CURVATURE FROM QUANTUM DEFORMATIONS

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I apologize but ...

This will be a very **commutative** talk ... (only one commutation rule throughout)

but a very **non-cocommutative** one,

because

Physics with **Lie group invariance** is deeply **non-cocommutative**

and, therefore, it becomes **noncommutative Physics** in the **dual**.

Main ideas

- **Hopf algebra symmetry** $\longrightarrow$ **Liouville (super)integrability**

- In general: noncommutativity $\longrightarrow$ curvature

At the "semiclassical" (**dual Poisson-Lie group**) level:
Hopf algebra deformations $\longrightarrow$ **noncocommutativity** $\longrightarrow$ **curvature**

Curved (pseudo)-riemannian spaces can be obtained from quantum deformations

The **(non-constant) curvature** is a smooth function of the deformation parameter

The coproduct allows this construction **for arbitrary dimension N**

- **ND curved spaces as geodesic flows on a (2+1) PL κ-Minkowski group**

Summary

1. DYNAMICAL COALGEBRA SYMMETRY AND (SUPER)INTEGRABILITY
2. HAMILTONIANS WITH $sl(2)$ COALGEBRA SYMMETRY
   - N-dimensional flat motion
3. HAMILTONIANS WITH $sl_z(2)$ COALGEBRA SYMMETRY
   - Deformed geodesic dynamics: the 2D case
4. QUANTUM DEFORMATIONS AND 2D CURVED SPACES
   - Spaces with non-constant curvature
   - Spaces with constant curvature
5. THE 3D CASE
   - Spaces with non-constant curvature
   - Spaces with constant curvature
6. CURVED SPACES AND PL DYNAMICS ON κ-MINKOWSKI GROUP.
7. CONCLUDING REMARKS

Poisson coalgebras

► A **coalgebra** (A, Δ) is a (unital, associative) algebra A endowed with a **coproduct map**

$$\Delta : A \to A \otimes A$$

which is both **coassociative**

$$(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta$$

and a **homomorphism** on A

$$\Delta(a\, b) = \Delta(a)\, \Delta(b) \qquad \forall a, b \in A.$$

therefore,

$$\Delta([a, b]_A) = [\Delta(a), \Delta(b)]_{A \otimes A} \qquad \forall a, b \in A$$

Moreover, if A is a Poisson algebra and

$$\boxed{\Delta(\{a, b\}_A) = \{\Delta(a), \Delta(b)\}_{A \otimes A} \qquad \forall a, b \in A}$$

we shall say that A is a **Poisson coalgebra**.

▶ The following diagram is **commutative**:

$$\begin{tikzcd}
 & A \otimes A \arrow[dr, "\Delta \otimes id"] & \\
A \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] & & A \otimes A \otimes A \\
 & A \otimes A \arrow[ur, "id \otimes \Delta"'] &
\end{tikzcd}$$

- **Example:** The UE algebra $U(\mathfrak{g})$ of any Lie algebra $\mathfrak{g}$ is a Hopf algebra:

$$\Delta_0(X) = 1 \otimes X + X \otimes 1,$$
$$\Delta_0(1) = 1 \otimes 1.$$

(This is a *non-conmutative* and *cocommutative* Hopf algebra)

- A 'two-fold way' for the definition of the **N-th coproduct** $\Delta^{(N)}$ by recurrence:

$$\Delta_R^{(N)} := (id \otimes \ldots^{N-2)} \otimes id \otimes \Delta^{(2)}) \circ \Delta^{(N-1)}.$$
$$\Delta^{(N)} := (\Delta^{(2)} \otimes id \otimes \ldots^{N-2)} \otimes id\,) \circ \Delta^{(N-1)}.$$

Poisson coalgebras and superintegrability

- **Lemma.**

$$\left[\Delta^{(m)}(C), \Delta^{(N)}(X_i)\right\}_{A\otimes A\otimes \ldots^{N)}\otimes A} = 0, \quad i = 1, \ldots, l, \quad 1 \leq m \leq N.$$

X_1	X_2	$\ldots$	X_l	$\mathcal{C}$
$\Delta^{(2)}(X_1)$	$\Delta^{(2)}(X_2)$	$\ldots$	$\Delta^{(2)}(X_l)$	$\Delta^{(2)}(\mathcal{C})$
$\Delta^{(3)}(X_1)$	$\Delta^{(3)}(X_2)$	$\ldots$	$\Delta^{(3)}(X_l)$	$\Delta^{(3)}(\mathcal{C})$
$\vdots$	$\vdots$		$\vdots$	$\vdots$
$\Delta^{(N)}(X_1)$	$\Delta^{(N)}(X_2)$	$\ldots$	$\Delta^{(N)}(X_l)$	$\Delta^{(N)}(\mathcal{C})$

- **Proposition.**[1] *The N-sites Hamiltonian*

$$H^{(N)} := \Delta^{(N)}(\mathcal{H}(X_1, \ldots, X_l)) = \mathcal{H}(\Delta^{(N)}(X_1), \ldots, \Delta^{(N)}(X_l)),$$

is in involution with the $C^{(m)}$ functions

$$\left[C^{(m)}, H^{(N)}\right\}_{A\otimes A\otimes \ldots^{N)}\otimes A} = 0, \qquad 1 \leq m \leq N,$$

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[1] A. B. and O. Ragnisco, *J. Phys. A: Math. Gen.* **31**, 3791 (1998).

where

$$C^{(m)} := \Delta^{(m)}(C(X_1, \ldots, X_l)) = C(\Delta^{(m)}(X_1), \ldots, \Delta^{(m)}(X_l)).$$

- **Corollary.** *The Casimirs* $C^{(m)}$ *are in involution*

$$\left[\Delta^{(k)}(C), \Delta^{(j)}(C)\right\} = 0, \qquad \forall\, k, j.$$

- **Functional independence** is guaranteed.

- These results hold simultaneously for higher rank coalgebras with **several functionally independent Casimir elements C_i**.

- The complete integrability of the specific Hamiltonians for arbitrary rank coalgebras depends on the dimensionality of the chosen **symplectic realization.**[2]

- The construction is also **valid for non-commutative coalgebras** (Lie coalgebras and quantum algebras).

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[2] A. B. and A. Blasco, *J. Phys. A: Math. Theor.*, to appear (2008).

• COALGEBRA SYMMETRY AND QMS: (2 N -3) integrals in involution [3]

“LEFT” set of $(N-1)$ integrals $C^{(m)}$:

$$
\begin{array}{lll}
C^{(2)} & \equiv \Delta^{(2)}(\mathcal{C}) & 1\otimes 2 \\
C^{(3)} & \equiv \Delta^{(3)}(\mathcal{C}) & 1\otimes 2\otimes 3 \\
\vdots & & \vdots \\
C^{(N-1)} & \equiv \Delta^{(N-1)}(\mathcal{C}) & 1\otimes 2\otimes\cdots\otimes(N-1) \\
C^{(N)} & \equiv \Delta^{(N)}(\mathcal{C}) & 1\otimes 2\otimes\cdots\otimes(N-1)\otimes N
\end{array}
$$

“RIGHT” set of $(N-1)$ integrals $C_{(m)}$:

$$
\begin{array}{llr}
C_{(2)} & \equiv \Delta_R^{(2)}(\mathcal{C}) & (N-1)\otimes N \\
C_{(3)} & \equiv \Delta_R^{(3)}(\mathcal{C}) & (N-2)\otimes(N-1)\otimes N \\
\vdots & & \vdots \\
C_{(N-1)} & \equiv \Delta_R^{(N-1)}(\mathcal{C}) & 2\otimes 3\otimes\cdots\otimes(N-1)\otimes N \\
C_{(N)} = C^{(N)} & \equiv \Delta_R^{(N)}(\mathcal{C}) & 1\otimes 2\otimes 3\otimes\cdots\otimes(N-1)\otimes N
\end{array}
$$

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[3] A. B., F.J. Herranz, F. Musso, and O. Ragnisco: in *Superintegrability in Classical and Quantum Systems*, CRM Proceedings and Lecture Notes, vol. 37, ed P. Tempesta *et al* (Providence, RI: AMS), 1 (2004).

