FINSLER GEOMETRY IN QG PHENO

Florian Girelli
In collaboration with
S. Liberati and L. Sindoni

Review soon to appear

July 8, 2008

QG PHENO THEORY: GOALS

⇝ **Construct an effective field theory describing encoding semi-classical Quantum Gravity (QG) effects on a flat space-time**

- Derive it from a full QG model.
    - Semi-classical limit from spinfoams where matter has been introduced...
    - Renormalization group applied to gravity.
    - Effective field theory technics applied to gravity.

⇝ **Modify Special Relativity to encode QG effects: effective flat semi-classical space-time.**

- Make Minkowski space-time non-commutative ⇝ DSR theories and its interpretations.
- ”New” type of geometrical structures: Finsler geometry as an effective geometry.

FROM SPECIAL RELATIVITY TO FLAT SEMI-CLASSICAL SPACE-TIME

⇝ Special Relativity can be deduced from:

A natural assumption:	⁕ Space-time as a manifold
Prescription to get numbers from:	⁕ Operational set up/measurement
A set of axioms:	⁕ Inertial observer
	⁕ Homogeneity of space-time
	⁕ Isotropy of space-time
	⁕ Pre-causality
	⁕ Relativity principle

FROM SPECIAL RELATIVITY TO FLAT SEMI-CLASSICAL SPACE-TIME

⋇ Space-time as a manifold
⋇ Operational set up/measurement
⋇ Inertial observer
⋇ Homogeneity of space-time
⋇ Isotropy of space-time
⋇ Pre-causality
⋇ Relativity principle

⇒ There is an invariant speed v_{inv}, and we have the Lorentz group.
Minkowski line element ds.

FROM SPECIAL RELATIVITY TO FLAT SEMI-CLASSICAL SPACE-TIME

※ Space-time as a manifold
※ Operational set up/measurement
※ Inertial observer
※ Homogeneity of space-time
※ Isotropy of space-time
※ Pre-causality
※ Relativity principle

⇒ (There is a maximum speed v_{max},
and we have the Lorentz group.
Minkowski line element ds.

⇝ Introducing a minimum length L_P can break homogeneity and/or differentiability. It can affect the measurement theory.

FURTHER THAN SPECIAL RELATIVITY?

$\rightsquigarrow$The 2d case contains a bit more than Minkowski metric if we weaken isotropy. Lalan (1936)
The most general transformation consistent with the other axioms is linear and leaves invariant the line element

$$ds = \frac{(dt^2 - dx^2)^{(1-b/c)/2}}{(dt+dx)^{b/c}}$$

New parameter b with dimension speed. New metric structure: a Finsler metric.
$\rightsquigarrow$Extension to the 4d case: ”Very Special Relativity”. The relativity group has only 4 generators + 4 translations (light-like preferred vector n^μ).

$$ds = \frac{(\eta_{\mu\nu} dx^\mu dx^\nu)^{(1-b/c)/2}}{(n_\mu dx^\mu)^{b/c}}$$

(Bogoslovsky grcq/0706.2621, Cohen and Glashow heph/0601236, Gibbons-Gomis-Pope hepth/07072174)

Element of length squared is not given as a small displacement squared.

In his Habilitationsvortrag of 1854, Riemann introduced a metric structure in a general space based on the element of arc

$$ds^2 = F^2(x_\mu, dx_\mu)$$

Here, F is a positive function on the tangent bundle TM and is homogeneous of degree one in dx_μ*. An important special case is when*

$$F^2 = g_{ij}(x)dx^i dx^j.$$

Finsler geometry is not a generalization of Riemannian geometry. It is better described as Riemannian geometry without the quadratic restriction.
Chern (Notices of AMS, September, 1996)

PROPOSAL

Semi-classical (flat) metric structure is effectively encoded in a Finsler metric.

Correct mathematical framework to talk about ”Rainbow metrics”.

FINSLER METRIC

Let $F(v)$ be a norm over the tangent space, that is $\forall v \in T_xM$

- $F(v) \neq 0$ if $v \neq \mathbf{0}$,
- $F(\lambda v) = |\lambda| F(v), \qquad \lambda \in \mathbb{R}$.

We can then define the ”Finsler metric” [required non degenerate and continuous]:

$$g_{\mu\nu}(v) = \frac{1}{2}\frac{\partial F^2}{\partial v^\mu \partial v^\nu} \qquad \Leftrightarrow F(v) = \sqrt{g_{\mu\nu}(v) v^\mu v^\nu}.$$

$\rightsquigarrow$A Finsler metric is not defined through a scalar product, but through an homogenous function on tangent space.

