κ-statistics

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M. Arzano, DB - *to appear (sooner or later)*

The symmetric group in QFT

Ladies and gentlemen, the symmetric group S_n.
S_n has a presentation in terms of a set of $n-1$ generators s_i for $i = 1, ..., n-1$ satisfying the relations

$$s_i s_j = s_j s_i \qquad \text{for } |i-j| > 1 \ , \tag{1}$$

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \qquad \text{for } 1 \leq i \leq n-2 \ , \tag{2}$$

$$s_i^2 = 1 \qquad \text{for } 1 \leq i \leq n-1 \ . \tag{3}$$

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The Fock space of QFT is built from direct sums of n-fold tensor products of irreps of the Poincaré group for $n = 0, .. + \infty$, all of which are in the same irrep of S_n

Possible because S_n intertwines the (cocommutative) Hopf algebra structure of Poincaré

QFT with deformed symmetries

QFT on noncommutative spacetime as a possible effective description of quantum gravity effects (Doplicher, Fredenhagen, Roberts; Freidel, Livine; ...)

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Not yet!

Canonical noncommutativity → twisted realization of S_n

κ-noncommutativity → open question with different answers
(Arzano, Marcianò; Daszkiewicz, Lukierski, Woronowicz; Young, Zegers)

– Try to understand what underlies different constructions –

Outline

- **Statistics in QFT on the Moyal plane**
- **κ-deformed QFT and κ-deformed statistics**
- **Twist structures in the momentum secor of κ-Poincaré**
- **Some words on covariant construction**
- **Conclusions and outlook**

The Moyal plane

By Moyal plane we mean the non-commutative spacetime characterized by

$$[x_\mu, x_\nu] = i\theta_{\mu\nu} \tag{4}$$

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This can be obtained by deforming the ordinary product among functions $m(f \otimes g)(x) = f(x) \cdot g(x)$ to a non-commutative $\star$-product via a “twist”

$$\mathcal{F}_\theta = \exp\left(-\frac{i}{2}\theta_{\mu\nu}\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial y_\nu}\right) \tag{5}$$

defining $m_\theta = m \circ \mathcal{F}_\theta^{-1}$

$$m_\theta(f \otimes g)(x) = \exp\left(\frac{i}{2}\theta_{\mu\nu}\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial y_\nu}\right) f(x) \cdot g(y)|_{y=x} = f(x) \star g(x)\,, \tag{6}$$

the vector space $\mathcal{C}(\mathbb{R}^{3+1})$ endowed with the non-commutative product m_θ becomes the non-commutative algebra $\mathcal{C}_\theta(\mathbb{R}^{3+1})$.

Twisted Poincaré

The Moyal plane is dual to the translation sector of a twisted Poincaré algebra which acts on it

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Reminder: given a Hopf algebra $\mathcal{A}$ with coproduct Δ and antipode S and given an element $\mathcal{F} \in \mathcal{A} \otimes \mathcal{A}$ satisfying appropriate conditions we can define a new Hopf algebra $\mathcal{A}_{\mathcal{F}}$ by twisting coproduct and antipode as

$$\Delta_{\mathcal{F}}(a) = \mathcal{F}\Delta(a)\mathcal{F}^{-1} \qquad \forall a \in \mathcal{A} \tag{7}$$

$$S_{\mathcal{F}}(a) = [m(id \otimes S)\mathcal{F}]S(a)[m(S \otimes id)\mathcal{F}^{-1}] \qquad \forall a \in \mathcal{A} \tag{8}$$

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Using $\mathcal{F}_\theta = \exp\left(\frac{i}{2}\theta_{\mu\nu}P^\mu P^\nu\right)$ as a twist on the (enveloping algebra of the) Poincaré algebra a new Hopf algebra was found by Oeckl, Chaichian et al., and Wess, which can be used as a symmetry algebra of QFT on the Moyal plane

It is a mild twist, in particular

$$\Delta_\theta(P_\mu) = \mathcal{F}_\theta\Delta_0(P_\mu)\mathcal{F}_\theta^{-1} = \Delta_0(P_\mu) \equiv P_\mu \otimes 1 + 1 \otimes P_\mu \tag{9}$$

The quasitriangular structure is non-trivial but still very simple. The $\mathcal{R}$-matrix is

$$\mathcal{R} = \mathcal{F}_\theta^{-2} = \exp\left(-i\theta_{\mu\nu}P^\mu \otimes P^\nu\right) \tag{10}$$