- $N=2$: one constant $\equiv$ **integrability**: $C^{(2)} \equiv C_{(2)}$
- $N=3$: three constants

$$\begin{array}{ccc} & C^{(3)} \equiv C_{(3)} & \\ C^{(2)} & & C_{(2)} \end{array}$$

- $N=4$: five constants

$$\begin{array}{ccccc} & & C^{(4)} \equiv C_{(4)} & & \\ & C^{(3)} & & C_{(3)} & \\ C^{(2)} & & & & C_{(2)} \end{array}$$

- N **arbitrary:** $(2N-3)$ **constants** $\longrightarrow$ **quasi-maximal superintegrability (QMS)**

$$\begin{array}{ccccccc} & & & C^{(N)} \equiv C_{(N)} & & & \\ & & C^{(N-1)} & & C_{(N-1)} & & \\ & & \vdots & & \vdots & & \\ & C^{(3)} & & & & C_{(3)} & \\ C^{(2)} & & & & & & C_{(2)} \end{array}$$

Hamiltonians with $sl(2)$–Poisson coalgebra symmetry

- **Poisson coalgebra**:

$$\{J_3, J_+\} = 2J_+ \qquad \{J_3, J_-\} = -2J_- \qquad \{J_-, J_+\} = 4J_3$$

$$\Delta^{(2)}(J_i) = 1 \otimes J_i + J_i \otimes 1$$

- **Casimir function**:

$$\mathcal{C} = J_3^2 - J_- J_+$$

- **Symplectic realization**: $\qquad D(\mathcal{C}) = -b$

$$J_-^{(1)} = D(J_-) = q_1^2 \qquad J_+^{(1)} = D(J_+) = p_1^2 + \frac{b_1}{q_1^2} \qquad J_3^{(1)} = D(J_3) = q_1 p_1$$

- **Hamiltonian function**:

$$\boxed{\mathcal{H} = \frac{1}{2} J_+}$$

$$H^{(1)} = D(\mathcal{H}) = \frac{1}{2} p_1^2 + \frac{b_1}{2q_1^2}.$$

- **2-particle realization**:

$$J_-^{(2)} = (D\otimes D)(\Delta^{(2)}(J_-)) = q_1^2 + q_2^2$$
$$J_+^{(2)} = (D\otimes D)(\Delta^{(2)}(J_+)) = p_1^2 + p_2^2 + \frac{b_1}{q_1^2} + \frac{b_2}{q_2^2}$$
$$J_3^{(2)} = (D\otimes D)(\Delta^{(2)}(J_3)) = q_1p_1 + q_2p_2$$

- **2-particle** Hamiltonian:

$$H^{(2)} = (D\otimes D)(\Delta^{(2)}(\mathcal{H})) = \frac{1}{2}J_+^{(2)} = \sum_{i=1}^{2}\left(\frac{1}{2}p_i^2 + \frac{b_i}{2q_i^2}\right)$$

- **Constant of the motion**:

$$C^{(2)} = (D\otimes D)(\Delta^{(2)}(\mathcal{C})) = -(q_1p_2 - q_2p_1)^2 - \left(b_1\frac{q_2^2}{q_1^2} + b_2\frac{q_1^2}{q_2^2}\right) - \sum_{i=1}^{2} b_i$$

- **k-particle realization**:

$$J_-^{(k)} = (D \otimes D \otimes \ldots^{k)} \otimes D)(\Delta^{(k)}(J_-)) = \sum_{i=1}^{k} q_i^2$$

$$J_+^{(k)} = (D \otimes D \otimes \ldots^{k)} \otimes D)(\Delta^{(k)}(J_+)) = \sum_{i=1}^{k} \left(p_i^2 + \frac{b_i}{q_i^2} \right)$$

$$J_3^{(k)} = (D \otimes D \otimes \ldots^{k)} \otimes D)(\Delta^{(k)}(J_3)) = \sum_{i=1}^{k} q_i p_i$$

- **N-particle** Hamiltonian:

$$H^{(N)} = \frac{1}{2} J_+^{(N)} = \frac{1}{2} (D \otimes D \otimes \ldots^{N)} \otimes D)(\Delta^{(N)}(J_+)) = \sum_{i=1}^{N} \left(\frac{1}{2} p_i^2 + \frac{b_i}{2 q_i^2} \right)$$

- **Constants of the motion (left)** in involution: $(m = 2, \ldots, N)$

$$C^{(m)} = (D \otimes D \otimes \ldots^{m)} \otimes D)(\Delta^{(m)}(\mathcal{C})) = \left(J_3^{(m)} \right)^2 - J_-^{(m)} J_+^{(m)}$$

$$C^{(m)} = -\sum_{i<j}^{m} \left\{ (q_i p_j - q_j p_i)^2 + \left(b_i \frac{q_j^2}{q_i^2} + b_j \frac{q_i^2}{q_j^2} \right) \right\} - \sum_{i=1}^{m} b_i = -\sum_{i<j}^{m} Q_{ij} - \sum_{i=1}^{m} b_i$$

Euclidean dynamics in N dimensions

- **Free Euclidean motion on $\mathbf{E}^N$**

$$\mathcal{H} = \frac{1}{2} J_+$$

under the symplectic realization $b_i = 0$:

$$\mathcal{H}^{(N)} = \frac{1}{2} \sum_{i=1}^{N} p_i^2 .$$

- **A more general superintegrable Hamiltonian**. Let us consider

$$\mathcal{H} = \frac{1}{2} J_+ + \mathcal{F}(J_-)$$

where $\mathcal{F}(J_-)$ is an **arbitrary smooth function** of J_-:

$$H^{(N)} = \frac{1}{2} J_+^{(N)} + \mathcal{F}(J_-^{(N)}) = \sum_{i=1}^{N} \left(\frac{1}{2} p_i^2 + \frac{b_i/2}{q_i^2} \right) + \mathcal{F}\left(\sum_{i=1}^{N} q_i^2 \right)$$

- Right integrals complete the set of $(2N-2)$ constants of the motion.

EXAMPLES OF QMS SYSTEMS WITH COALGEBRA SYMMETRY:

- **ND Evans system.**[4] Arbitrary $\mathcal{F}(J_-)$:

$$H^{(N)} = \frac{1}{2}J_+^{(N)} + \mathcal{F}(J_-^{(N)}) = \sum_{i=1}^{N}\left(\frac{1}{2}p_i^2 + \frac{b_i/2}{q_i^2}\right) + \mathcal{F}\left(\sum_{i=1}^{N} q_i^2\right)$$

- **ND Smorodinsky–Winternitz system.**[5] $\mathcal{F}\left(J_-\right) = \omega^2 J_-$:

$$H^{(N)} = \sum_{i=1}^{N}\left(\frac{1}{2}p_i^2 + \frac{b_i/2}{q_i^2}\right) + \omega^2\left(\sum_{i=1}^{N} q_i^2\right)$$

- **ND degenerate ($a_i = \omega^2$) Garnier system.**[6] [7] $\mathcal{F}\left(J_-\right) = \omega^2 J_- + J_-^2$:

$$H^{(N)} = \sum_{i=1}^{N}\left(\frac{1}{2}p_i^2 + \frac{b_i/2}{q_i^2}\right) + \omega^2\left(\sum_{i=1}^{N} q_i^2\right) + \left(\sum_{i=1}^{N} q_i^2\right)^2$$

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[4] N.W. Evans, J. Math. Phys. 32, 3369 (1991).

[5] J. Fris, V. Mandrosov, Ya A. Smorodinsky, M. Uhlir, P. Winternitz, Phys. Lett. 16, 354 (1965). Note that one more integral is left, and it cannot be obtained from coalgebra symmetry.

[6] R. Garnier, Rend. Circ. Math. Palermo 43, 155 (1919).

[7] S. Wojciechowski, Physica Scripta 31, 433 (1985).