$\rightsquigarrow$The inverse of $g_{\mu\nu}(v)$ is not the dual metric! The dual metric is obtained by a Legendre transform.

SOME EXAMPLES OF FINSLER METRICS

- Consider the norm $F(v) = \sqrt{A_{\mu\nu} v^\mu v^\nu}$, with $A_{\mu\nu}$ symmetric, then the Finsler metric is given by:

  $$g_{\mu\nu} = A_{\mu\nu}.$$

  A Riemannian metric is therefore a specific example of Finsler metric.

- Consider the norm $F(v) = \sqrt{B_{\mu\nu\alpha\beta} v^\mu v^\nu v^\alpha v^\beta}$ with $B_{\mu\nu\alpha\beta}$ symmetric. The Finsler metric is given by:

  $$g_{\mu\nu} = \frac{B_{\mu\nu\alpha\beta} v^\alpha v^\beta}{\sqrt{B_{\mu\nu\alpha\beta} v^\mu v^\nu v^\alpha v^\beta}} - \frac{B_{\mu\nu\alpha\beta} v^\nu v^\alpha v^\beta B_{\mu\nu\alpha\beta} v^\mu v^\alpha v^\beta}{(B_{\mu\nu\alpha\beta} v^\mu v^\nu v^\alpha v^\beta)^{3/2}}.$$

  It is called the Berwald-Moore metric (appears in Analogue models, sound waves...).

SOME EXAMPLES OF FINSLER METRICS

- Consider the norm $F(v) = \sqrt{A_{\mu\nu} v^\mu v^\nu}$, with $A_{\mu\nu}$ symmetric, then the Finsler metric is given by:

  $$g_{\mu\nu} = A_{\mu\nu}.$$

  A Riemannian metric is therefore a specific example of Finsler metric.

- Consider the norm $F(v) = \sqrt{B_{\mu\nu\alpha\beta} v^\mu v^\nu v^\alpha v^\beta}$ with $B_{\mu\nu\alpha\beta}$ symmetric. The Finsler metric is given by:

  $$g_{\mu\nu} = \frac{B_{\mu\nu\alpha\beta} v^\alpha v^\beta}{\sqrt{B_{\mu\nu\alpha\beta} v^\mu v^\nu v^\alpha v^\beta}} - \frac{B_{\mu\nu\alpha\beta} v^\nu v^\alpha v^\beta B_{\mu\nu\alpha\beta} v^\mu v^\alpha v^\beta}{(B_{\mu\nu\alpha\beta} v^\mu v^\nu v^\alpha v^\beta)^{3/2}}.$$

  It is called the Berwald-Moore metric (appears in Analogue models, sound waves...).

FINSLER GEOMETRY

- Given a Finsler geometry, we can define the geodesic equation.

$$\int \sqrt{g_{\nu\rho}(x,\dot{x})\dot{x}^\nu \dot{x}^\rho} d\tau \quad \rightarrow \quad \ddot{x}^\mu + \Gamma^\mu_{\nu\rho}(x,\dot{x})\,\dot{x}^\nu \dot{x}^\rho = 0$$

$$\Gamma^\mu_{\nu\rho}(x,\dot{x}) = \frac{1}{2} g^{\mu\sigma}(x,\dot{x})(-\partial_\sigma g_{\nu\rho}(x,\dot{x}) + \partial_\nu g_{\rho\sigma}(x,\dot{x}) + \partial_\rho g_{\sigma\nu}(x,\dot{x})).$$

- We can define Killing vectors ξ

$$g_{\rho\mu}(x,\dot{x})\partial_\nu \xi^\rho + g_{\nu\rho}(x,\dot{x})\partial_\mu \xi^\rho + 2C_{\rho\mu\nu}(x,\dot{x})\frac{\partial \xi^\rho}{\partial x^\sigma}\dot{x} + \frac{\partial g_{\mu\nu}}{\partial x^\rho}\xi^\rho = 0,$$

  where $C_{\mu\nu\rho} = \frac{1}{2}\frac{\partial g_{\mu\nu}(x,\dot{x})}{\partial \dot{x}^\rho}$ is the "Cartan tensor".
  Symmetries can be vector dependent [would be analog to non-linear realization of the symmetries] or not [symmetry breaking].

- The set of Finsler geometries is very large. Mathematicians explore these structures that can be pretty complicated. We are interested here at Finsler metrics which look almost the same as a Riemannian metric: introduce perturbations in Planck scale.

