Quantum (anti)de Sitter deformations: spatial isotropy, non-commutative spacetimes and zero-curvature limit

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Some basics on quantum algebras

Lie bialgebras

• A **Lie bialgebra** (g, δ) is a Lie algebra g endowed with a cocommutator $\delta : g \to g \otimes g$ such that

i) δ is a 1–cocycle, i.e.,

$$\delta([X, Y]) = [\delta(X),\, 1 \otimes Y + Y \otimes 1] + [1 \otimes X + X \otimes 1,\, \delta(Y)], \quad \forall X, Y \in g.$$

ii) The dual map $\delta^* : g^* \otimes g^* \to g^*$ is a Lie bracket on g^*.

• A Lie bialgebra (g, δ) is called a **coboundary bialgebra** if there exists an element $r \in g \otimes g$, **the classical r–matrix**, such that

$$\delta(X) = [1 \otimes X + X \otimes 1,\, r], \qquad \forall X \in g.$$

• All Lie bialgebras associated to semisimple Lie algebras are coboundary (e.g., $so(p, q)$) so all of them come from classical r-matrices.

• Let g be a Lie algebra and let r be an element of $g \wedge g$. The cocomutator $\delta : g \to g \wedge g$ given by

$$\delta(X) = [1 \otimes X + X \otimes 1,\, r], \qquad X \in g,$$

defines a **coboundary Lie bialgebra** $(g, \delta(r))$ if and only if r fulfills the **modified classical Yang–Baxter equation** (YBE)

$$[X \otimes 1 \otimes 1 + 1 \otimes X \otimes 1 + 1 \otimes 1 \otimes X, [[r, r]]\,] = 0, \qquad X \in g,$$

where $[[r, r]]$ is the **Schouten bracket** defined by

$$[[r, r]] := [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}],$$

and, if $r = r^{ij} X_i \otimes X_j$, we have denoted $r_{12} = r^{ij} X_i \otimes X_j \otimes 1$, $r_{13} = r^{ij} X_i \otimes 1 \otimes X_j$ and $r_{23} = r^{ij} 1 \otimes X_i \otimes X_j$.

• When the r–matrix is such that $[[r, r]] = 0$, that is the **classical YBE**, we shall say that $(g, \delta(r))$ is a **non-standard or triangular** Lie bialgebra.

• On the contrary, a solution r of the **modified classical YBE** with $[[r, r]] \neq 0$ gives rise to a **standard or quasitriangular** Lie bialgebra.

If g is the Lie algebra of a Lie group G, the (unique) **Poisson–Lie structure** on $C^\infty(G)$ linked to a fixed bialgebra $(g, \delta(r))$ is given by the Sklyanin bracket

$$\{\Psi, \Phi\} = r^{\alpha\beta} \left(X^L_\alpha \Psi X^L_\beta \Phi - X^R_\alpha \Psi X^R_\beta \Phi \right), \qquad \Psi, \Phi \in C^\infty(G), \tag{1}$$

where X^L_α and X^R_τ are the left and right invariant vector fields of G, respectively.

• **Each Lie bialgebra underlies a quantum deformation of the Lie algebra** (at the first order in the deformation parameter).

• **Classifications of Lie bialgebras provide all possible quantum deformations**.

• This task has been done mainly for lower dimensional Lie algebras: Heisenberg–Weyl h_3 or (1+1) Galilei algebra [1], the 2D Euclidean algebra [2], the harmonic oscillator h_4 algebra [3], the (1+1) extended Galilei algebra [4], the $gl(2)$ algebra [5], the Schrödinger algebra[6] and for higher dimensions, the (3+1) Poincaré algebra [7].

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[1] Kupershmidt B A 1993 *J. Phys. A* **26** L929; Hussin V, Lauzon A and Rideau G 1994 *Lett. Math. Phys.* **31** 159; Ballesteros A, Herranz F J and Parashar P 1997 *J. Phys. A* **30** L149; Kowalczyk E 1997 *Acta Phys. Pol. B* **28** 1893

[2] Sobczyk J 1996 *J. Phys. A* **29** 2887

[3] Hussin V, Lauzon A and Rideau G 1996 *J. Phys. A* **29** 4105; Ballesteros A and Herranz F J 1996 *J. Phys. A* **29** 4307

[4] Opanowicz A 1998 *J. Phys. A* **31** 8387; Ballesteros A, Celeghini E and Herranz F J 2000 *J. Phys. A* **33** 3431

[5] Kupershmidt B A 1994 *J. Phys. A* **27** L47; Ballesteros A, Herranz F J and Parashar P 1999 *J. Phys. A* **32** 2369

[6] Ballesteros A, Herranz F J and Parashar P 2000 *J. Phys. A* **33** 3445

[7] Zakrzewski S 1995 in *Quantum Groups, Formalism and Applications* ed J Lukierski *et al* 433; Podlés P and Woronowicz S L 1996 *Commun. Math. Phys.* **178** 61

Quantum algebras

A **quantum deformation of the Lie algebra** g endowed with a Hopf structure, that is, a **quantum algebra** $U_z(g)$ is an algebra of formal power series in (at least) a deformation parameter z ($q = \mathrm{e}^z$) with coefficients in $U(g)$ (the universal enveloping algebra of g) and with some compatible maps (counit, antipode and coproduct).

• **Deformed commutation rules** which under the limit $z \to 0$ ($q \to 1$) reproduces the Lie brackets of g.

• A **coproduct** $\Delta : U_z(g) \to U_z(g) \otimes U_z(g)$ which is a homomorphism,

$$\text{if} \quad [X, Y] = Z \quad \text{then} \quad \Delta([X, Y]) = [\Delta X, \Delta Y] = \Delta Z$$

and verifies the coassosiativity condition

$$(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta.$$

The first order of Δ gives the cocommutator δ:

$$\Delta = \sum_{l=0}^{\infty} \Delta_{(l)} = \sum_{l=0}^{\infty} z^l \delta_{(l)}, \qquad \delta = \Delta_{(1)} - \sigma \circ \Delta_{(1)}.$$

where $\sigma(X \otimes Y) = Y \otimes X$ is the flip or exchange operator.

Universal quantum R-matrices

Given a quantum algebra $(U_z(g), \Delta)$, if we are able to find an invertible element $\mathcal{R}$ in $U_z(g) \otimes U_z(g)$ such that

$$\mathcal{R}\Delta(X)\mathcal{R}^{-1} = \sigma \circ \Delta(X) \quad \forall\, X \in U_z(g),$$

then this $\mathcal{R}$ will provide a direct construction of the corresponding quantum group $Fun_z(G)$ via the FRT approach[8]. Moreover, if $\mathcal{R}$ fulfills the relations

$$(\Delta \otimes id)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23} \qquad (id \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12} \tag{2}$$

(where, if $\mathcal{R} = \sum_i a_i \otimes b_i$, we denote $\mathcal{R}_{12} \equiv \sum_i a_i \otimes b_i \otimes 1$, $\mathcal{R}_{13} \equiv \sum_i a_i \otimes 1 \otimes b_i$, $\mathcal{R}_{23} \equiv \sum_i 1 \otimes a_i \otimes b_i$), then the element $\mathcal{R}$ is a solution of the **Quantum Yang–Baxter Equation** (QYBE):

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}. \tag{3}$$

In this case, the pair $(U_z(g), \mathcal{R})$ is called a **quasitriangular Hopf algebra**. In particular, if the R-matrix verifies $\sigma \circ \mathcal{R} = \mathcal{R}^{-1}$, we shall say that $(U_z(g), \mathcal{R})$ is a **triangular Hopf algebra**[9].