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Because of the non-trivial quasitriangular structure the representations of S_n must be twisted accordingly

$$\pi_\theta(s_1) = \mathcal{F}_\theta \pi_0(s_1) \mathcal{F}_\theta^{-1} \qquad \text{for } n = 2 \tag{11}$$

where $\pi_0(s_1)$ is the standard realization as simple permutation

$$\pi_0(s_1)|p_1\rangle \otimes |p_2\rangle = \tau |p_1\rangle \otimes |p_2\rangle = |p_2\rangle \otimes |p_1\rangle \tag{12}$$

The action of $\pi_\theta(s_1)$ is a simple permutation times a (momentum-dependent) phase factor,

$$\pi_\theta(s_1)|p_1\rangle \otimes |p_2\rangle = e^{i\theta_{\mu\nu} p_1^\mu p_2^\nu} |p_2\rangle \otimes |p_1\rangle \tag{13}$$

reflecting the extremely simple quasi-triangular structure.

Easily extended to $n > 2$. The "twisted permutations" are still a realization of S_n.

κ-deformed QFT

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In the construction of Arzano and Marcianò the usual "symmetrized" two-particle state

$$1/\sqrt{2}(|\vec{p}_1\rangle \otimes |\vec{p}_2\rangle + |\vec{p}_2\rangle \otimes |\vec{p}_1\rangle) \tag{15}$$

is replaced by

$$|p_1 p_2\rangle_\kappa = \frac{1}{\sqrt{2}}\left[|\vec{p}_1\rangle \otimes |\vec{p}_2\rangle + |e^{-\omega(\vec{p}_1)/\kappa}\vec{p}_2\rangle \otimes |e^{\omega(e^{-\omega(\vec{p}_1)/\kappa}\vec{p}_2)/\kappa}\vec{p}_1\rangle\right] \tag{16}$$

so that it is still an eigenvalue of $\Delta(P_j) = P_j \otimes 1 + e^{-P_0/\kappa} \otimes P_j$ and
$\Delta(P_0) = P_0 \otimes 1 + 1 \otimes P_0$.
Extended to all n, it forms a non standard realization of S_n, in which flip operation affects the labels of the states.

Peculiar to $m = 0$ and (asymmetric) bicrossproduct base.

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$$\Delta(P_j) = P_j \otimes e^{\frac{1-\beta}{2\kappa} P_0} + e^{-\frac{1+\beta}{2\kappa} P_0} \otimes P_j \tag{18}$$

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We can re-express it in terms of the starndard cocommutative coproduct in a "twist" style

$$\Delta(P_\mu) = \mathcal{F}\Delta_0(P_\mu)\mathcal{F}^{-1} \ , \tag{19}$$

where

$$\mathcal{F} = e^{\frac{1}{2\kappa}(P_0 \otimes P_j \frac{\partial}{\partial P_j} - P_j \frac{\partial}{\partial P_j} \otimes P_0) + \frac{\beta}{2\kappa}(P_0 \otimes P_j \frac{\partial}{\partial P_j} + P_j \frac{\partial}{\partial P_j} \otimes P_0)} \tag{20}$$

Note that strictly speaking it is not a twist unless we extend the algebra.

Star-product in κ-Minkowski

The "twist" map we introduced is essentially related to a one-parameter family of possible $\star$-products on $U^*_\kappa(T_4)$, which by construction is κ-Minkowski:

$$[\hat{x}^\mu, \hat{x}^\nu] = \frac{i}{\kappa}(\delta^\mu_0 \delta^\nu_j - \delta^\nu_0 \delta^\mu_j)\hat{x}^j \ . \tag{21}$$

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Symmetric part of the product does not enter the commutator so we can write for example

$$x^\mu \star x^\nu = x^\mu x^\nu + \frac{i}{2\kappa}(\delta^\mu_0 \delta^\nu_j - \delta^\nu_0 \delta^\mu_j)x^j + \beta \frac{i}{2\kappa}(\delta^\mu_0 \delta^\nu_j + \delta^\nu_0 \delta^\mu_j)x^j \ , \tag{22}$$

with $\beta \in \mathbb{R}$, and which can be written as $m(\mathcal{F}^{-1} \triangleright (x^\mu \otimes x^\nu))$ with

$$\mathcal{F}^{-1} = e^{\frac{i}{2\kappa}\left(\frac{\partial}{\partial x^0} \otimes x^j \frac{\partial}{\partial x^j} - x^j \frac{\partial}{\partial x^j} \otimes \frac{\partial}{\partial x^0}\right) + \beta \frac{i}{2\kappa}\left(\frac{\partial}{\partial x^0} \otimes x^j \frac{\partial}{\partial x^j} + x^j \frac{\partial}{\partial x^j} \otimes \frac{\partial}{\partial x^0}\right)} \ . \tag{23}$$