FINSLER GEOMETRY

- Given a Finsler geometry, we can define the geodesic equation.

$$\int \sqrt{g_{\nu\rho}(x,\dot{x})\dot{x}^\nu \dot{x}^\rho} d\tau \quad \rightarrow \quad \ddot{x}^\mu + \Gamma^\mu_{\nu\rho}(x,\dot{x})\,\dot{x}^\nu \dot{x}^\rho = 0$$

$\Gamma^\mu_{\nu\rho}(x,\dot{x}) = \frac{1}{2} g^{\mu\sigma}(x,\dot{x})(-\partial_\sigma g_{\nu\rho}(x,\dot{x}) + \partial_\nu g_{\rho\sigma}(x,\dot{x}) + \partial_\rho g_{\sigma\nu}(x,\dot{x}))$.

- We can define Killing vectors ξ

$$g_{\rho\mu}(x,\dot{x})\partial_\nu \xi^\rho + g_{\nu\rho}(x,\dot{x})\partial_\mu \xi^\rho + 2C_{\rho\mu\nu}(x,\dot{x})\frac{\partial \xi^\rho}{\partial x^\sigma}\dot{x} + \frac{\partial g_{\mu\nu}}{\partial x^\rho}\xi^\rho = 0,$$

where $C_{\mu\nu\rho} = \frac{1}{2}\frac{\partial g_{\mu\nu}(x,\dot{x})}{\partial \dot{x}^\rho}$ is the ”Cartan tensor”.
Symmetries can be vector dependent [would be analog to non-linear realization of the symmetries] or not [symmetry breaking].

- The set of Finsler geometries is very large. Mathematicians explore these structures that can be pretty complicated. We are interested here at Finsler metrics which look almost the same as a Riemannian metric: introduce perturbations in Planck scale.

FINSLER GEOMETRY

- Given a Finsler geometry, we can define the geodesic equation.

$$\int \sqrt{g_{\nu\rho}(x,\dot{x})\dot{x}^{\nu}\dot{x}^{\rho}}d\tau \quad \rightarrow \quad \ddot{x}^{\mu} + \Gamma^{\mu}_{\nu\rho}(x,\dot{x})\,\dot{x}^{\nu}\dot{x}^{\rho} = 0$$

  $\Gamma^{\mu}_{\nu\rho}(x,\dot{x}) = \frac{1}{2}g^{\mu\sigma}(x,\dot{x})(-\partial_{\sigma}g_{\nu\rho}(x,\dot{x}) + \partial_{\nu}g_{\rho\sigma}(x,\dot{x}) + \partial_{\rho}g_{\sigma\nu}(x,\dot{x}))$.

- We can define Killing vectors ξ

$$g_{\rho\mu}(x,\dot{x})\partial_{\nu}\xi^{\rho} + g_{\nu\rho}(x,\dot{x})\partial_{\mu}\xi^{\rho} + 2C_{\rho\mu\nu}(x,\dot{x})\frac{\partial\xi^{\rho}}{\partial x^{\sigma}}\dot{x} + \frac{\partial g_{\mu\nu}}{\partial x^{\rho}}\xi^{\rho} = 0,$$

  where $C_{\mu\nu\rho} = \frac{1}{2}\frac{\partial g_{\mu\nu}(x,\dot{x})}{\partial\dot{x}^{\rho}}$ is the "Cartan tensor".
  Symmetries can be vector dependent [would be analog to non-linear realization of the symmetries] or not [symmetry breaking].

- The set of Finsler geometries is very large. Mathematicians explore these structures that can be pretty complicated. We are interested here at Finsler metrics which look almost the same as a Riemannian metric: introduce perturbations in Planck scale.

MODIFIED MASS SHELL AND FINSLER

⇝A modified mass shell can originate from a metric momentum dependent (Rainbow metric).

$$E^2 = m^2 + p^2 + \sum_{n=1}^{\infty} \alpha_n(p, M_P) \Leftrightarrow g_{\mu\nu}(p, M_P, m) p^\mu p^\nu = m^2.$$

⇝Statement:

⁎ A momentum dependent metric should be treated geometrically as a Finsler metric: Rainbow geometries should be seen as Finsler geometries.

⁎ A particle with modified mass shell is a geodesic in a Finsler metric.

