mathcal{M}$; $k/4\pi = 1/\beta$.

Integrating over A_0 gives the constraint

$$\frac{k}{2\pi}\mathsf{F}_S\delta(\partial\mathcal{M}) - \sum_{i=1}^{n}\mathsf{D}_i\delta^{(3)}(x - x_{(i)})d^3x = 0$$

The path integral is nonzero only if the connection A_S satisfies this constraint.

If the particle path $P \notin \partial\mathcal{M}$ this constraint forces $\mathsf{D}_i = 0$. But since the charge D_i is conserved the path must either fully belong to the boundary, or give zero contribution.

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The conclusion

$Z^{(0)}$ provides quantum Chern–Simons theory with $\mathrm{SO}(4,1)$ gauge group on the boundary manifold of the form $S^2_n \times \mathbb{R}$. The charged punctures of S^2_n correspond to particles, described by $\mathrm{SO}(4,1)$ charges D_i.

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The action

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Following Alekseev & Malkin instead of the action lets consider the associated symplectic form

$$\Omega = \int_{S_n^2} \frac{k}{4\pi} \langle \delta \mathrm{A}_S \wedge \delta \mathrm{A}_S \rangle + \sum_{i=1}^{n} \delta \left\langle \mathrm{D}_i \, , \, h_i^{-1} \delta h_i \right\rangle$$

where A_S is subject to the constraint

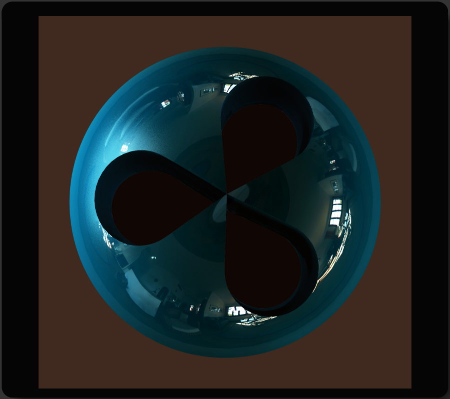
$$\frac{k}{2\pi} \mathrm{F}_S \delta(\partial \mathcal{M}) = \sum_{i=1}^{n} \mathrm{D}_i \delta^{(2)}(x - x_{(i)}) d^2 x$$

We must identify regions where connection is simple.

$$\frac{k}{2\pi} \mathrm{F}_S \delta(\partial\mathcal{M})$$

$$= \sum_{i=1}^{n} \mathrm{D}_i \delta^{(2)}(x - x_{(i)}) d^2x$$

Step 1. Draw the holonomies starting at common point P_0 that surround the particles.



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$$\frac{k}{2\pi} F_S \delta(\partial\mathcal{M})$$

$$= \sum_{i=1}^{n} D_i \delta^{(2)}(x - x_{(i)}) d^2 x$$

Step 2. Remove the plaquettes.

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$$\frac{k}{2\pi}\mathsf{F}_S\delta(\partial\mathcal{M})$$

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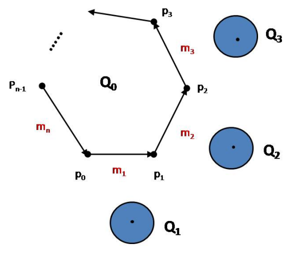
Step 3. Cut off at P_0

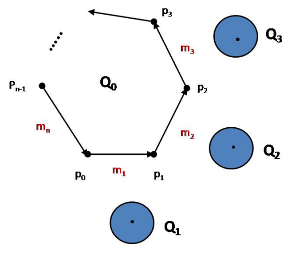


We must identify regions where connection is simple.

$$\frac{k}{2\pi} \mathsf{F}_S \delta(\partial \mathcal{M})$$

$$= \sum_{i=1}^{n} \mathsf{D}_i \delta^{(2)}(x - x_{(i)}) d^2 x$$

And thats what we get
...



On Q_0: $As|_{Q_0} = \gamma_0 ds \gamma_0^{-1}$

On Q_i:

$$As|_{Q_i} = \gamma_i ds \gamma_i^{-1}$$

$$+\frac{1}{k}\gamma_i D_i d\phi_i \gamma_i^{-1}$$

It is easy to see that Q_i satisfies the constraint.