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As illustration of the relation between twist and $\star$-product think of the well known example of plane waves

$$m(\mathcal{F}^{-1} \triangleright (e^{ip_\mu x^\mu} \otimes e^{iq_\mu x^\mu})) = \exp\left\{-i(p^0 + q^0)x^0 + i(p^j \ e^{\frac{1-\beta}{2\kappa}q^0} + q^j \ e^{-\frac{1+\beta}{2\kappa}p^0})x^j\right\}$$

$$\langle P_\mu, m(\mathcal{F}^{-1} \triangleright (e^{ip_\mu x^\mu} \otimes e^{iq_\mu x^\mu}))\rangle = (p_\mu \dot{+} q_\mu) = \langle \mathcal{F}\Delta_0(P_\mu)\mathcal{F}^{-1}, e^{ip_\mu x^\mu} \otimes e^{iq_\mu x^\mu}\rangle$$

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For $n > 2$ we have

$$\Delta^{n-1}(P_\mu) \equiv (\Delta \otimes \underbrace{id \otimes ...id}_{n-1})\Delta^{n-2}(P_\mu) = \mathcal{F}_{12...n}\Delta_0^{n-1}(P_\mu)\mathcal{F}_{12...n}^{-1} \tag{24}$$

where

$$\begin{aligned} &\mathcal{F}_{12} = [\mathcal{F} \otimes 1][(\Delta_0 \otimes id)\mathcal{F}] \\ &\mathcal{F}_{123} = [\mathcal{F} \otimes 1 \otimes 1][(\Delta_0 \otimes id \otimes id)\mathcal{F} \otimes 1][(\Delta_0 \otimes id \otimes id)(\Delta_0 \otimes id)\mathcal{F}] \\ &... \end{aligned} \tag{25}$$

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Then define

$$\pi(s_i) = \mathcal{F}_{12...n}\pi_0(s_i)\mathcal{F}_{12...n}^{-1} = \mathcal{F}_i\pi_0(s_i)\mathcal{F}_i^{-1} \ , \qquad (26)$$

where

$$\mathcal{F}_i = \underbrace{1 \otimes ...1}_{i-1} \otimes \mathcal{F} \otimes \underbrace{1 \otimes ...1}_{n-i-1} \ . \qquad (27)$$

$\pi(s_i)$, $i \in \{1, ...n\}$, form a realization of S_n

The symmetric group in κ-Fock space (continued)

Explicitly on an n-fold tensor product of single particle states we have

$$\pi(s_i)|\vec{p}_1\rangle \otimes ...|\vec{p}_i\rangle \otimes |\vec{p}_{i+1}\rangle \otimes ...|\vec{p}_n\rangle = |\vec{p}_1\rangle \otimes ...|e^{-\omega(\vec{p}_i)/\kappa}\,\vec{p}_{i+1}\rangle \otimes |e^{\omega(\vec{p}_{i+1})/\kappa}\vec{p}_i\rangle \otimes ...|\vec{p}_n\rangle$$

- Same as AM symmetrization but off-shell
    - Need to go off-shell is a result of the impossibility of writing the total momentum as abelian sum of on-shell momenta
- Valid for any $\beta \in \mathbb{R}$
    - Symmetric part of the twist does not contribute to symmetrization. One more step towards base independence.
- Valid for massive case

creation/annihilation operators

As usual we define a vacuum state $|0\rangle$ such that $a^\dagger_{\vec{p}}|0\rangle = |\vec{p}\rangle$ and $a_{\vec{p}}|0\rangle = 0$. With the symmetrization given above we find

$$a^\dagger_{\vec{p}}\, a^\dagger_{\vec{q}} - a^\dagger_{e^{-\omega(\vec{p})/\kappa}\vec{q}}\, a^\dagger_{e^{\omega(\vec{q})/\kappa}\vec{p}} = a^\dagger_{\vec{p}}\, a^\dagger_{\vec{q}} - \mathcal{R}^{-1} a^\dagger_{\vec{q}}\, a^\dagger_{\vec{p}} = 0\ . \tag{28}$$