[8]N.Y. Reshetikhin, L.A. Takhtadzhyan and L.D. Faddeev, *Leningrad Math. J.* **1** (1990) 193.
[9]V.G. Drinfel'd in "Proceedings of the International Congress of Mathematics", (MRSI Berkeley 1986), p. 798.

If we expand $\mathcal{R}$ as a formal power series in the deformation parameter

$$\mathcal{R} = 1 \otimes 1 + z\, r + o(z^2)$$

the fact that $\mathcal{R}$ **is a solution of the QYBE implies that r is a solution of the CYBE**, $[[r, r]] = 0$.

• Such triangular r **always exists** for **non-standard deformations**, and higher orders of the quantum R-matrices can be available from different methods.

• On the contrary, if we are dealing with a **standard deformation** the underlying skewsymmetric r-matrix is **not** a solution of the CYBE (although it is of the modified CYBE) and, therefore, it cannot be the first order of the quantum R-matrix.

• Sometimes this problem can be circumvented by taking into account that any r-matrix, r, generates the same Lie bialgebra as $r' = r + \eta$, with η being any $Ad^{\otimes 2}$–invariant element of $g \otimes g$, this is,

$$[X \otimes 1 + 1 \otimes X, \eta] = 0 \qquad \forall X \in g.$$

For some Lie bialgebras, an η element can be found in such a way that r' is a solution of the cassical YBE, but in general this is not the case.

In that situation we cannot even guarantee the existence of the first order of the solution of the QYBE we are looking for. This is the essential point to explain, for instance, all the difficulties that have been encountered with the (standard) κ-Poincaré in 1+1 and 3+1 dimensions.[10]

Some properties for the quantum deformation (in all orders in the deformation parameters) of both types of Lie bialgebras are:

• **Non-standard or triangular deformations**:
– There always exists a quantum universal R-matrix, so that the (dual) quantum group can be computed by applying the FRT procedure.
– The deformed commutation rules can be written as nondeformed through a nonlinear change of basis, so the deformation only appears in the coproduct (twist).

• **Standard or quasi-triangular deformations**:
– The above properties are not ensured in general.

[10] A. Ballesteros, F.J.H. and C.M. Pereña in "New Symmetries in the Theories of Fundamental Interactions", eds. J. Lukierski and M. Mozrzymas PWN, (1997) 3.

Quasitriangular deformation: κ-Poincaré in (1+1)D

The Lie brackets of the (1+1)D Poincaré algebra, $\mathcal{P}(1+1)$, written in terms of the **boost** K, **time translation** $H = P_0$ and **space translation** $P = P_1$ generators are

$$[K, H] = P \qquad [K, P] = H \qquad [H, P] = 0.$$

The following element is a **standard classical r-matrix** for $\mathcal{P}(1+1)$

$$r = zK \wedge H = z(K \otimes H - H \otimes K),$$

that generates the following **cocommutator**:

$$\delta(P) = 0 \qquad \delta(H) = zH \wedge P \qquad \delta(K) = zK \wedge P.$$

Note that the deformation parameter $z = 1/\kappa$.

This **quantum Poincaré algebra** can be obtained in the form:

$$\begin{aligned} &\Delta(P) = 1 \otimes P + P \otimes 1 \\ &\Delta(H) = 1 \otimes H + H \otimes e^{zP} \quad \Delta(K) = 1 \otimes K + K \otimes e^{zP} \end{aligned}$$

$$[K, H] = \frac{1}{2z}(e^{2zP} - 1) + \frac{z}{2}H^2 \qquad [K, P] = H \qquad [H, P] = 0.$$

In order to study the obtention of a solution of the CYBE starting from the skewsymmetric $r = zK \wedge H$ we first compute the most general $Ad^{\otimes 2}$–invariant element η; it reads

$$\eta = \tau_1(P \otimes P - H \otimes H) + \tau_2(P \otimes H - H \otimes P).$$

Notice that the first term is directly related to the Poincaré Casimir $\mathcal{C} = P^2 - H^2$. Recall that $r' = r + \eta$ gives rise to the same cocommutator δ as r and thus to the same deformation.

If we impose now $r' = r + \eta$ to fulfill the CYBE, $[[r', r']] = 0$, we obtain that z **has to be zero**. Therefore there is **no** $r' \in \mathcal{P}(1+1) \otimes \mathcal{P}(1+1)$ that can serve as the first order to construct the universal R-matrix for this quantum algebra.

• In the very special case of (2+1)D there **does** exist such an r' and the universal R-matrix.[11]

• In (3+1)D, the corresponding standard classical r-matrix **cannot** be transformed by adding a term η in order to become a solution of the CYBE. Recall that the κ-Poincaré group was obtained[12] by means of a Weyl quantization of its associated Poisson–Lie structure.

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[11] E. Celeghini, R. Giachetti, E. Sorace and M. Tarlini, *J. Math. Phys.* **32** (1991) 1159.

[12] P. Maślanka, *J. Phys. A* **26** (1993) L1251; J. Lukierski and H. Ruegg, *Phys. Lett. B* **329** (1994) 189.

Triangular deformation: null-plane Poincaré in (1+1)D

The $\mathcal{P}(1+1)$ can als be expressed in a light-cone or null-plane basis by changing the translations as $P_\pm = H \pm P$; hence the commutation rules read

$$[K, P_+] = P_+ \qquad [K, P_-] = -P_- \qquad [P_+, P_-] = 0.$$

A second possible deformation for $\mathcal{P}(1+1)$ naturally arises in this basis by considering the **non-standard classical r-matrix**

$$r = zK \wedge P_+ = zK \wedge (H + P).$$

The **commutator** is given by

$$\delta(K) = zK \wedge P_+ \qquad \delta(P_+) = 0 \qquad \delta(P_-) = zP_- \wedge P_+.$$

The **quantum null-plane deformation** corresponds to

$$\Delta(P_+) = 1 \otimes P_+ + P_+ \otimes 1,$$
$$\Delta(K) = 1 \otimes K + K \otimes e^{zP_+}, \qquad \Delta(P_-) = 1 \otimes P_- + P_- \otimes e^{zP_+},$$

$$[K, P_+] = \frac{e^{zP_+} - 1}{z}, \qquad [K, P_-] = -P_-, \qquad [P_+, P_-] = 0.$$

The Casimir and the universal quantum R-matrix of this quantum algebra are

$$C_z = \frac{1 - e^{-zP_+}}{z} P_-$$
$$R = \exp\{-zP_+ \otimes K\} \exp\{zK \otimes P_+\}.$$

Some remarks.