Hamiltonians with $sl_z(2)$–Poisson coalgebra symmetry

- **Poisson coalgebra**: a Poisson non-standard deformation of $sl(2)$[8]

$$\begin{aligned}\{J_3, J_+\} &= 2J_+ \cosh zJ_- \\ \{J_3, J_-\} &= -2\frac{\sinh zJ_-}{z} \\ \{J_-, J_+\} &= 4J_3\end{aligned}$$

$$\begin{aligned}\Delta(J_-) &= 1 \otimes J_- + J_- \otimes 1 \\ \Delta(J_+) &= e^{-zJ_-} \otimes J_+ + J_+ \otimes e^{zJ_-} \\ \Delta(J_3) &= e^{-zJ_-} \otimes J_3 + J_3 \otimes e^{zJ_-}\end{aligned}$$

- **Casimir function**:

$$\mathcal{C}_z = J_3^2 - \frac{\sinh zJ_-}{z} J_+$$

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[8] This is a Poisson version of C. Ohn, Lett. Math. Phys. 25, 85 (1992). Their universal quantum R-matrix was given in A.B. and F.J. Herranz, J. Phys. A: Math. Gen. 29, L311 (1996).

- **Symplectic realization**: $C_z^{(1)} = D_z(\mathcal{C}_z) = -b_1$

$$
\begin{aligned}
J_-^{(1)} &= D_z(J_-) = q_1^2 \\
J_+^{(1)} &= D_z(J_+) = \frac{\sinh z q_1^2}{z q_1^2} p_1^2 + \frac{z b_1}{\sinh z q_1^2} \\
J_3^{(1)} &= D_z(J_3) = \frac{\sinh z q_1^2}{z q_1^2} q_1 p_1
\end{aligned}
$$

- **Hamiltonian function**: let us take again

$$
\boxed{\mathcal{H} = \frac{1}{2} J_+}
$$

- **One-particle Hamiltonian**:

$$
H_z^{(1)} = D_z(\mathcal{H}) = \frac{1}{2} J_+^{(1)} = \frac{1}{2} \frac{\sinh z q_1^2}{z q_1^2} p_1^2 + \frac{1}{2} \frac{z b_1}{\sinh z q_1^2}
$$

- **2-particle realization**:

$$J_-^{(2)} = (D_z \otimes D_z)(\Delta_z^{(2)}(J_-)) = q_1^2 + q_2^2$$

$$J_+^{(2)} = (D_z \otimes D_z)(\Delta_z^{(2)}(J_+)) = \sum_{i=1}^{2} \left(\frac{\sinh z q_i^2}{z q_i^2} p_i^2 + \frac{z b_i}{\sinh z q_i^2} \right) e^{z K_i^{(2)}(q^2)}$$

$$J_3^{(2)} = (D_z \otimes D_z)(\Delta_z^{(2)}(J_3)) = \sum_{i=1}^{2} \frac{\sinh z q_i^2}{z q_i^2} q_i p_i e^{z K_i^{(2)}(q^2)}$$

where $K_1^{(2)}(q^2) = q_2^2$ and $K_2^{(2)}(q^2) = -q_1^2$

- **2-particle** Hamiltonian:

$$H_z^{(2)} = (D_z \otimes D_z)(\Delta^{(2)}(\mathcal{H})) = \frac{1}{2} J_+^{(2)}$$

$$H_z^{(2)} = \sum_{i=1}^{2} \frac{1}{2} \left(\frac{\sinh z q_i^2}{z q_i^2} p_i^2 + \frac{z b_i}{\sinh z q_i^2} \right) e^{z K_i^{(2)}(q^2)}$$

Note that the **deformation destroys separability**[9].

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[9] A. B. and F.J. Herranz, *J. Phys. A: Math. Gen.* **32**, 8851 (1999).

- **Constant of the motion**:

$$C_z^{(2)} = (D_z \otimes D_z)(\Delta^{(2)}(\mathcal{C})) = (J_3^{(2)})^2 - \frac{\sinh z J_-^{(2)}}{z} J_+^{(2)}$$

$$\begin{aligned} C_z^{(2)} = & -\frac{\sinh z q_1^2}{z q_1^2}\frac{\sinh z q_2^2}{z q_2^2} (q_1 p_2 - q_2 p_1)^2 \, e^{z(q_2^2 - q_1^2)} - b_1 e^{2z q_2^2} - b_2 e^{-2z q_1^2} \\ & - \left(b_1 \frac{\sinh z q_2^2}{\sinh z q_1^2} + b_2 \frac{\sinh z q_1^2}{\sinh z q_2^2} \right) e^{z(q_2^2 - q_1^2)} . \end{aligned}$$

- **m-th coproduct**:

$$\begin{aligned} \Delta^{(m)}(J_-) = & \, J_- \otimes 1 \otimes 1 \otimes \ldots^{m-1)} \otimes 1 \\ & +1 \otimes J_- \otimes 1 \otimes \ldots^{m-2)} \otimes 1 + \ldots \\ & +1 \otimes 1 \otimes \ldots^{m-1)} \otimes 1 \otimes J_- \end{aligned}$$

$$\begin{aligned} \Delta^{(m)}(J_i) = & \, J_i \otimes e^{z\,J_-} \otimes e^{z\,J_-} \otimes \ldots^{m-1)} \otimes e^{z\,J_-} \\ & +e^{-z\,J_-} \otimes J_i \otimes e^{z\,J_-} \otimes \ldots^{m-2)} \otimes e^{z\,J_-} + \ldots \\ & +e^{-z\,J_-} \otimes e^{-z\,J_2} \otimes \ldots^{m-1)} \otimes e^{-z\,J_-} \otimes J_i , \qquad i = +, 3. \end{aligned}$$

- m**-particle realization**:

$$J_-^{(m)} = \sum_{i=1}^{m} q_i^2$$

$$J_+^{(m)} = \sum_{i=1}^{m} \left(\frac{\sinh z q_i^2}{z q_i^2} p_i^2 + \frac{z b_i}{\sinh z q_i^2} \right) e^{z K_i^{(m)}(q^2)}$$

$$J_3^{(m)} = \sum_{i=1}^{m} \frac{\sinh z q_i^2}{z q_i^2} q_i p_i e^{z K_i^{(m)}(q^2)}$$

where $$K_i^{(m)}(x) = -\sum_{k=1}^{i-1} x_k + \sum_{l=i+1}^{m} x_l$$ and

$$K_{ij}^{(m)}(x) = K_i^{(m)}(x) + K_j^{(m)}(x) = -2 \sum_{k=1}^{i-1} x_k - x_i + x_j + 2 \sum_{l=j+1}^{m} x_l \qquad (i < j)$$

- N**-particle** Hamiltonian:

$$H_z^{(N)} = \frac{1}{2} J_+^{(N)} = \sum_{i=1}^{N} \frac{1}{2} \left(\frac{\sinh z q_i^2}{z q_i^2} p_i^2 + \frac{z b_i}{\sinh z q_i^2} \right) e^{z K_i^{(N)}(q^2)}$$

- **Constants of the motion** in involution:

$$C_z^{(m)} = -\sum_{i<j}^{m} Q_{ij}^z \, e^{zK_{ij}^{(m)}(q^2)} - \sum_{i=1}^{m} b_i e^{2zK_i^{(m)}(q^2)}$$

where

$$Q_{ij}^z = \left\{ \frac{\sinh zq_i^2}{zq_i^2} \frac{\sinh zq_j^2}{zq_j^2} (q_i p_j - q_j p_i)^2 + \left(b_i \frac{\sinh zq_j^2}{\sinh zq_i^2} + b_j \frac{\sinh zq_i^2}{\sinh zq_j^2} \right) \right\} .$$

- A more general **deformed system** with $\mathcal{F}(J_-)$ an **arbitrary potential** function:

$$\mathcal{H} = \frac{1}{2} J_+ + \mathcal{F}(J_-)$$

$$H_z^{(N)} = \frac{1}{2} J_+^{(N)} + \mathcal{F}(J_-^{(N)}) = \sum_{i=1}^{N} \frac{1}{2} \left(\frac{\sinh zq_i^2}{zq_i^2} p_i^2 + \frac{zb_i}{\sinh zq_i^2} \right) e^{zK_i^{(N)}(q^2)} + \mathcal{F}\left(\sum_{i=1}^{N} q_i^2 \right)$$

- Thus we have obtained N-dimensional **QMS deformations** of the **Evans, Smorodinski-Winternitz and degenerate Garnier** systems.