$$a_{\vec{p}}\, a_{\vec{q}} - a_{e^{\omega(\vec{p})/\kappa}\vec{q}}\, a_{e^{-\omega(\vec{q})/\kappa}\vec{p}} = a_{\vec{p}}\, a_{\vec{q}} - \mathcal{R} a_{\vec{q}}\, a_{\vec{p}} = 0 \tag{29}$$

$$a_{\vec{p}}\, a^\dagger_{\vec{q}} - a^\dagger_{e^{\omega(\vec{p})/\kappa}\vec{q}}\, a_{e^{\omega(\vec{q})/\kappa}\vec{p}} = a_{\vec{p}}\, a^\dagger_{\vec{q}} - \mathcal{F}_{21}\mathcal{F} a^\dagger_{\vec{q}}\, a_{\vec{p}} = \delta_{pq}\ . \tag{30}$$

Similar to the relations found by Daszkiewicz, Lukierski, Woronowicz.
We have cast them in a form which resembles the case of standard quantum groups, having used the "R-matrix"

$$\mathcal{R} = \mathcal{F}_{21}\mathcal{F}^{-1} = e^{-\frac{1}{\kappa}P_0 \wedge P_j \frac{\partial}{\partial P_j}}\ . \tag{31}$$

The quantum Yang-Baxter equation $\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}$ is trivially satisfied by (31) because $[P_0, P_j \frac{\partial}{\partial P_j}] = 0$
This together with $\mathcal{R}^{-1} = \mathcal{R}_{21}$ gives us another demonstration of the fact that our twisted permutations (26) are a realizations of S_n, once we rewrite them as

$$\pi(s_i) = \pi_0(s_i)\mathcal{R}_{ii+1}\ . \tag{32}$$

The "covariant" construction

A symmetrization that besides momenta and rotations intertwines also with boosts is being constructed perturbatively in $1/\kappa$ by Young and Zegers

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To second order the resulting “flip” in the case of $n = 2$ particles can be interpreted as

$$\pi_\kappa(s_1)|p_1\rangle \otimes |p_2\rangle = \pi_0(s_1)\mathcal{R}|p_1\rangle \otimes |p_2\rangle$$

with

$$\mathcal{R} = e^{\frac{1}{\kappa}(\bar{N}_j \otimes \bar{P}_j - \bar{P}_j \otimes \bar{N}_j)} + O(\frac{1}{\kappa^3}) = e^{\frac{1}{\kappa}((N_j + \frac{\epsilon_{jlm}}{2\kappa} M_l P_m) e^{\frac{P_0}{2\kappa}} \wedge P_j e^{\frac{P_0}{2\kappa}})} + O(\frac{1}{\kappa^3}) \ .$$

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As first shown by Zakrzewski $r = \bar{N}_j \otimes \bar{P}_j - \bar{P}_j \otimes \bar{N}_j$ is the classical r-matrix associated with κ-Poincaré but it does not satisfy the classical Yang-Baxter equation (it does satisfy the modified one),

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→ hence $\mathcal{R}$ does not satisfy the quantum YB equation (it is not a quasitriangular structure),

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with

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As first shown by Zakrzewski $r = \bar{N}_j \otimes \bar{P}_j - \bar{P}_j \otimes \bar{N}_j$ is the classical r-matrix associated with κ-Poincaré but it does not satisfy the classical Yang-Baxter equation (it does satisfy the modified one),
→ hence $\mathcal{R}$ does not satisfy the quantum YB equation (it is not a quasitriangular structure),
→ hence the naive n-particle extension $\pi_\kappa(s_i) = \pi_0(s_i)\mathcal{R}_{ii+1}$ does not satisfy the braid relation of S_n (precisely at order $1/\kappa^2$).
This is of course in agreement with the finding of a one-paramter family of realizations of S_3 by Young and Zegers.

Some comments on covariance of multiparticle states

Questions remain open on the covariant symmetrization:

- Is it extendible to all orders?
- Is the ambiguity for $n > 2$ solvable?
- Is $\pi_\kappa(s_i)\pi_\kappa(s_j) = \pi_\kappa(s_j)\pi_\kappa(s_i)$ for $|i - j| > 1$ satisfied?

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It would be nice to have a concrete model shedding some light on the physics of non-cocommutativty (similar to holonomies in $2d$)

Conclusions and outlook

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- Highlighted relations with different realizations of $\star$-product
- Using twist we constructed a symmetrization valid for different bases and massive case

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- Some progress but problems remain open as usual