• These results can be extended to **higher dimensions**[13].

• In both types of deformations the nondeformed generator, either H or P_+, arises in the coproduct and commutation relations through exponentials.

• Hence **dimensions of the deformation parameter** are inherit from the nondeformed generator since $[zH] = 1$ or $[zP_+] = 1$.

• First order **non-commutative Minkowskian spacetimes**. Let $\hat{x}_0$, $\hat{x}_1$, $\hat{x}_\pm$ the dual coordinates to H, P and $P_\pm$, then

$$\text{Standard:} \quad \delta(H) = zH \wedge P \ \Rightarrow\ [\hat{x}_0, \hat{x}_1] = -z\hat{x}_1.$$

$$\text{Non-standard:} \quad \delta(P_-) = zP_- \wedge P_+ \ \Rightarrow\ [\hat{x}_-, \hat{x}_+] = -z\hat{x}_+.$$

[13] A. Ballesteros, F.J.H., M.A. del Olmo and M. Santander, *Phys. Lett. B* **351** (1995) 137; A. Ballesteros, F.J.H. and C.M. Pereña, *Phys. Lett. B* **391** (1997) 71

Some ideas and motivation

► Some approaches to the problem of unification of Quantum Mechanics and General Relativity have led to arguments that suggest a **modification of Lorentz symmetry at the Planck scale**.

► It has been proposed that rotation/boost transformations between inertial observers might be characterized by **two observer-independent scales.**[14] In addition to the velocity scale, c, one introduces a length (momentum) scale possibly related with the Planck scale l_P.

► There have been proposed different ways to introduce the two observer-independent scales[15] which are being called **"Doubly Special Relativity" theories**.

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[14] G. Amelino-Camelia, Int. J. Mod. Phys. D **11** (2002) 35; Phys. Lett. B **510** (2001) 255; gr-qc/0106004. In Karpacz 2001, New developments in fundamental interaction theories 137.
N. R. Bruno, G. Amelino-Camelia and J. Kowalski-Glikman, Phys. Lett. B **522** (2001) 133
N. R. Bruno, Phys. Lett. B **547** (2002) 109

[15] J. Magueijo and L. Smolin, Phys. Rev. Lett. 88 (2002) 190403.
J. Kowalski-Glikman, arXiv:hep-th/0209264.
J. Lukierski and A. Nowicki: Int. J. Mod. Phys. A **18** (2003) 7.
L. Freidel, J. Kowalski-Glikman and L. Smolin, Phys. Rev. D **69**, 044001 (2004)
F. Girelli, E. R. Livine and D. Oriti, Nucl. Phys. B **708**, 411 (2005)

▶ The **kappa-Poincaré algebra**[16] has arised as an appropriate structure supporting such DSR approaches.

▶ It has been shown in[17] that the perturbations of the Kodama state, *i.e.* the **vacuum state of a Chern-Simons quantum gravity theory** with cosmological constant Λ, at least in **(2+1)D**, are invariant under transformations that close the **q-(anti)de Sitter algebra**. The low energy regime/zero-curvature limit of such a q-(anti)de Sitter ends being kappa-Poincaré.

▶ If Lorentz symmetry has to be modified at the Planck scale and a non-zero curvature/cosmological constant seems to be physically relevant, it is necessary to understand the role that curvature could play in these scenarios: search for higher dimensional **quantum (anti)de Sitter algebras**.

[16] J. Lukierski, A. Nowicki, H. Ruegg, and V.N. Tolstoy, Phys. Lett. B264 (1991) 331.
S. Giller, P. Kosinski, J. Kunz, M. Majewski and P. Maslanka: Phys. Lett. B **286** (1992) 57.
J. Lukierski, H. Ruegg and A. Nowicky: Phys. Lett. B **293** (1992) 344.
J. Lukierski, H. Ruegg and W. J. Zakrzewski, Annals Phys. **243** (1995) 90

[17] G. Amelino-Camelia, L. Smolin, A. Starodubtsev, Class. Quant. Grav. 21 (2004) 3095.

(2+1)D quantum (anti)de Sitter algebras

The $(2+1)$D Lie algebras of the three relativistic spacetimes of constant curvature ω are denoted $so_\omega(2,2)$. In terms of the generators of rotations, time and space translations, and boosts, $\{J, P_0, P_i, K_i\}$, the commutation relations of $so_\omega(2,2)$ read

$$\begin{array}{lll} [J,P_i]=\epsilon_{ij}P_j, & [J,K_i]=\epsilon_{ij}K_j, & [J,P_0]=0, \\ [P_i,K_j]=-\delta_{ij}P_0, & [P_0,K_i]=-P_i, & [K_1,K_2]=-J, \\ [P_0,P_i]=\omega K_i, & [P_1,P_2]=-\omega J, & \end{array}$$

where $i,j=1,2$ and ϵ_{ij} is a skewsymmetric tensor such that $\epsilon_{12}=1$.

According to the sign of ω we find that these Lie brackets reproduce:

• The **AdS** algebra, $so(2,2)$, when $\omega=+1/R^2>0$ and where R is the AdS radius.

• The **dS** algebra, $so(3,1)$, when $\omega=-1/R^2<0$ and where R is the dS radius.

• And the **Poincaré** algebra, $iso(2,1)$, when $\omega=0$; this case also corresponds to the flat limit/contraction $R\to\infty$ such that $so(2,2)\to iso(2,1)\leftarrow so(3,1)$.

The **Hopf structure** of a standard quantum deformation $U_z(so_\omega(2,2))$ introduced in[18] reads

$$\Delta(P_0) = 1 \otimes P_0 + P_0 \otimes 1, \qquad \Delta(J) = 1 \otimes J + J \otimes 1,$$

$$\begin{aligned}
\Delta(P_i) &= \mathrm{e}^{-\frac{z}{2}P_0} \cosh(\tfrac{z}{2}\rho J) \otimes P_i + P_i \otimes \mathrm{e}^{\frac{z}{2}P_0} \cosh(\tfrac{z}{2}\rho J) \\
&\quad + \rho\, \mathrm{e}^{-\frac{z}{2}P_0} \sinh(\tfrac{z}{2}\rho J) \otimes \epsilon_{ij} K_j - \rho\, \epsilon_{ij} K_j \otimes \mathrm{e}^{\frac{z}{2}P_0} \sinh(\tfrac{z}{2}\rho J), \\
\Delta(K_i) &= \mathrm{e}^{-\frac{z}{2}P_0} \cosh(\tfrac{z}{2}\rho J) \otimes K_i + K_i \otimes \mathrm{e}^{\frac{z}{2}P_0} \cosh(\tfrac{z}{2}\rho J) \\
&\quad - \mathrm{e}^{-\frac{z}{2}P_0} \frac{\sinh(\frac{z}{2}\rho J)}{\rho} \otimes \epsilon_{ij} P_j + \epsilon_{ij} P_j \otimes \mathrm{e}^{\frac{z}{2}P_0} \frac{\sinh(\frac{z}{2}\rho J)}{\rho},
\end{aligned}$$