Deformed geodesic dynamics: the 2D case

- Since in the following we shall consider **only free motion**, we take $b_i = 0$.
- **One-particle symplectic realization** of $sl_z(2)$:

$$J_-^{(1)} = q_1^2$$

$$J_+^{(1)} = \frac{\sinh zq_1^2}{zq_1^2}\, p_1^2$$

$$J_3^{(1)} = \frac{\sinh zq_1^2}{zq_1^2}\, q_1 p_1$$

where $\mathcal{C}_z^{(1)} = 0$. Hence dimensions of z are $[z] = [q_1]^{-2} = [J_-]^{-1}$.

- **Two-particle symplectic realization**:

$$J_-^{(2)} = q_1^2 + q_2^2$$

$$J_+^{(2)} = \frac{\sinh zq_1^2}{zq_1^2}\, \mathrm{e}^{zq_2^2} p_1^2 + \frac{\sinh zq_2^2}{zq_2^2}\, \mathrm{e}^{-zq_1^2} p_2^2$$

$$J_3^{(2)} = \frac{\sinh zq_1^2}{zq_1^2}\, \mathrm{e}^{zq_2^2} q_1 p_1 + \frac{\sinh zq_2^2}{zq_2^2}\, \mathrm{e}^{-zq_1^2} q_2 p_2$$

- **Two-particle Casimir**:

$$\mathcal{C}_z^{(2)} = \Delta_z(\mathcal{C}_z) = \frac{\sinh zq_1^2}{zq_1^2}\,\frac{\sinh zq_2^2}{zq_2^2}\,\mathrm{e}^{-zq_1^2}\mathrm{e}^{zq_2^2}\,(q_1p_2 - q_2p_1)^2\,,$$

which is a constant of the motion for any Hamiltonian $\mathcal{H}_z^{(2)} = \Delta_z(\mathcal{H}) = \mathcal{H}(J_-^{(2)}, J_+^{(2)}, J_3^{(2)})$.

- The most general ***integrable*** (and quadratic in the momenta) **deformation of the free motion on E^2** with $(sl_z(2), \Delta_z)$-symmetry is obtained through

$$\boxed{\mathcal{H} = \frac{1}{2} J_+ f(zJ_-)}$$

where f is any smooth function such that $\lim_{z\to 0} f(zJ_-) = 1$.

- The simplest choice:

$$\boxed{\mathcal{H} = \frac{1}{2} J_+}$$

$$\mathcal{H}_z^{\mathrm{I}} = \frac{1}{2}\left(\frac{\sinh zq_1^2}{zq_1^2}\,\mathrm{e}^{zq_2^2}p_1^2 + \frac{\sinh zq_2^2}{zq_2^2}\,\mathrm{e}^{-zq_1^2}p_2^2\right).$$

- **A *(maximally superintegrable)* deformation of the 2D free motion** is given by

$$\boxed{\mathcal{H} = \frac{1}{2} J_+ \mathrm{e}^{zJ_-}}$$

$$\mathcal{H}_z^{\mathrm{S}} = \frac{1}{2}\left(\frac{\sinh z q_1^2}{z q_1^2}\, \mathrm{e}^{z q_1^2} \mathrm{e}^{2 z q_2^2} p_1^2 + \frac{\sinh z q_2^2}{z q_2^2}\, \mathrm{e}^{z q_2^2} p_2^2 \right) .$$

since there exists an additional (and independent) constant of the motion:[10]

$$\mathcal{I}_z = \frac{\sinh z q_1^2}{2 z q_1^2}\, \mathrm{e}^{z q_1^2} p_1^2 .$$

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[10] A. B. and F.J. Herranz, *J. Phys. A: Math. Gen.* **32**, 8851 (1999).

Quantum deformations and 2D curved spaces

1. The case $\mathcal{H} = \frac{1}{2} J_+$: non-constant curvature

- The **kinetic energy** coming from this Hamiltonian is[11]

$$\mathcal{T}_z^{\mathrm{I}}(q_i, \dot{q}_i) = \frac{1}{2}\left(\frac{z q_1^2}{\sinh z q_1^2}\, \mathrm{e}^{-z q_2^2} \dot{q}_1^2 + \frac{z q_2^2}{\sinh z q_2^2}\, \mathrm{e}^{z q_1^2} \dot{q}_2^2 \right),$$

that defines a geodesic flow on a 2D Riemannian space with signature diag$(+,+)$:

$$\boxed{\mathrm{d}s_I^2 = \frac{2 z q_1^2}{\sinh z q_1^2}\, \mathrm{e}^{-z q_2^2}\, \mathrm{d}q_1^2 + \frac{2 z q_2^2}{\sinh z q_2^2}\, \mathrm{e}^{z q_1^2}\, \mathrm{d}q_2^2}$$

- The **Gaussian curvature** K is **non-constant and *negative***:

$$K = -z \sinh\left(z (q_1^2 + q_2^2) \right).$$

Thus, the 2D space generated through the geodesic flow given by the **coproduct** is of **hyperbolic type** and endowed with a **“radial” symmetry**.

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[11] A. B., F.J. Herranz and O. Ragnisco, *Phys. Lett. B* **610**, 107 (2005).

- **A change of coordinates** including another parameter $\lambda_2 \neq 0$:

$$\cosh(\lambda_1 \rho) = \exp\left\{z(q_1^2 + q_2^2)\right\}, \qquad \sin^2(\lambda_2 \theta) = \frac{\exp\left\{2zq_1^2\right\} - 1}{\exp\left\{2z(q_1^2 + q_2^2)\right\} - 1},$$

where both $\lambda_1 = \sqrt{z}$ and λ_2 can take either a real or a pure imaginary value.

- **First-order terms** of the expansion in z:

$$\rho^2 \simeq 2(q_1^2 + q_2^2), \qquad \sin^2(\lambda_2 \theta) \simeq \frac{q_1^2}{q_1^2 + q_2^2}.$$

Thus ρ is a **radial coordinate** and θ is a **either circular or hyperbolic angle**.

- The metric is :

$$\boxed{ds_I^2 = \frac{1}{\cosh(\lambda_1 \rho)} \left(d\rho^2 + \lambda_2^2 \frac{\sinh^2(\lambda_1 \rho)}{\lambda_1^2} d\theta^2 \right) = \frac{1}{\cosh(\lambda_1 \rho)} ds_0^2}$$

- Where ds_0^2 is just the **metric of the 2D Cayley-Klein spaces** in terms of geodesic polar coordinates,[12] [13] provided that we identify $z = \lambda_1^2 \equiv -\kappa_1$ and $\lambda_2^2 \equiv \kappa_2$.

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[12] F.J. Herranz, Ph.D. Thesis, Universidad de Valladolid (1995).

[13] F.J. Herranz, R. Ortega and M. Santander, *J. Phys. A: Math. Gen.* **33**, 4525 (2000).

▶ We find the **following spaces**:

- When λ_2 is real, we get a 2D **“deformed sphere”** $\mathbf{S}^2_z$ $(z < 0)$, and a **deformed hyperbolic or Lobachewski space** $\mathbf{H}^2_z$ $(z > 0)$.
- When λ_2 is imaginary, we obtain a **“deformation of the (1+1)D anti-de Sitter spacetime”** $\mathbf{AdS}^{1+1}_z$ $(z < 0)$ and of the **de Sitter one** $\mathbf{dS}^{1+1}_z$ $(z > 0)$.
- In the non-deformed case $z \to 0$, the Euclidean space $\mathbf{E}^2$ (λ_2 real) and Minkowskian spacetime $\mathbf{M}^{1+1}$ (λ_2 imaginary) are recovered.

▶ The “additional” parameter λ_2 provides the signature of the metric.