LEGENDRE TRANSFORMATION

Let's be given a mass shell, start from the particle action in the hamiltonian formalism and perform a Legendre transform, assuming $\{x_\mu, p^\nu\} = \delta_\mu^\nu$.

$$\mathcal{S} = \int \left(dx^\mu p_\mu - \lambda(g_{\mu\nu}(p, M_P, m) p^\mu p^\nu - m^2) \right)$$

Hamilton equations provide the relation between momentum and speed.

$$\dot{x} = \lambda\{x, g_{\mu\nu}(p, M_P, m) p^\mu p^\nu\}.$$

We must invert this relation: this is not always possible! **In general, it will not be invertible everywhere: we are looking on the sector for momentum smaller than Planck scale.**

We obtain

$$p = f(\dot{x}, \lambda, M_P, m),$$

then we vary over look at the equation of motion for λ and insert this value in the action.

$$\mathcal{S} = \int \Big(dx^\mu p_\mu - \lambda (g_{\mu\nu}(p, M_P, m) p^\mu p^\nu - m^2) \Big)$$
$$\downarrow$$
$$\mathcal{S} = m \int d\tau F(\dot{x}^\mu, m, M_P)$$

The outcome is a complicated norm, mass-dependent...

AN EXAMPLE: METRIC VS MULTI-METRIC

A 2d particle with mass shell $p_0^2 - p_1^2 - \frac{1}{M_P} p_1^3 = m^2$ is a geodesic of the metric arising from

$$F_{(m,M_P)}(\dot{x}) \sim \sqrt{\dot{t}^2 - \dot{\mathrm{x}}^2} - \frac{m}{2M_P} \frac{\dot{\mathrm{x}}^3}{\dot{t}^2 - \dot{\mathrm{x}}^2}.$$

- A multi-metric structure appears...
- A possible way out: introduce a IR cut-off Λ together with the UV cut-off M_P.

$$F_{(\Lambda,M_P)}(\dot{x}) \sim \sqrt{\dot{t}^2 - \dot{\mathrm{x}}^2} - \frac{\Lambda}{2M_P} \frac{\dot{\mathrm{x}}^3}{\dot{t}^2 - \dot{\mathrm{x}}^2} \rightarrow p_0^2 - p_1^2 - \frac{1}{m} \frac{\Lambda}{M_P} p_1^3 = m^2.$$

CAUSALITY STRUCTURE

Look for the "light cones": $F_m(\dot{x}) = 0$ or $\mathcal{M}(p, M_P) = E^2 - g(p, M_P) = 0$?

$$v = \frac{\dot{x}}{\dot{t}} = \frac{dE}{dp} = \frac{1}{2}\frac{g'(p)}{\sqrt{g(p) + m^2}}$$

therefore the massless particle is the fastest. Construct the light in the tangent space:

- take the mass-shell $\mathcal{M}(p, M_P) = 0$.
- use the Hamilton equation $\dot{x} = \lambda \frac{\partial \mathcal{M}}{\partial p}$
- $\mathcal{M}(p, M_P) = 0 \rightarrow \mathcal{M}(\dot{x}) = 0$ is the light cone structure (and not $F(\dot{x}) = 0$).

FINSLER IN PHYSICS?

Do we have already evidences of Finsler metrics in known physics?

Waves propagating in a medium can be described using Finsler metrics.

SOUND WAVES IN CRYSTALS

⇝The deformation w.r.t. equilibrium configuration is described by a vector field $u_i(x)$

⇝The strain tensor ε_{ij} which quantify the strain of an object undergoing a small deformation $u_i(x)$ is

$$\varepsilon_{ij} = \frac{\partial u_i}{\partial x^j} + \frac{\partial u_j}{\partial x^i}$$

Let us note σ_{ij} the stress tensor. The Hooke's law tell us that for linear deformations

$$\sigma_{ij} = \lambda_{ijmn} \varepsilon_{mn},$$

where λ_{ijmn} depends on the structure of the crystal.
Force applied per unit volume:

$$F_i = \frac{\partial \sigma_{ij}}{\partial x^j}$$

(Landau and Lifshitz, vol. 7, Theory of elasticity)

⇝Equation of motion for the small deformation u_i

$$\rho \ddot{u}_i = F_i = \lambda_{ijmn} \partial_j \varepsilon_{mn}$$

Eikonal approximation : $u^i = \epsilon^i \exp(-i(\omega t - \mathbf{k} \cdot \mathbf{x}))$.

$$\left(\rho \omega^2 \delta_{il} - \lambda_{ijlm} k_j k_m\right) \epsilon_l = 0$$

Fresnel equation:

$$\det\left(\rho \omega^2 \delta_{il} - \lambda_{ijlm} k_j k_m\right) = 0$$

In 3d, this is an algebraic equation of degree six. The geometrical content is described by the tensor λ. We obtain a Berwald-Moore metric (similar to Analog models for Gravity).