$$[J, P_i] = \epsilon_{ij} P_j, \qquad [J, K_i] = \epsilon_{ij} K_j, \qquad [J, P_0] = 0,$$

$$[P_i, K_j] = -\delta_{ij} \frac{\sinh(z P_0)}{z} \cosh(z\rho J), \qquad [P_0, K_i] = -P_i,$$

$$[K_1, K_2] = -\cosh(z P_0) \frac{\sinh(z\rho J)}{z\rho}, \qquad [P_0, P_i] = \omega K_i,$$

$$[P_1, P_2] = -\omega \cosh(z P_0) \frac{\sinh(z\rho J)}{z\rho},$$

where we have written $\omega = \rho^2$

[18] A. Ballesteros, F.J.H., M.A. del Olmo, and M. Santander, J. Phys. A **27** (1994) 1283.

This is exactly the q-(anti)de Sitter algebra consided in the Chern-Simons quantum gravity theory!

The **first-order deformation** is provided by the Lie brackets of $so_\omega(2,2)$, classical r-matrix

$$r = z(K_1 \wedge P_1 + K_2 \wedge P_2),$$

and cocommutator

$$\begin{aligned} &\delta(P_0) = 0, \qquad \delta(J) = 0, \\ &\delta(P_i) = z(P_i \wedge P_0 - \omega \epsilon_{ij} K_j \wedge J), \\ &\delta(K_i) = z(K_i \wedge P_0 + \epsilon_{ij} P_j \wedge J). \end{aligned}$$

By duality we find that the first-order **non-commutative AdS, Minkowskian and dS spacetimes** are simultaneously defined by the $(2+1)$D κ-Minkowski space

$$[\hat{x}_0, \hat{x}_i] = -z\hat{x}_i, \qquad [\hat{x}_1, \hat{x}_2] = 0,$$

where x_μ, are the non-commutative group coordinates dual to P_μ.

When higher orders in the quantum coordinates are considered the resulting non-commutative spaces generalize the κ-Minkowski one since corrections depending on the curvature appear.

SUMMARY

1. **Some possible deformed (anti)de Sitter kinematics**
2. **Dimensions of the deformation parameters**
3. **Non-commutative (anti)de Sitter spacetimes**
4. **Spatial isotropy**
   - **4.1. Zero-curvature limit: deformed Poincaré symmetries**
   - **4.2. Non-relativistic limit: deformed Newtonian and Galilean symmetries**
5. **Other possible (anti)de Sitter deformations**
6. **Concluding remarks**

1. Some possible deformed (anti)de Sitter kinematics

▶ **(3+1)D (anti)de Sitter Lie algebras**: 10 generators of time and space translations P_0 and P_i, boosts K_i and rotations J_i $(i = 1, 2, 3)$. Commutation rules of $so_\omega(3, 2)$:

$$
\begin{array}{lll}
[J_i, J_j] = \varepsilon_{ijk} J_k, & [J_i, P_j] = \varepsilon_{ijk} P_k, & [J_i, K_j] = \varepsilon_{ijk} K_k, \\
[P_i, P_j] = -\dfrac{\omega}{c^2}\, \varepsilon_{ijk} J_k, & [P_i, K_j] = -\dfrac{1}{c^2}\, \delta_{ij} P_0, & [K_i, K_j] = -\dfrac{1}{c^2}\, \varepsilon_{ijk} J_k, \\
[P_0, P_i] = \omega K_i, & [P_0, K_i] = -P_i, & [P_0, J_i] = 0.
\end{array}
$$

• $\omega = 6\Lambda$ is the **curvature** (Λ **cosmological constant**) and c is the **speed of light**.

• If R is the (anti)de Sitter radius, $so_\omega(3, 2)$ reduce to the **anti-de Sitter** algebra $so(3, 2)$ when $\omega = +1/R^2$, and to the **de Sitter** one $so(4, 1)$ when $\omega = -1/R^2$.

• The limit $\omega \to 0$ $(R \to \infty)$ in $so_\omega(3, 2)$ leads to the **Poincaré** algebra.

• $c \to \infty$ corresponds to the non-relativistic contraction giving rise to **Galilei** $(\omega = 0)$ and **Newton–Hooke** algebras $(\omega \neq 0)$.

▶ **(3+1)D (anti)de Sitter Lie bialgebras**: *Any* possible quantum deformation of $so_\omega(3,2)$ is provided by a **classical r-matrix**, whose most general form depends on **45 deformation parameters** r^{ij} (Y_i are the generators of $so_\omega(3,2)$):

$$r = r^{ij} Y_i \wedge Y_j = r^{ij}(Y_i \otimes Y_j - Y_j \otimes Y_i).$$

Quantum (anti)de Sitter algebras are determined, at the first-order in all the deformation parameters r^{ij}, by the **cocommutator** δ coming from the r-matrix through

$$\delta(Y_i) = [Y_i \otimes 1 + 1 \otimes Y_i, r] = f_i^{jk} Y_j \wedge Y_k.$$

▶ **Assumptions**:

(1) We require that, at least, one **deformation parameter** plays the role of a **fundamental energy/Planck length**; the energy/time translation generator P_0 must be:

$$\delta(P_0) = 0, \qquad \Delta(P_0) = P_0 \otimes 1 + 1 \otimes P_0.$$

- Dimensions of the resulting deformation parameters are inherited from P_0.
- Uncertainty/dispersion relations could further be fitted through corrections on P_0.
- Under $\delta(P_0) = 0$, the initial 45 deformation parameters are reduced to **15** ones.

(2) By taking into account known (2+1)D quantum (anti)de Sitter algebras[19], we impose that another generator which *does* commute with P_0 remains **non-deformed**: one of the three **rotation** generators, say J_3.

- **Spatial isotropy** seems to be **broken** from the very beginning in our approach!
- The second constraint $\delta(J_3) = 0$ leaves a five-parametric candidate r-matrix

$$
\begin{aligned}
r = z_1(K_1 \wedge P_1 + K_2 \wedge P_2) + z_2(P_1 \wedge P_2 + \omega K_1 \wedge K_2) \\
+ z_3 P_0 \wedge J_3 + z_4 K_3 \wedge P_3 + z_5 J_1 \wedge J_2,
\end{aligned}
$$

where z_i $(i = 1, \ldots, 5)$ are **5 deformation parameters** $(z_i = 1/\kappa_i = \ln q_i)$.