▶ In the new coordinates, the **Gaussian curvature** reads

$$K(\rho) = -\frac{1}{2}\lambda_1^2\,\frac{\sinh^2(\lambda_1\rho)}{\cosh(\lambda_1\rho)} = -\frac{1}{2}\,z\,\frac{\sinh^2(\sqrt{z}\rho)}{\cosh(\sqrt{z}\rho)},$$

Table 1: **Metric and curvature of the curved spaces for different values of the deformation parameter $z = \lambda_1^2$ and signature parameter λ_2.**

2D deformed Riemannian spaces	$(1+1)$D deformed relativistic spacetimes
• **Deformed sphere $\mathbf{S}_z^2$**	• **Deformed anti-de Sitter spacetime $\mathbf{AdS}_z^{1+1}$**
$z = -1;\ (\lambda_1, \lambda_2) = (\mathrm{i}, 1)$	$z = -1;\ (\lambda_1, \lambda_2) = (\mathrm{i}, \mathrm{i})$
$\mathrm{d}s^2 = \dfrac{1}{\cos\rho}\left(\mathrm{d}\rho^2 + \sin^2\rho\, \mathrm{d}\theta^2\right)$	$\mathrm{d}s^2 = \dfrac{1}{\cos\rho}\left(\mathrm{d}\rho^2 - \sin^2\rho\, \mathrm{d}\theta^2\right)$
$K = -\dfrac{\sin^2\rho}{2\cos\rho}$	$K = -\dfrac{\sin^2\rho}{2\cos\rho}$
• Euclidean space $\mathbf{E}^2$	• Minkowskian spacetime $\mathbf{M}^{1+1}$
$z = 0;\ (\lambda_1, \lambda_2) = (0, 1)$	$z = 0;\ (\lambda_1, \lambda_2) = (0, \mathrm{i})$
$\mathrm{d}s^2 = \mathrm{d}\rho^2 + \rho^2 \mathrm{d}\theta^2$	$\mathrm{d}s^2 = \mathrm{d}\rho^2 - \rho^2 \mathrm{d}\theta^2$
$K = 0$	$K = 0$
• **Deformed hyperbolic space $\mathbf{H}_z^2$**	• **Deformed de Sitter spacetime $\mathbf{dS}_z^{1+1}$**
$z = 1;\ (\lambda_1, \lambda_2) = (1, 1)$	$z = 1;\ (\lambda_1, \lambda_2) = (1, \mathrm{i})$
$\mathrm{d}s^2 = \dfrac{1}{\cosh\rho}\left(\mathrm{d}\rho^2 + \sinh^2\rho\, \mathrm{d}\theta^2\right)$	$\mathrm{d}s^2 = \dfrac{1}{\cosh\rho}\left(\mathrm{d}\rho^2 - \sinh^2\rho\, \mathrm{d}\theta^2\right)$
$K = -\dfrac{\sinh^2\rho}{2\cosh\rho}$	$K = -\dfrac{\sinh^2\rho}{2\cosh\rho}$

- **Kinetic energy term** of $\mathcal{H}_z^{\rm I}$ in the new coordinates (ρ, θ):

$$\mathcal{T}_z^{\rm I}(\rho,\theta;\dot\rho,\dot\theta) = \frac{1}{2\cosh(\lambda_1\rho)}\left(\dot\rho^2 + \lambda_2^2\,\frac{\sinh^2(\lambda_1\rho)}{\lambda_1^2}\,\dot\theta^2\right).$$

- **New Hamiltonian**

$$\widetilde{H}_z^{\rm I} = \frac{1}{2}\cosh(\lambda_1\rho)\left(p_\rho^2 + \frac{\lambda_1^2}{\lambda_2^2\sinh^2(\lambda_1\rho)}\,p_\theta^2\right).$$

The corresponding constant of the motion is

$$\mathcal{C}_z^{(2)} \equiv \widetilde{C}_z = p_\theta^2$$

which allows the usual reduction:

$$\widetilde{H}_z^{\rm I} = \frac{1}{2}\cosh(\lambda_1\rho)\,p_\rho^2 + \frac{\lambda_1^2\cosh(\lambda_1\rho)}{2\lambda_2^2\sinh^2(\lambda_1\rho)}\,\widetilde{C}_z.$$

- **The explicit integration of the geodesic motion** on all these spaces can be explicitly performed in terms of elliptic integrals.

The constant curvature case

- **The superintegrable Hamiltonian**

$$\boxed{\mathcal{H} = \frac{1}{2} J_+ \mathrm{e}^{zJ_-}}$$

gives the kinetic energy

$$\mathcal{T}_z^{\mathrm{S}}(q_i, \dot{q}_i) = \frac{1}{2}\left(\frac{z q_1^2}{\sinh z q_1^2}\, \mathrm{e}^{-z q_1^2} \mathrm{e}^{-2 z q_2^2}\, \dot{q}_1^2 + \frac{z q_2^2}{\sinh z q_2^2}\, \mathrm{e}^{-z q_2^2}\, \dot{q}_2^2 \right).$$

The **metric** is now

$$\boxed{\mathrm{d}s_S^2 = \frac{2 z q_1^2}{\sinh z q_1^2}\, \mathrm{e}^{-z q_1^2} \mathrm{e}^{-2 z q_2^2}\, \mathrm{d}q_1^2 + \frac{2 z q_2^2}{\sinh z q_2^2}\, \mathrm{e}^{-z q_2^2}\, \mathrm{d}q_2^2}$$

- The **Gaussian curvature** K **for** $\mathrm{d}s_S^2$ **is *constant* and** $K = z$.
- Under the previous change of coordinates, $(q_1, q_2) \to (\rho, \theta)$, the metric is written as

$$\mathrm{d}s_S^2 = \frac{1}{\cosh^2(\lambda_1 \rho)} \left(\mathrm{d}\rho^2 + \lambda_2^2 \frac{\sinh^2(\lambda_1 \rho)}{\lambda_1^2}\, \mathrm{d}\theta^2 \right) = \frac{1}{\cosh^2(\lambda_1 \rho)}\, \mathrm{d}s_0^2,$$

- **A new radial coordinate:**

$$r = \int_0^{\rho} \frac{\mathrm{d}x}{\cosh(\lambda_1 x)},$$

By making use of the functional relations

$$\tanh\left(\lambda_1 \frac{\rho}{2}\right) = \tan\left(\lambda_1 \frac{r}{2}\right), \qquad \cosh(\lambda_1 \rho) = \frac{1}{\cos(\lambda_1 r)}, \qquad \sinh(\lambda_1 \rho) = \tan(\lambda_1 r),$$

we finally obtain

$$\mathrm{d}s_S^2 = \mathrm{d}r^2 + \lambda_2^2 \frac{\sin^2(\lambda_1 r)}{\lambda_1^2}\,\mathrm{d}\theta^2,$$

which is **just the CK metric written in geodesic polar coordinates** (r, θ) and provided that $z = \lambda_1^2 \equiv \kappa_1$ and $\lambda_2^2 \equiv \kappa_2$.

- The geodesic motion can be reduced to a "radial" 1D system from which the geodesic curves can be obtained. Namely,

$$\widetilde{H}_z^{\mathrm{S}} = \frac{1}{2}\,p_r^2 + \frac{\lambda_1^2}{2\lambda_2^2 \sin^2(\lambda_1 r)}\,\widetilde{C}_z.$$

Table 2: **Metric and curvature of the six spaces of constant curvature for $\lambda_1 = \sqrt{z} \in \{1, 0, \mathrm{i}\}$ and $\lambda_2 \in \{1, \mathrm{i}\}$**