And now, plugging these expressions into symplectic form we find that the symplectic form is built from to blocks, the free and interaction one $\Omega = \Omega_{free} + \Omega_{int}$.

$$\Omega_{free} = \frac{k}{4\pi}\sum_{i=1}^{n}\langle C_i g_i^{-1}\delta g_i C_i^{-1} \wedge g_i^{-1}\delta g_i\rangle$$

where

$C_i = \exp\left(\frac{2\pi}{k} D_i\right)$ is a SO(4, 1) group element carrying information about the particles masses and spins;

$g_i \in SO(4,1)$ carry information about particles momenta and positions

And now, plugging these expressions into symplectic form we find that the symplectic form is built from to blocks, the free and interaction one $\Omega = \Omega_{free} + \Omega_{int}$.

$$\Omega_{free} = \frac{k}{4\pi} \sum_{i=1}^{n} \langle C_i g_i^{-1} \delta g_i C_i^{-1} \wedge g_i^{-1} \delta g_i \rangle$$

where
$C_i = \exp\left(\frac{2\pi}{k} D_i\right)$ is a $SO(4,1)$ group element carrying information about the particles masses and spins;
$g_i \in SO(4,1)$ carry information about particles momenta and positions

$$\Omega_{int} = -\frac{k}{4\pi}\sum_{i=1}^{n}\langle \delta K_i K_i^{-1} \wedge \delta K_{i-1} K_{i-1}^{-1}\rangle$$

where

$K_i = (M_i M_{i-1}\cdots M_1)^{-1}$ are products of holonomies

$M_i = g_i C_i^{-1} g_i^{-1}$.

Undeformed

Deformed

Undeformed	Deformed
One dimensionful parameter ℓ (cosmic length scale)	Two dimensionful parameters ℓ & κ (Planck mass)

Undeformed

One dimensionful parameter ℓ (cosmic length scale)

$$\Omega \sim \langle [D, g^{-1}\delta g] \wedge g^{-1}\delta g \rangle$$

$$D = m\,\ell\,\gamma^{04}$$

Deformed

Two dimensionful parameters ℓ & κ (Planck mass)

$$\Omega \sim \ell^2\kappa^2\, \langle Cg^{-1}\delta g\, C^{-1} \wedge g^{-1}\delta g \rangle$$

$$C = \exp\left(\frac{m}{\ell\kappa^2}\gamma^{04}\right)$$

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Finite ℓ corresponds to finite cosmological constant and the particle on de Sitter space. Minkowski space corresponds to $\ell \to \infty$.

$g \in \mathrm{SO}(4,1)$ and can be decomposed into translational and Lorentz parts (Cartan decomposition):

$$g = \mathcal{T}\,\mathcal{L} \qquad \mathcal{T} = \exp\left(\frac{q^a}{\ell}\,\gamma^{a4}\right)$$

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Deformed

Instead we have to do with group valued momentum

$$\Pi = \mathcal{L}\exp\left(\frac{m}{\ell\kappa^2}\gamma^{04}\right)\mathcal{L}^{-1}$$

which is an element of de Sitter space of momenta.

Undeformed

Deformed

Undeformed

Deformed

then in the limit $\ell \to \infty$

$$\Omega \sim \delta q^a \wedge \delta p_a$$

Undeformed

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Deformed

However, in the limit $\ell \to \infty$ again

$$\Omega \sim \delta q^a \wedge \delta p_a$$

The final result in the deformed case is

$$\Omega = \Omega_{free} + \Omega_{int} = \sum_{i=1}^{n} \left(\delta p^{(i)}{}_a \wedge \delta q^{(i)a} + \frac{1}{\kappa^2} \delta p^{(i)}{}_a \wedge \sum_{j=1}^{i-1} \delta p^{(j)a} \right)$$

So the Lagrangian reads

$$L = \sum_{i=1}^{n} \left(p^{(i)} \cdot \frac{d}{d\tau} q^{(i)} + \lambda^{(i)} \left[p^{(i)2} + m^{(i)2} \right] + \frac{1}{\kappa^2} p^{(i)}{}_a \cdot \sum_{j=1}^{i-1} \frac{d}{d\tau} p^{(j)a} \right)$$

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What went wrong?

Does finite ℓ improve anything?

The free term remain still undeformed (corresponding to particle moving on de Sitter space). We have not been able to calculate the interaction term.

Perhaps we could see the global structure if we change variables before taking the limit (an analog of semidualization)?

But even if so, how to justify this change physically?

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Summary

1. 3D versus 4D
   - Results in 3D gravity
   - 4D gravity as a constrained BF theory
   - Coupling to particles
2. Perturbative quantization and holography
   - Perturbative quantum gravity
   - The holography
3. Chern–Simons and particles deformation
   - Alekseev–Malkin procedure
   - Deformed vs undeformed
   - The result