▶ **Modified classical Yang–Baxter equation**: We find five equations:

$$
\begin{aligned}
& z_1 z_2 = 0, \qquad c^2(z_1 - z_4)z_5 - \omega z_2 z_4 = 0, \\
& c^2 z_2 z_5 + z_1(z_1 - z_4) + \omega z_2^2 = 0, \\
& c^2 z_2 z_5 - z_4(z_1 - z_4) = 0, \\
& c^2 z_5^2 + \omega \left(z_2 z_5 - \frac{1}{c^2} z_1 z_4 \right) = 0,
\end{aligned}
$$

[19] A. Ballesteros, F.J.H., M.A. del Olmo, M. Santander, J. Phys. A 27 (1994) 1283; J. Phys. A 28 (1995) 941. G. Amelino-Camelia, L. Smolin, A. Starodubtsev, Class. Quant. Grav. 21 (2004) 3095.

which lead to **two families** of **two-parametric (anti)de Sitter Lie bialgebras**:

$$r_{z_1,z_3} = z_1 \left(K_1 \wedge P_1 + K_2 \wedge P_2 + K_3 \wedge P_3 \pm \frac{\sqrt{\omega}}{c^2} J_1 \wedge J_2 \right) + z_3 P_0 \wedge J_3,$$
$$r_{z_2,z_3} = z_2 \left(P_1 \wedge P_2 + \omega K_1 \wedge K_2 - \frac{\omega}{c^2} J_1 \wedge J_2 \pm \sqrt{\omega} P_3 \wedge K_3 \right) + z_3 P_0 \wedge J_3.$$

▶ **The Schouten bracket** of r_{z_l,z_3} $(l = 1, 2)$ reads

$$[[r_{z_l,z_3}, r_{z_l,z_3}]] = A_l \left(\frac{\omega}{c^2} J_1 \wedge J_2 \wedge J_3 - \frac{1}{2} \varepsilon_{ijk} \left(\omega J_i \wedge K_j \wedge K_k + J_i \wedge P_j \wedge P_k \right) \right)$$
$$+ A_l \sum_{i=1}^{3} K_i \wedge P_i \wedge P_0, \qquad \text{where} \quad A_1 = \frac{z_1^2}{c^2}, \quad A_2 = \frac{z_2^2 \omega}{c^2}.$$

• If $z_1 \neq 0$ and $z_2 \neq 0$ both r-matrices would give rise to **standard or quasitriangular** quantum (anti)de Sitter algebras.

• If if $z_1 = 0$ and $z_2 = 0$, both r-matrices reduce to $r_{z_3} = z_3 P_0 \wedge J_3$, which is a solution of the classical Yang–Baxter equation, $[[r_{z_3}, r_{z_3}]] = 0$, thus providing a **triangular or non-standard** deformation supported by a Reshetikhin twist.

Summing up,

• Each two-parametric bialgebra can be decomposed as a sum of one **standard** and another **non-standard** term:

$$r_{z_l,z_3} = r_{z_l} + r_{z_3}, \qquad \delta_{z_l,z_3} = \delta_{z_l} + \delta_{z_3}, \qquad l = 1, 2.$$

• The bialgebra associated to r_{z_1} was already obtained by other methods. [20] The construction of the two-parametric quantum (anti)de Sitter algebras (in all orders in z_l and z_3) is an **open problem**.

• Once a one-parametric quantum algebra is obtained for the "pure" standard component, $U_{z_l}(so_\omega(3,2))$, with coproduct Δ_{z_l} and deformed commutation rules $[Y_i, Y_j]_{z_l}$, the complete two-parametric deformation would be provided by the following **twisting element** as:

$$\Delta_{z_l,z_3} = \mathcal{F}_{z_3} \Delta_{z_l} \mathcal{F}_{z_3}^{-1}, \qquad \mathcal{F}_{z_3} = \exp\{-z_3 P_0 \otimes J_3\}, \qquad [Y_i, Y_j]_{z_l,z_3} \equiv [Y_i, Y_j]_{z_l}.$$

We display each (anti)de Sitter bialgebra, where hereafter capital **Latin indices** run as $I, J = 1, 2$.

[20] A. Ballesteros, N.A. Gromov, F.J.H., M.A. del Olmo, M. Santander, J. Math. Phys. 36 (1995) 5916.

Standard 1: $r_{z_1} = z_1 \left(K_1 \wedge P_1 + K_2 \wedge P_2 + K_3 \wedge P_3 \pm \frac{\sqrt{\omega}}{c^2} J_1 \wedge J_2 \right)$

$$\delta_{z_1}(P_I) = \frac{z_1}{c^2} \left(P_I \wedge P_0 - \omega \varepsilon_{Ijk} J_j \wedge K_k \pm \sqrt{\omega}\, J_I \wedge P_3 \right)$$

$$\delta_{z_1}(P_3) = \frac{z_1}{c^2} \left(P_3 \wedge P_0 - \omega \varepsilon_{3jk} J_j \wedge K_k \mp \sqrt{\omega}\, (J_1 \wedge P_1 + J_2 \wedge P_2) \right)$$

$$\delta_{z_1}(K_I) = \frac{z_1}{c^2} \left(K_I \wedge P_0 + \varepsilon_{Ijk} J_j \wedge P_k \pm \sqrt{\omega}\, J_I \wedge K_3 \right)$$

$$\delta_{z_1}(K_3) = \frac{z_1}{c^2} \left(K_3 \wedge P_0 + \varepsilon_{3jk} J_j \wedge P_k \mp \sqrt{\omega}\, (J_1 \wedge K_1 + J_2 \wedge K_2) \right)$$

$$\delta_{z_1}(J_I) = \pm \frac{z_1}{c^2} \sqrt{\omega}\, J_I \wedge J_3$$

Standard 2: $r_{z_2} = z_2 \left(P_1 \wedge P_2 + \omega K_1 \wedge K_2 - \frac{\omega}{c^2} J_1 \wedge J_2 \pm \sqrt{\omega} P_3 \wedge K_3 \right)$

$$\delta_{z_2}(P_I) = \frac{z_2\, \omega}{c^2} \left(J_3 \wedge P_I - J_I \wedge P_3 + \varepsilon_{IJ3} K_J \wedge P_0 \pm \sqrt{\omega}\, \varepsilon_{IJ3} J_J \wedge K_3 \right)$$

$$\delta_{z_2}(K_I) = \frac{z_2}{c^2} \left(\omega J_3 \wedge K_I - \omega J_I \wedge K_3 - \varepsilon_{IJ3} P_J \wedge P_0 \mp \sqrt{\omega}\, \varepsilon_{IJ3} J_J \wedge P_3 \right)$$

$$\delta_{z_2}(P_3) = \pm \frac{z_2 \sqrt{\omega}}{c^2} P_0 \wedge P_3 \qquad \delta_{z_2}(K_3) = \pm \frac{z_2 \sqrt{\omega}}{c^2} P_0 \wedge K_3$$

$$\delta_{z_2}(J_I) = z_2 \left(\frac{\omega}{c^2} J_3 \wedge J_I + P_I \wedge P_3 + \omega K_I \wedge K_3 \pm \sqrt{\omega}\, \varepsilon_{IJ3} (K_J \wedge P_3 - P_J \wedge K_3) \right)$$