2D Riemannian spaces	$(1+1)$D relativistic spacetimes
• **Sphere** $\mathbf{S}^2$: $(\lambda_1, \lambda_2) = (1, 1)$ $\mathrm{d}s^2 = \mathrm{d}r^2 + \sin^2 r \, \mathrm{d}\theta^2$ $K = 1$	• **Anti-de Sitter spacetime** $\mathbf{AdS}^{1+1}$: $(\lambda_1, \lambda_2) = (1, \mathrm{i})$ $\mathrm{d}s^2 = \mathrm{d}r^2 - \sin^2 r \, \mathrm{d}\theta^2$ $K = 1$
• **Euclidean space** $\mathbf{E}^2$: $(\lambda_1, \lambda_2) = (0, 1)$ $\mathrm{d}s^2 = \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2$ $K = 0$	• **Minkowskian spacetime** $\mathbf{M}^{1+1}$: $(\lambda_1, \lambda_2) = (0, \mathrm{i})$ $\mathrm{d}s^2 = \mathrm{d}r^2 - r^2 \mathrm{d}\theta^2$ $K = 0$
• **Hyperbolic space** $\mathbf{H}^2$: $(\lambda_1, \lambda_2) = (\mathrm{i}, 1)$ $\mathrm{d}s^2 = \mathrm{d}r^2 + \sinh^2 r \, \mathrm{d}\theta^2$ $K = -1$	• **De Sitter spacetime** $\mathbf{dS}^{1+1}$: $(\lambda_1, \lambda_2) = (\mathrm{i}, \mathrm{i})$ $\mathrm{d}s^2 = \mathrm{d}r^2 - \sinh^2 r \, \mathrm{d}\theta^2$ $K = -1$

The 3D case

- **Three-particle symplectic realization** (with all $b_i = 0$):

$$
\begin{aligned}
J_-^{(3)} &= q_1^2 + q_2^2 + q_3^2 \equiv \mathbf{q}^2 \\
J_+^{(3)} &= \frac{\sinh z q_1^2}{z q_1^2}\, p_1^2\, \mathrm{e}^{z q_2^2} \mathrm{e}^{z q_3^2} + \frac{\sinh z q_2^2}{z q_2^2}\, p_2^2\, \mathrm{e}^{-z q_1^2} \mathrm{e}^{z q_3^2} + \frac{\sinh z q_3^2}{z q_3^2}\, p_3^2\, \mathrm{e}^{-z q_1^2} \mathrm{e}^{-z q_2^2} \\
J_3^{(3)} &= \frac{\sinh z q_1^2}{z q_1^2}\, q_1 p_1\, \mathrm{e}^{z q_2^2} \mathrm{e}^{z q_3^2} + \frac{\sinh z q_2^2}{z q_2^2}\, q_2 p_2\, \mathrm{e}^{-z q_1^2} \mathrm{e}^{z q_3^2} + \frac{\sinh z q_3^2}{z q_3^2}\, q_3 p_3\, \mathrm{e}^{-z q_1^2} \mathrm{e}^{-z q_2^2} .
\end{aligned}
$$

- **Three-particle Casimir**

$$
\begin{aligned}
\mathcal{C}_z^{(3)} = & \frac{\sinh z q_1^2}{z q_1^2} \frac{\sinh z q_2^2}{z q_2^2} \left(q_1 p_2 - q_2 p_1\right)^2 \mathrm{e}^{-z q_1^2} \mathrm{e}^{z q_2^2} \mathrm{e}^{2 z q_3^2} \\
& + \frac{\sinh z q_1^2}{z q_1^2} \frac{\sinh z q_3^2}{z q_3^2} \left(q_1 p_3 - q_3 p_1\right)^2 \mathrm{e}^{-z q_1^2} \mathrm{e}^{z q_3^2} \\
& + \frac{\sinh z q_2^2}{z q_2^2} \frac{\sinh z q_3^2}{z q_3^2} \left(q_2 p_3 - q_3 p_2\right)^2 \mathrm{e}^{-2 z q_1^2} \mathrm{e}^{-z q_2^2} \mathrm{e}^{z q_3^2}
\end{aligned}
$$

which Poisson-commutes with the three-particle generators $\{J_-^{(3)}, J_+^{(3)}, J_3^{(3)}\}$.

3D spaces with non-constant curvature

- The Hamiltonian $\mathcal{H}_z^{\mathrm{I}} = \frac{1}{2} J_+^{(3)}$ gives[14]

$$\mathcal{T}_z^{\mathrm{I}}(q_i, \dot{q}_i) = \frac{1}{2}\left(\frac{zq_1^2}{\sinh zq_1^2}\, \mathrm{e}^{-zq_2^2}\mathrm{e}^{-zq_3^2}\dot{q}_1^2 + \frac{zq_2^2}{\sinh zq_2^2}\, \mathrm{e}^{zq_1^2}\mathrm{e}^{-zq_3^2}\dot{q}_2^2 + \frac{zq_3^2}{\sinh zq_3^2}\, \mathrm{e}^{zq_1^2}\mathrm{e}^{zq_2^2}\dot{q}_3^2 \right)$$

that defines a **geodesic flow on a 3D Riemannian space** with the $(+,+,+)$ metric

$$\mathrm{d}s_I^2 = \frac{2zq_1^2}{\sinh zq_1^2}\, \mathrm{e}^{-zq_2^2}\mathrm{e}^{-zq_3^2}\, \mathrm{d}q_1^2 + \frac{2zq_2^2}{\sinh zq_2^2}\, \mathrm{e}^{zq_1^2}\mathrm{e}^{-zq_3^2}\, \mathrm{d}q_2^2 + \frac{2zq_3^2}{\sinh zq_3^2}\, \mathrm{e}^{zq_1^2}\mathrm{e}^{zq_2^2}\, \mathrm{d}q_3^2 .$$

- The **sectional curvatures** K_{ij} are

$$\begin{aligned} K_{12} &= \frac{z}{4}\, \mathrm{e}^{-z\mathbf{q}^2} \left(1 + \mathrm{e}^{2zq_3^2} - 2\mathrm{e}^{2z\mathbf{q}^2} \right) \\ K_{13} &= \frac{z}{4}\, \mathrm{e}^{-z\mathbf{q}^2} \left(2 - \mathrm{e}^{2zq_3^2} + \mathrm{e}^{2zq_2^2}\mathrm{e}^{2zq_3^2} - 2\mathrm{e}^{2z\mathbf{q}^2} \right) \\ K_{23} &= \frac{z}{4}\, \mathrm{e}^{-z\mathbf{q}^2} \left(2 - \mathrm{e}^{2zq_2^2}\mathrm{e}^{2zq_3^2} - 2\mathrm{e}^{2z\mathbf{q}^2} \right) \end{aligned}$$

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[14] A. B., F.J. Herranz and O. Ragnisco, math-ph/0508038. Czech. J. Phys. **55**, 1327 (2005).

- K is the **scalar curvature**, which reads

$$K_{12} + K_{13} + K_{23} = -\frac{5}{2}\, z \, \sinh(z\, \mathbf{q}^2) = K/2$$

- **New canonical coordinates (ρ, θ, ϕ):**

$$\begin{aligned}
\cosh^2(\lambda_1 \rho) &= \mathrm{e}^{2z\mathbf{q}^2} \\
\sinh^2(\lambda_1 \rho)\cos^2(\lambda_2 \theta) &= \mathrm{e}^{2zq_1^2}\mathrm{e}^{2zq_2^2}\left(\mathrm{e}^{2zq_3^2} - 1\right) \\
\sinh^2(\lambda_1 \rho)\sin^2(\lambda_2 \theta)\cos^2(\phi) &= \mathrm{e}^{2zq_1^2}\left(\mathrm{e}^{2zq_2^2} - 1\right) \\
\sinh^2(\lambda_1 \rho)\sin^2(\lambda_2 \theta)\sin^2(\phi) &= \left(\mathrm{e}^{2zq_1^2} - 1\right)
\end{aligned}$$

where $z = \lambda_1^2$ and $\lambda_2 \neq 0$ is the additional parameter.

- **The metric is transformed into**

$$\mathrm{d}s_I^2 = \frac{1}{\cosh(\lambda_1 \rho)}\left(\mathrm{d}\rho^2 + \lambda_2^2 \frac{\sinh^2(\lambda_1 \rho)}{\lambda_1^2}\left(\mathrm{d}\theta^2 + \frac{\sinh^2(\lambda_2 \theta)}{\lambda_2^2}\,\mathrm{d}\phi^2\right)\right)$$

- This is the metric of the 3D Riemannian and relativistic spacetimes written in geodesic polar coordinates and multiplied by a global factor $1/\cosh(\lambda_1\rho)$ that encodes the information concerning the variable curvature of the space.