LIGHT IN CRYSTALS

Consider Maxwell equations in an homogeneous medium:

$$\begin{array}{ll} \nabla \cdot \mathbf{B} = 0; & \nabla \times \mathbf{B} = \epsilon \cdot \frac{\partial}{\partial t}\mathbf{E} \\ \nabla \cdot \mathbf{E} = 0; & \nabla \times \mathbf{E} + \frac{\partial}{\partial t}\mathbf{B} = 0 \end{array}$$

ϵ_{ij} encode anisotropies of the crystal. Let ϵ_i be its eigenvalues

- isotropic: $\epsilon_{ij} = \epsilon\delta_{ij}$
- uniaxial: $\epsilon_1 = \epsilon_2 \neq \epsilon_3$
- biaxial: $\epsilon_1 \neq \epsilon_2 \neq \epsilon_3$

We are interested in waves propagating in the crystal.

$$\mathbf{E} = \mathbf{E}_0 \exp(-i\omega t + i\mathbf{k}\cdot\mathbf{x}); \quad \mathbf{B} = \mathbf{B}_0 \exp(-i\omega t + i\mathbf{k}\cdot\mathbf{x})$$

Introducing the quantity

$$\mathbf{n} = c\frac{\mathbf{k}}{\omega}.$$

we obtain the Fresnel equation

$$\det(n^2\delta_{ij} - n_i n_j - \epsilon_{ij}) = 0$$

FINSLER IN DSR?

- In a couple of papers, Liberati et al introduced a stochastic component to the tetrad induced by QG effects. (gr-qc/0511031, gr-qc/0607024)
- A particle will probe the fluctuations of the tetrad up to the scale L $E^{\mu}{}_{\alpha}(L) =< e^{\mu}{}_{\alpha} >_L$. This scale can be naturally identified with the particle's de Broglie length $L = \hbar/p_0$.
- In this case, the effective reference frame becomes momentum dependent $E^{\mu}{}_{\alpha}(p_0)$.

$$e^{\mu}{}_{\alpha} \rightarrow E^{\mu}{}_{\alpha} = \mathcal{U}(e^{\mu}{}_{\alpha}, \pi_{\mu}, M_P).$$

For example:

$$\mathcal{U}(e^{\mu}{}_{\alpha}, \pi_{\mu}, M_P) = \frac{1}{\sqrt{1 - \frac{\pi_{\mu} e^{\mu}{}_{0}}{M_P}}} e^{\mu}{}_{\alpha}, \quad p_{\alpha} = \frac{1}{\sqrt{1 - \frac{\pi_{\mu} e^{\mu}{}_{0}}{M_P}}} e^{\mu}{}_{\alpha} \pi_{\mu}$$

The fundamental variable is the frame ${e^\mu}_\alpha$. We have then a non-linear transformation of the effective frame:

$$p_\alpha = {E^\mu}_\alpha \pi_\mu \xrightarrow{\Lambda} p'_\alpha = \mathcal{U}\left(\Lambda \cdot \mathcal{U}^{-1}(E)\right)\pi_\mu.$$

Such frame will be associated to a Finsler metric.

$$\eta^{\alpha\beta}{E^\mu}_\alpha(\pi){E^\nu}_\beta(\pi)\pi_\mu\pi_\nu = g^{\mu\nu}(\pi)\pi_\mu\pi_\nu$$

- A similar result is obtained in the renormalization group approach (Girelli, Liberati, Percacci, Rahmede, gr-qc/0607030).
- It is however not clear how this extends in more general situations: multi-particles case...
- The key question to answer is how the reference frame becomes system dependent and the way to encode symmetries.

OUTLOOK

⇝Finsler metrics are ”generalizations” or ”relaxations” of the usual (pseudo-)Riemannian metrics. This is the correct mathematical framework to describe ”Rainbow metrics”.

⇝They appear naturally as intermediate effective structures in analogs models for gravity.

⇝They are naturally associated with higher order derivatives field theories.

⇝They seem to be a natural geometric framework to describe effectively semi-classical space-times.

⇝Use Finsler metrics in the General Relativity context: equivalence principle...

⇝Use Finsler metrics in Quantum Field Theory?