Non-standard: $r_{z_3} = z_3 P_0 \wedge J_3$

$$\delta_{z_3}(P_i) = z_3 (\omega J_3 \wedge K_i + \varepsilon_{ij3} P_j \wedge P_0)$$

$$\delta_{z_3}(K_i) = -z_3 (J_3 \wedge P_i - \varepsilon_{ij3} K_j \wedge P_0)$$

$$\delta_{z_3}(J_I) = z_3 \varepsilon_{IJ3} J_J \wedge P_0$$

2. Dimensions of the deformation parameters

• **Dimensional analysis** of the above bialgebras shows that the three deformation parameters z_i have the following dimensions coming from P_0,

$$[z_1] = c^2[P_0]^{-1}, \qquad [z_2] = c^2[P_0]^{-2}, \qquad [z_3] = [P_0]^{-1},$$

provided that

$$[P_0] = \sqrt{\omega} = \sqrt{6\Lambda}.$$

• These dimensionful deformation parameters could play the role of **fundamental scales**.

• Since, such kinds of symmetries are expected to appear at the Planck-energy regime, these could be related to the **Planck length** l_{P} as

$$z_1 \simeq l_{\mathrm{P}}, \qquad z_2 \simeq l_{\mathrm{P}}^2, \qquad z_3 \simeq l_{\mathrm{P}}$$

by considering units with $c = \hbar = 1$.

Next, we can take contact with some **quantum gravity models**[21] by rescaling the deformation parameters as

$$z_l = \frac{z'_l}{(\sqrt{\omega})^l}, \qquad z_3 = \frac{z'_3}{\sqrt{\omega}}, \qquad l = 1, 2.$$

Therefore we can set

$$\ln q_i = z'_i = f_i(l_{\rm P}\sqrt{\omega}) = f_i(l_{\rm P}\sqrt{6\Lambda}), \qquad i = 1, 2, 3,$$

where the functions f_i could further be fixed according to the particular dynamical quantum gravity model under consideration.

The latter form of the deformation parameters will be useful when we will study the **low energy and zero-curvature limits** of the (anti)de Sitter deformations.

We stress that the solutions here obtained enable the possibility of working with two possible quantum (anti)de Sitter deformations, each of them with **two observer-independent scales** besides c.

[21] G. Amelino-Camelia, L. Smolin, A. Starodubtsev, Class. Quant. Grav. 21 (2004) 3095. L. Smolin, hep-th/0501091.

3. Non-commutative (anti)de Sitter spacetimes

If $\hat{y}^i$ denotes the **non-commutative coordinate** dual to the generator Y_i such that

$$\langle \hat{y}^i | Y_j \rangle = \delta^i_j,$$

Lie bialgebra duality[22] provides the **first-order quantum group** dual to the chosen bialgebra

$$\delta(Y_i) = f_i^{jk} Y_j \wedge Y_k \quad \Rightarrow \quad [\hat{y}^j, \hat{y}^k] = f_i^{jk} \hat{y}^i.$$

• Let $\{\hat{\theta}_i, \hat{x}_0, \hat{x}_i, \hat{\xi}_i\}$ be the **non-commutative coordinates** dual to $\{J_i, P_0, P_i, K_i\}$.

• The **non-commutative (anti)de Sitter spaces** associated to the two families of bialgebras arise as the commutation rules involving the quantum coordinates $\hat{x}_\mu$:

$$r_{z_1,z_3}: \qquad [\hat{x}_0, \hat{x}_i] = -\frac{z_1}{c^2}\,\hat{x}_i + z_3\,\varepsilon_{ij3}\,\hat{x}_j, \qquad [\hat{x}_i, \hat{x}_j] = 0.$$

$$r_{z_2,z_3}: \qquad [\hat{x}_0, \hat{x}_I] = -\frac{z_2}{c^2}\,\varepsilon_{IJ3}\,\hat{\xi}_J + z_3\,\varepsilon_{IJ3}\,\hat{x}_J, \quad [\hat{x}_0, \hat{x}_3] = \pm\frac{z_2}{c^2}\sqrt{\omega}\,\hat{x}_3,$$

$$[\hat{x}_I, \hat{x}_3] = z_2\hat{\theta}_I, \qquad [\hat{x}_1, \hat{x}_2] = 0.$$

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[22] V. Chari, A. Pressley, A Guide to Quantum Groups, Cambridge Univ. Press, Cambridge, 1994.
S. Majid, Foundations of Quantum Group Theory, Cambridge Univ. Press, Cambridge, 1995.

• The **kappa-Minkowski space**[23] appears within the first family when $z_3 = 0$ and such a first-order structure is simultaneously shared by the (anti)de Sitter and Poincaré cases as there is no dependence on the curvature ω.

$$r_{z_1,z_3=0}: \qquad [\hat{x}_0, \hat{x}_i] = -\frac{z_1}{c^2}\,\hat{x}_i, \qquad [\hat{x}_i, \hat{x}_j] = 0.$$

However, further **corrections** depending on the **curvature** ω can be expected to appear in the complete quantum (anti)de Sitter spaces similarly to what happens in the $(2+1)$D case[24]

$$[\hat{x}_0, \hat{x}_1] = -\frac{z_1}{c^2}\,\frac{\tanh(\sqrt{\omega}\hat{x}_1)}{\sqrt{\omega}\cosh^2(\sqrt{\omega}\hat{x}_2)}, \quad [\hat{x}_0, \hat{x}_2] = -\frac{z_1}{c^2}\,\frac{\tanh(\sqrt{\omega}\hat{x}_2)}{\sqrt{\omega}}, \quad [\hat{x}_1, \hat{x}_2] = 0.$$

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[23] P. Maslanka, J. Phys. A 26 (1993) L1251
S. Majid, H. Ruegg, Phys. Lett. B 334 (1994) 348
S. Zakrzewski, J. Phys. A 27 (1994) 2075
J. Lukierski, H. Ruegg, Phys. Lett. B 329 (1994) 189.
J. Lukierski, A. Nowicki, W.J. Zakrzewski, Ann. Phys. 243 (1995) 90

[24] F.J.H., A. Ballesteros, N.R. Bruno, in Symmetry Methods in Physics, C. Burdik, O. Navrátil, S. Posta (Eds.), JINR, Dubna, 2004; hep-th/0409295

4. Spatial isotropy

So far we have obtained a classification of the possible deformations of the (anti)de Sitter algebras by requiring P_0 to remain primitive together with the additional condition to keep one of the three rotation generators primitive as well.

Nevertheless, **neither the algebra nor the coalgebra have a symmetrical form in the three spatial directions**.