- **Sectional and scalar curvatures read**

$$K_{12}=K_{13}=-\frac{1}{2}\lambda_1^2\,\frac{\sinh^2(\lambda_1\rho)}{\cosh(\lambda_1\rho)}\qquad K_{23}=K_{12}/2\qquad K=-\frac{5}{2}\lambda_1^2\,\frac{\sinh^2(\lambda_1\rho)}{\cosh(\lambda_1\rho)}.$$

- Therefore, we have a **quantum group deformation of the 3D sphere** $(i,1)$**, hyperbolic** $(1,1)$**, de Sitter** $(1,i)$ **and anti-de Sitter** (i,i) **spaces**.

- The “classical” limit $z\to 0$ corresponds to a zero-curvature limit: Euclidean $(0,1)$ and Minkowskian $(0,i)$ spaces.

- **Geodesic motion Hamiltonian**[15]:

$$H_z^{\mathrm{I}}=\frac{1}{2}\cosh(\lambda_1\rho)\left(p_\rho^2+\frac{\lambda_1^2}{\lambda_2^2\sinh^2(\lambda_1\rho)}\left(p_\theta^2+\frac{\lambda_2^2}{\sin^2(\lambda_2\theta)}\,p_\phi^2\right)\right),$$

$$C_z^{(2)}=p_\phi^2,\qquad\qquad C_z^{(3)}=p_\theta^2+\frac{\lambda_2^2}{\sin^2(\lambda_2\theta)}\,p_\phi^2.$$

---

[15] provided that $H_z^{\mathrm{I}}=2\mathcal{H}_z^{\mathrm{I}}$, $C_z^{(2)}=4\mathcal{C}_z^{(2)}$ and $C_z^{(3)}=4\lambda_2^2\mathcal{C}_z^{(3)}$.

3D spaces with constant curvature

- **The maximally superintegrable Hamiltonian**

$$\boxed{\mathcal{H}_z^{\mathrm{S}} = \frac{1}{2} J_+^{(3)}\, \mathrm{e}^{z J_-^{(3)}}}$$

has four (functionally independent) constants of motion, namely $\mathcal{C}_z^{(2)}$, $\mathcal{C}_z^{(3)}$, $\mathcal{I}_z^{(2)}$ and

$$\mathcal{I}_z^{(3)} = \frac{\sinh zq_1^2}{2zq_1^2}\, \mathrm{e}^{zq_1^2}\, \mathrm{e}^{2zq_2^2} p_1^2 + \frac{\sinh zq_2^2}{2zq_2^2}\, \mathrm{e}^{zq_2^2} p_2^2 .$$

- The associated **kinetic energy** is

$$\mathcal{T}_z^{\mathrm{S}}(q_i, \dot{q}_i) = \frac{1}{2}\left(\frac{zq_1^2}{\sinh zq_1^2}\, \mathrm{e}^{zq_1^2} \dot{q}_1^2 + \frac{zq_2^2}{\sinh zq_2^2}\, \mathrm{e}^{2zq_1^2} \mathrm{e}^{zq_2^2} \dot{q}_2^2 + \frac{zq_3^2}{\sinh zq_3^2}\, \mathrm{e}^{2zq_1^2} \mathrm{e}^{2zq_2^2} \mathrm{e}^{zq_3^2} \dot{q}_3^2 \right)$$

- **The metric reads**

$$\mathrm{d}s_S^2 = \frac{2zq_1^2}{\sinh zq_1^2}\, \mathrm{e}^{zq_1^2}\, \mathrm{d}q_1^2 + \frac{2zq_2^2}{\sinh zq_2^2}\, \mathrm{e}^{2zq_1^2} \mathrm{e}^{zq_2^2}\, \mathrm{d}q_2^2 + \frac{2zq_3^2}{\sinh zq_3^2}\, \mathrm{e}^{2zq_1^2} \mathrm{e}^{2zq_2^2} \mathrm{e}^{zq_3^2}\, \mathrm{d}q_3^2 .$$

and note that

$$\mathcal{T}_z^{\mathrm{S}} = \mathcal{T}_z^{\mathrm{I}}\, \mathrm{e}^{z\mathbf{q}^2} \qquad \text{and} \qquad \mathrm{d}s_{\mathrm{S}}^2 = \mathrm{d}s_{\mathrm{I}}^2\, \mathrm{e}^{z\mathbf{q}^2} .$$

- The space is **Riemannian** with sectional and scalar curvatures given by

$$K_{ij} = z \qquad K = 6z.$$

- Through the change of coordinates and by introducing a **new radial coordinate**

$$r = \int_0^\rho \frac{\mathrm{d}x}{\cosh(\lambda_1 x)}$$

(i.e. $\cosh(\lambda_1\rho) = 1/\cos(\lambda_1 r)$) we find that $\mathrm{d}s^2_{\mathrm{S}}$ is just transformed into the 3D CK metric written in terms of geodesic polar coordinates

$$\mathrm{d}s^2_{\mathrm{S}} = \mathrm{d}r^2 + \lambda_2^2 \frac{\sin^2(\lambda_1 r)}{\lambda_1^2} \left(\mathrm{d}\theta^2 + \frac{\sin^2(\lambda_2\theta)}{\lambda_2^2}\, \mathrm{d}\phi^2 \right).$$

- According to the values of (λ_1, λ_2), this metric provides the non-deformed 3D spaces

sphere $(1, 1)$
Euclidean $(0, 1)$
hyperbolic $(i, 1)$
anti-de Sitter $(1, i)$
Minkowskian $(0, i)$
de Sitter (i, i).

- The maximally superintegrable **Hamiltonian**, $H_z^{\mathrm{S}} = 2\mathcal{H}_z^{\mathrm{S}}$, is found to be

$$H_z^{\mathrm{S}} = \frac{1}{2}\left(p_r^2 + \frac{\lambda_1^2}{\lambda_2^2 \sin^2(\lambda_1 r)}\left(p_\theta^2 + \frac{\lambda_2^2}{\sin^2(\lambda_2\theta)}\, p_\phi^2\right)\right),$$

- We have **four functionally independent constants of the motion**:

$$C_z^{(2)} = p_\phi^2,$$

$$C_z^{(3)} = p_\theta^2 + \frac{\lambda_2^2}{\sin^2(\lambda_2\theta)}\, p_\phi^2,$$

$$I_z^{(2)} = \left(\lambda_2 \sin(\lambda_2\theta) \sin\phi\, p_r + \frac{\lambda_1 \cos(\lambda_2\theta)\sin\phi}{\tan(\lambda_1 r)}\, p_\theta + \frac{\lambda_1\lambda_2\cos\phi}{\tan(\lambda_1 r)\sin(\lambda_2\theta)}\, p_\phi\right)^2,$$

$$I_z^{(3)} = \left(\lambda_2 \sin(\lambda_2\theta)\, p_r + \frac{\lambda_1\cos(\lambda_2\theta)}{\tan(\lambda_1 r)}\, p_\theta\right)^2 + \lambda_1^2\lambda_2^2\left(\frac{\tan^2(\lambda_1 r)\sin^2(\lambda_2\theta) + 1}{\tan^2(\lambda_1 r)\sin^2(\lambda_2\theta)}\right) p_\phi^2,$$

where $I_z^{(2)} = 4\lambda_2^2\mathcal{I}_z^{(2)}$ and $I_z^{(3)} = 4\lambda_2^2\mathcal{I}_z^{(3)}$.

- Notice that the **two sets** $\{H_z^{\mathrm{S}}, C_z^{(2)}, C_z^{(3)}\}$ and $\{H_z^{\mathrm{S}}, I_z^{(2)}, I_z^{(3)}\}$ consist of three functions in involution.

Integrable potentials on curved spaces with coalgebra symmetry

- We can consider **more general integrable Hamiltonians**[16] [17]

$$\mathcal{H}_z = \frac{1}{2} J_+ \, f(zJ_-) + \mathcal{U}(zJ_-)$$

$$\text{such that} \qquad \lim_{z\to 0} \mathcal{U}(zJ_-) = V \qquad \text{and} \qquad \lim_{z\to 0} f(zJ_-) = 1.$$

Obviously, V has to be an integrable potential on the flat space.

- The function $f(zJ_-)$ gives us the type of **curved space**.