Since we are describing a constant-curvature algebra of symmetries, even if deformed, one could expect that the deformation would not cause a breaking in the spatial isotropy. This is at the origin of a conceptual problem which makes the proposals for (3+1)D quantum (anti)de Sitter algebras less appealing in view of its applications at the Planck scale physics.

Recall that this problem does not appear in (2+1)D deformations.

Let us show how it is possible to restore spatial isotropy, at least for the family of solutions r_{z_1,z_3}.

Let us consider the following **invertible change of basis** where $\widetilde{Y}_i$ and Y_i denote the new and the original generators of $so(3,2)_\omega$:

$$
\begin{aligned}
\widetilde{Y}_1 &= -\frac{2}{\sqrt{6}} Y_2 + \frac{1}{\sqrt{3}} Y_3, & Y_1 &= \frac{1}{\sqrt{2}}\left(\widetilde{Y}_2 - \widetilde{Y}_3\right),\\
\widetilde{Y}_2 &= \frac{1}{\sqrt{2}} Y_1 + \frac{1}{\sqrt{6}} Y_2 + \frac{1}{\sqrt{3}} Y_3, & Y_2 &= \frac{1}{\sqrt{6}}\left(-2\widetilde{Y}_1 + \widetilde{Y}_2 + \widetilde{Y}_3\right),\\
\widetilde{Y}_3 &= -\frac{1}{\sqrt{2}} Y_1 + \frac{1}{\sqrt{6}} Y_2 + \frac{1}{\sqrt{3}} Y_3, & Y_3 &= \frac{1}{\sqrt{3}}\left(\widetilde{Y}_1 + \widetilde{Y}_2 + \widetilde{Y}_3\right),\\
&\text{for}\quad Y_i \in \{J_i, P_i, K_i\}, & \widetilde{P}_0 &= P_0.
\end{aligned}
$$

The **commutation relations** of $so(3,2)_\omega$ **remain** formally **the same** in this new basis.

When the change of basis is applied to the classical r-matrix r_{z_1,z_3} and to its corresponding cocommutator, we obtain the (anti)de Sitter bialgebra now expressed in a completely symmetric form.

We denote the transformed deformation parameter $\tilde{z}_3$ as

$$\tilde{z}_3 = z_3/\sqrt{3}.$$

$$r_{z_1,\tilde{z}_3} = z_1 \left(\tilde{K}_1 \wedge \tilde{P}_1 + \tilde{K}_2 \wedge \tilde{P}_2 + \tilde{K}_3 \wedge \tilde{P}_3 \pm \frac{\sqrt{\omega}}{\sqrt{3}\, c^2} \left(\tilde{J}_1 \wedge \tilde{J}_2 + \tilde{J}_2 \wedge \tilde{J}_3 + \tilde{J}_3 \wedge \tilde{J}_1 \right) \right) + \tilde{z}_3 \tilde{P}_0 \wedge \left(\tilde{J}_1 + \tilde{J}_2 + \tilde{J}_3 \right),$$

$$\delta_{z_1,\tilde{z}_3}(\tilde{P}_0) = 0,$$

$$\delta_{z_1,\tilde{z}_3}(\tilde{P}_i) = \frac{z_1}{c^2} \left(\tilde{P}_i \wedge \tilde{P}_0 - \omega \varepsilon_{ijk} \tilde{J}_j \wedge \tilde{K}_k \pm \sqrt{\frac{\omega}{3}} \left(\tilde{J}_i \wedge (\tilde{P}_1 + \tilde{P}_2 + \tilde{P}_3) - \sum_{s=1}^{3} \tilde{J}_s \wedge \tilde{P}_s \right) \right) + \tilde{z}_3 \left(\omega (\tilde{J}_1 + \tilde{J}_2 + \tilde{J}_3) \wedge \tilde{K}_i + \varepsilon_{ijk} \tilde{P}_j \wedge \tilde{P}_0 \right),$$

$$\delta_{z_1,\tilde{z}_3}(\tilde{K}_i) = \frac{z_1}{c^2} \left(\tilde{K}_i \wedge \tilde{P}_0 + \varepsilon_{ijk} \tilde{J}_j \wedge \tilde{P}_k \pm \sqrt{\frac{\omega}{3}} \left(\tilde{J}_i \wedge (\tilde{K}_1 + \tilde{K}_2 + \tilde{K}_3) - \sum_{s=1}^{3} \tilde{J}_s \wedge \tilde{K}_s \right) \right) - \tilde{z}_3 \left((\tilde{J}_1 + \tilde{J}_2 + \tilde{J}_3) \wedge \tilde{P}_i - \varepsilon_{ijk} \tilde{K}_j \wedge \tilde{P}_0 \right),$$

$$\delta_{z_1,\tilde{z}_3}(\tilde{J}_i) = \pm \frac{z_1}{c^2} \sqrt{\frac{\omega}{3}}\, \tilde{J}_i \wedge (\tilde{J}_1 + \tilde{J}_2 + \tilde{J}_3) + \tilde{z}_3 \varepsilon_{ijk} \tilde{J}_j \wedge \tilde{P}_0,$$

Then the associated **non-commutative (anti)de Sitter spaces** turns out to be also symmetric

$$[\hat{X}_0, \hat{X}_i] = -\frac{z_1}{c^2}\,\hat{X}_i + \tilde{z}_3\,\varepsilon_{ijk}\,\hat{X}_j, \qquad [\hat{X}_i, \hat{X}_j] = 0,$$

where $\hat{X}_\mu$ is the quantum coordinate dual to the new momentum $\tilde{P}_\mu$.

In fact, the non-commutative spacetimes in terms of $\hat{x}_\mu$ and $\hat{X}_\mu$ are related through the **change of quantum coordinates**:

$$\begin{aligned}
\hat{X}_0 &= \hat{x}_0,\\
\hat{X}_1 &= -\frac{2}{\sqrt{6}}\,\hat{x}_2 + \frac{1}{\sqrt{3}}\,\hat{x}_3,\\
\hat{X}_2 &= \frac{1}{\sqrt{2}}\,\hat{x}_1 + \frac{1}{\sqrt{6}}\,\hat{x}_2 + \frac{1}{\sqrt{3}}\,\hat{x}_3,\\
\hat{X}_3 &= -\frac{1}{\sqrt{2}}\,\hat{x}_1 + \frac{1}{\sqrt{6}}\,\hat{x}_2 + \frac{1}{\sqrt{3}}\,\hat{x}_3.
\end{aligned}$$

Although we have obtained all the expressions written in a completely symmetrical form with respect to the three spatial directions, their former form is more appropriate to calculate the complete (all orders in z_1, z_3) Hopf algebra and the change of basis ensures that spatial isotropy will not actually be lost in the resulting quantum (anti)de Sitter algebras.

4.1. Zero-curvature limit: deformed Poincaré symmetries

In order to investigate the Poincaré limit we rescale the deformation parameters as

$$z'_i = f_i(l_{\rm P}\sqrt{\omega}), \qquad i = 1, 2, 3,$$

thus allowing them to take part of the contraction process.

- If $z'_1 \cong \sqrt{\omega} l_{\rm P}$, thus taking f_1 linear and $f_3 = 0$, in the limit of zero-curvature we obtain

$$\lim_{\omega \mapsto 0} r_{z_1,\tilde{z}_3} = l_{\rm P}\left(\tilde{K}_1 \wedge \tilde{P}_1 + \tilde{K}_2 \wedge \tilde{P}_2 + \tilde{K}_3 \wedge \tilde{P}_3\right);$$
$$\lim_{\omega \mapsto 0} \delta_{z_1,\tilde{z}_3}(\tilde{P}_0) = 0, \qquad \lim_{\omega \mapsto 0} \delta_{z_1,\tilde{z}_3}(\tilde{P}_i) = l_{\rm P} \tilde{P}_i \wedge \tilde{P}_0,$$
$$\lim_{\omega \mapsto 0} \delta_{z_1,\tilde{z}_3}(\tilde{K}_i) = l_{\rm P}\left(\tilde{K}_i \wedge \tilde{P}_0 + \varepsilon_{ijk}\tilde{J}_j \wedge \tilde{P}_k\right), \qquad \lim_{\omega \mapsto 0} \delta_{z_1,\tilde{z}_3}(\tilde{J}_i) = 0.$$

The form of the f_i is suggested by loop quantum gravity models[25] and leads exactly to the **kappa-Poincaré bialgebra**[26]

[25] G. Amelino-Camelia, L. Smolin, A. Starodubtsev, Class. Quant. Grav. 21 (2004) 3095. L. Smolin, hep-th/0501091.

[26] J. Lukierski, A. Nowicki, W.J. Zakrzewski, Ann. Phys. 243 (1995) 90

- If $z_1' \cong (\sqrt{\omega} l_{\rm P})^\alpha$, with $\alpha > 1$ and $f_3 = 0$, then we find that

$$\lim_{\omega \mapsto 0} r_{z_1,\tilde{z}_3} = 0;$$
$$\lim_{\omega \mapsto 0} \delta_{z_1,\tilde{z}_3}(\tilde{P}_0) = 0, \qquad \lim_{\omega \mapsto 0} \delta_{z_1,\tilde{z}_3}(\tilde{P}_i) = 0,$$
$$\lim_{\omega \mapsto 0} \delta_{z_1,\tilde{z}_3}(\tilde{K}_i) = 0, \qquad \lim_{\omega \mapsto 0} \delta_{z_1,\tilde{z}_3}(\tilde{J}_i) = 0,$$

which gives rise to the **classical Poincaré**.

In both casis one can formulate hypothesis on f_3 **different from zero**. In such casis one obtains a **twisted bialgebra**, which affects changes only in the coalgebra sector and not in the algebra sector.

Since r_{z_3} does not depend on ω, even in the case of zero-curvature ($\omega = 0$), which corresponds for r_{z_1} to kappa-Poincaré, one can apply the twisting operator

$$\Delta_{z_l,z_3} = \mathcal{F}_{z_3} \Delta_{z_l} \mathcal{F}_{z_3}^{-1}, \qquad \mathcal{F}_{z_3} = \exp\{-z_3 P_0 \otimes J_3\}, \qquad [Y_i, Y_j]_{z_l,z_3} \equiv [Y_i, Y_j]_{z_l}.$$

so that a **biparametric version of kappa-Poincaré**, that does not imply changes on the algebra sector, can be obtained supporting a generalization of the **kappa-Minkowski space**:

$$[\hat{X}_0, \hat{X}_i] = -\frac{z_1}{c^2} \hat{X}_i + \tilde{z}_3 \, \varepsilon_{ijk} \, \hat{X}_j, \qquad [\hat{X}_i, \hat{X}_j] = 0.$$

4.2. Non-relativistic limit: deformed Newtonian and Galilean symmetries

The $c \to \infty$ limit of any deformed relativistic symmetry should lead to a **non-deformed or "trivial" structure**, since it is quite natural to think that any quantum deformation is an intrinsically high-energy relativistic effect.

Recall that the Planck length l_{P} is defined as

$$l_{\mathrm{P}} = \sqrt{\frac{\hbar G}{c^3}},$$

so it goes to 0 both in the $\hbar \to 0$ limit and in the $c \to \infty$ one.

Under the limit $c \to \infty$ we obtain

$$\begin{aligned}
&\lim_{c\mapsto\infty} r_{z_1,\tilde{z}_3} = z_1 \left(\tilde{K}_1 \wedge \tilde{P}_1 + \tilde{K}_2 \wedge \tilde{P}_2 + \tilde{K}_3 \wedge \tilde{P}_3 \right);\\
&\lim_{c\mapsto\infty} \delta_{z_1,\tilde{z}_3}(\tilde{P}_0) = 0, \qquad \lim_{c\mapsto\infty} \delta_{z_1,\tilde{z}_3}(\tilde{P}_i) = 0,\\
&\lim_{c\mapsto\infty} \delta_{z_1,\tilde{z}_3}(\tilde{K}_i) = 0, \qquad \lim_{c\mapsto\infty} \delta_{z_1,\tilde{z}_3}(\tilde{J}_i) = \tilde{z}_3 \varepsilon_{ijk} \tilde{J}_j \wedge \tilde{P}_0,
\end{aligned}$$

which is the Galilei algebra with a possible residual twist when $\tilde{z}_3 \neq 0$.

5. Other possible (anti)de Sitter deformations

What about to set the **three** rotation generators non-deformed from the beginning (as it happens in kappa-Poincaré)?

From the initial 45-parameters r-matrix, the conditions

$$\delta(J_1) = \delta(J_2) = \delta(J_3) = 0$$

gives a 3-parametric candidate (obviously completely symmetric):

$$\begin{aligned} r_{z_1,z_2,z_3} = {} & z_1 \left(K_1 \wedge P_1 + K_2 \wedge P_2 + K_3 \wedge P_3\right) \\ & + z_1 \left(J_1 \wedge P_1 + J_2 \wedge P_2 + J_3 \wedge P_3\right) \\ & + z_3 \left(J_1 \wedge K_1 + J_2 \wedge K_2 + J_3 \wedge K_3\right). \end{aligned}$$

When the modified classical Yang–Baxter equation is imposed, we find that the three deformation parameters z_i must vanishes whenever the curvature $\omega \neq 0$. Thus **there does not exist a quantum (anti)de Sitter algebra with a non-deformed rotation subalgebra**.

6. Concluding remarks

• We have found all possible quantum deformations for the $(3+1)$ (anti)de Sitter algebras which keep P_0 and J_3 non-deformed.

• There are only two possible deformations that depend on two deformation parameters.

• One of them can be written in a symmetric form with respect to the rotation generators.

• Furthermore this has as its zero-curvature limit a generalization of kappa-Poincaré depending on an additional deformation parameter related with a twist.

• One restriction of such deformation provides the $(2+1)$D case considered in quantum gravity.