- **Harmonic oscillator on these spaces**: certain $\mathcal{U}$ such that

$$\lim_{z\to 0} \mathcal{U} = \beta_0 J_-$$

- **The Kepler–Coulomb (KC) potential**: certain $\mathcal{U}$ such that

$$\lim_{z\to 0} \mathcal{U} = -\gamma/\sqrt{J_-}$$

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[16] A. B., F.J. Herranz and O. Ragnisco, J. Phys. A: Math. Gen. **38**, 7129 (2005).

[17] A.B., A. Enciso, F. J. Herranz, and O. Ragnisco, *Physica D* **237**, 505 (2008).

All this is Poisson-Lie dynamics on κ-Minkowski group!

- Let us consider the following element g_z of a 3D Lie group M_z:

$$g_z = \begin{pmatrix} e^{2\,z\,A} & 0 & 0 \\ 0 & e^{2\,z\,A} & 0 \\ B & C & 1 \end{pmatrix}$$

The group parameters are A, B and C.

- But the Lie algebra of M_z is

$$\hat{x}_0 = \begin{pmatrix} 2\,z & 0 & 0 \\ 0 & 2\,z & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \hat{x}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \hat{x}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

which is just **the (2+1) κ–Minkowski space!**

$$[\hat{x}_0, \hat{x}_1] = -2\,z\,\hat{x}_1 \qquad [\hat{x}_0, \hat{x}_2] = -2\,z\,\hat{x}_2 \qquad [\hat{x}_1, \hat{x}_2] = 0$$

and

$$g_z = e^{B\,\hat{x}_1}\, e^{C\,\hat{x}_2}\, e^{A\,\hat{x}_0}$$

- ► Therefore, M_z can be called the **(2+1) κ–Minkowski group**. As a Lie algebra, the κ–Minkowski algebra is sometimes called "book algebra"[18] or "Borel algebra".[19] Note that the $z \to 0$ limit of M_z is the three dimensional abelian group.

- ► A natural **coalgebra structure on Fun(M_z)** is given by the **group product**:

$$\Delta(g_z) = \begin{pmatrix} \Delta(e^{2\,z\,A}) & 0 & 0 \\ 0 & \Delta(e^{2\,z\,A}) & 0 \\ \Delta(B) & \Delta(C) & \Delta(1) \end{pmatrix} :=$$

$$:= g_z \dot{\otimes} g_z = \begin{pmatrix} e^{2\,z\,A} & 0 & 0 \\ 0 & e^{2\,z\,A} & 0 \\ B & C & 1 \end{pmatrix} \dot{\otimes} \begin{pmatrix} e^{2\,z\,A} & 0 & 0 \\ 0 & e^{2\,z\,A} & 0 \\ B & C & 1 \end{pmatrix} .$$

Such a coproduct takes the following values on the group entries:

$$\begin{aligned} &\Delta(1) = 1 \otimes 1 \\ &\Delta(e^{2\,z\,A}) = e^{2\,z\,A} \otimes e^{2\,z\,A} \longleftarrow \Delta(A) = 1 \otimes A + A \otimes 1 \\ &\Delta(B) = 1 \otimes B + B \otimes e^{2\,z\,A} \\ &\Delta(C) = 1 \otimes C + C \otimes e^{2\,z\,A}. \end{aligned}$$

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[18] J.F. Cariñena, A. Ibort, G. Marmo and A. Perelomov, J. Phys. A: Math. Gen. **27**, 7425 (1994).

[19] L. Freidel, J. Kowalski–Glickman, S. Nowak, *Phys. Lett. B* **648**, 70 (2007).

▶ Now let us consider the following **change of dynamical variables**

$$A \equiv J_- \qquad B \equiv J_+\, e^{zJ_-} \qquad C \equiv J_3\, e^{zJ_-}$$

We obtain

$$\begin{aligned}
\Delta(J_-) &= 1 \otimes J_- + J_- \otimes 1 \\
\Delta(J_+) &= e^{-zJ_-} \otimes J_+ + J_+ \otimes e^{zJ_-} \\
\Delta(J_3) &= e^{-zJ_-} \otimes J_3 + J_3 \otimes e^{zJ_-}
\end{aligned}$$

which is just the **coproduct** of $sl_z(2)$, that can be thought of as the **group multiplication** for M_z in certain dynamical variables.

▶ But we know that the Poisson structure

$$\begin{aligned}
\{J_3, J_+\} &= 2J_+ \cosh zJ_- \\
\{J_3, J_-\} &= -2\frac{\sinh zJ_-}{z} \\
\{J_-, J_+\} &= 4J_3
\end{aligned}$$

is compatible with such coproduct. Therefore, **the Poisson coalgebra $sl_z(2)$ is just a Poisson–Lie structure on the (2+1) κ–Minkowski group**.

▶ In the “manifestly covariant” basis (A, B, C) The PL structure reads

$$
\begin{aligned}
\{A, B\} &= -4\, C \\
\{A, C\} &= 4\, \frac{e^{2\, z\, A} - 1}{2\, z} \\
\{B, C\} &= -4\, B + 4\, z\, C^2
\end{aligned}
$$

and the Casimir function is

$$\mathcal{C}_z = C^2\, e^{-2\, z\, A} + B\, \frac{1 - e^{2\, z\, A}}{2\, z}.$$

▶ Hence, **all the previous curved spaces can be interpreted as geodesic flows generated by Hamiltonians defined on $(2, 3, \dots)$ copies of the PL group M_z.**

• ND spaces with **non–constant curvature**

$$\boxed{\mathcal{H} = \frac{1}{2}\, J_+ = \frac{1}{2} B\, \mathrm{e}^{-z\, A}}$$

Geodesic flow associated to $(\hat{x}_0, \hat{x}_1)$

• ND spaces with **constant curvature**

$$\boxed{\mathcal{H} = \frac{1}{2}\, J_+\, \mathrm{e}^{z\, J_-} = \frac{1}{2} B}$$

dS/AdS spaces as geodesic flows associated to the nc translation $\hat{x}_1$

Concluding remarks

- Other **examples of Poisson-Lie dynamics on (2+1) M_z** can be found in [20].
- The **Poisson geometry** of the κ–Minkowski group deserves some attention:

  One nc "point": Poisson manifold $M_z \longrightarrow$ one–dimensional dynamics

  Two nc "points": $M_z \otimes M_z \longrightarrow$ 2D curved space

  $\vdots$

  N nc "points": $M_z \otimes M_z \otimes \ldots^{(N)} \otimes M_z \longrightarrow$ ND curved space
- **Generalization to arbitrary dimensions** from the N-th symplectic realization:

$$\mathcal{H}_z^{\mathrm{I},(N)} = \frac{1}{2} J_+ = \frac{1}{2} \sum_{i=1}^{N} \frac{\sinh z q_i^2}{z q_i^2} p_i^2 \exp\left(-z \sum_{k=1}^{i-1} q_k^2 + z \sum_{l=i+1}^{N} q_l^2 \right).$$

The exponentials coming from the coproduct are the objects that generate the sectional curvatures. The geometric characterization of the underlying N-dimensional curved spaces generalizes the results for the 3D case.

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[20] A. B., and O. Ragnisco, J. Phys. A: Math. Gen. **36**, 10505 (2003).

- Many **other possible choices** for a deformed kinetic energy:

$$\mathcal{H} = \frac{1}{2} J_+ f(zJ_-),$$

and therefore for the "dynamical" generation of deformed spaces.

General expression for the 2D Gaussian curvature:

$$K(x) = \frac{z}{f(x)} \left(f(x) f'(x) \cosh x + \left(f(x) f''(x) - f^2(x) - f'^2(x) \right) \sinh x \right),$$

where $x \equiv zJ_- = z(q_1^2 + q_2^2) = zq^2$, $f' = \frac{\mathrm{d}f(x)}{\mathrm{d}x}$ and $f'' = \frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2}$.

- The study of the **free motion of a quantum mechanical particle on these curved spaces** should provide a geometric interpretation of the non-standard quantum deformation of $sl(2)$.

- Open problem: the same construction for **(3+1) κ–Minkowski group**.

- The main problem:

To understand the geometry (in any sense) of κ–Minkowski space.