A MOTIVIC VERSION OF p-ADIC INTEGRATION

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1. Motivic measures

I will discuss how one can specialize motivic integration to p-adic integration via point counting. We use $\mathcal{M}_k$ to denote the Grothendieck ring of varieties over k, localized at $\mathbb{L}$, the class of the affine line.

Let $\mathcal{X}$ be a scheme of finite type, defined over a complete discrete valuation ring with residue field k. In the theory of geometric motivic integration, as developed by Denef and Loeser [DL02] [Loo02] in the case when the discrete valuation ring is of equal characteristic, and by Sebag [Seb04] in the mixed characteristic case, one defines the space of arcs on $\mathcal{X}$, which we will denote $\mathcal{X}_\infty$. For our purposes, this is just a set, and in the case when the discrete valuation ring is absolutely unramified of mixed characteristic, which is the case we are mainly interested in, it equals $\mathcal{X}\big(\mathbf{W}(\overline{k})\big)$, where $\mathbf{W}$ is the ring scheme of Witt vectors and $\overline{k}$ is an algebraic closure of k. For every positive integer n there is a k-variety $\mathcal{X}_n$, with the property that, in the mixed characteristic case, $\mathcal{X}_n(K) = \mathcal{X}(\mathbf{W}_n(K))$ for every k-algebra K. This is called the variety of n-jets; the arc space is the projective limit of the set of $\overline{k}$-points of these.

The subsets of $\mathcal{X}_\infty$ that are defined by a finite truncation of arcs, meaning that they are specified already by subschemes of $\mathcal{X}_n$ for some n, are called *stable*. The collection of stable subsets is a Boolean algebra; on this algebra one defines an additive measure, $\tilde{\mu}_\mathcal{X}$, taking valuses in $\mathcal{M}_k$. When we work over $\mathbb{Z}_p$, so that $k = \mathbb{F}_p$, there is a ring homomorphism $\mathrm{C}_p \colon \mathcal{M}_k \to \mathbb{Q}$, induced by counting $\mathbb{F}_p$-points on varieties. If $\mathcal{X}$ is smooth, and $A \subset \mathcal{X}_\infty$ is stable, then we may specialize $\tilde{\mu}_\mathcal{X}(A)$ to the ordinary p-adic measure by applying C_p. For example, if $\mathcal{X} = \mathbb{A}^d_{\mathbb{Z}_p}$, then $\mathcal{X}_\infty = (\mathbb{Z}_p^{\mathrm{unram}})^d$, and $\mathrm{C}_p\,\tilde{\mu}_\mathcal{X}(A) = \mu_{\mathrm{Haar}}(A \cap \mathbb{Z}_p^n)$, *cf.* [LS03], Lemma 4.6.2.

When defining the geometric motivic measure theory, one uses these stable sets as building blocks, using coverings of them to define generel measurable sets. This procedure involves taking limits, and so the measure of A, $\mu_\mathcal{X}(A)$ now lies in a completion of $\mathcal{M}_k$. The standard choice is to use $\widehat{\mathcal{M}_k}$, the completion with respect to the dimension filtration. However, for our purpose of specializing to p-adic integration, this is not a suitable choice. The reason is that C_p is not continuous with respect to the dimension filtration. (To see this, note for example that the sequence $x_n = p^n/\mathbb{L}^n \in \mathcal{M}_{\mathbb{F}_p}$ tends to 0 with respect to dimension, but $\mathrm{C}_p\, x_n = 1$ for every n.) For this reason, C_p does not have a natural extension to $\widehat{\mathcal{M}_k}$ (which would be by continuity).

In fact, this has been the main problem so far in adapting the theory of geometric motivic integration to fit my purposes. The main part of this talk will be devoted to a discussion of which topology to use. Before that, let me just show with an example how the specialization works.

Let $\overline{\mathrm{K}_0}(\mathsf{Var}_k)$ be the "correct" completion of $\mathcal{M}_k$, one on which C_p is defined and continuous when $k = \mathbb{F}_p$. We define the measurable subsets of $\mathcal{X}_\infty$ in the standard way, see e.g., [DL02] [Seb04]. If $A \subset \mathcal{X}_\infty$ is measurable, then its measure, $\mu_\mathcal{X}(A)$, is an element of $\overline{\mathrm{K}_0}(\mathsf{Var}_k)$. Now, suppose for example, that $\mathcal{X} = \mathbb{A}^d_{\mathbb{Z}_p}$, so that $k = \mathbb{F}_p$. With the definition of the arc space that I use, we then have $\mathcal{X}_\infty = \mathbf{W}(\overline{\mathbb{F}}_p)^d$ where $\mathbf{W}$ is the ring scheme of Witt vectors. $\mathbf{W}(\overline{\mathbb{F}}_p)$ may be identified with $\mathbb{Z}_p^{\mathrm{unram}}$, the integers in an unramified closure of $\mathbb{Q}_p$, and it contains $\mathbb{Z}_p$ as a subring. Suppose now that $A \subset (\mathbb{Z}_p^{\mathrm{unram}})^d$ is measurable, so that $\mu_\mathcal{X}(A)$ lies in $\overline{\mathrm{K}_0}(\mathsf{Var}_k)$. Then, similarly as in the case of a stable set, we have

$$\mathrm{C}_p\,\mu_\mathcal{X}(A) = \mu_{\mathrm{Haar}}(A \cap \mathbb{Z}_p^d).$$

More generally, if q is a power of p, we will have

$$\mathrm{C}_q\,\mu_\mathcal{X}(A) = \mu_{\mathrm{Haar}}(A \cap \mathbf{W}(\mathbb{F}_q)^d).$$

Example 1.1. *Let $\mathcal{X} = \mathbb{A}^1_{\mathbb{Z}_p}$. Define the function $|\cdot|\colon \mathcal{X}_\infty \to \overline{\mathrm{K}_0}(\mathsf{Var}_k)$ as $x \mapsto \mathbb{L}^{-\operatorname{ord} x}$. Within the framework that I discuss in this talk one proves that*

$$\int_{\mathcal{X}_\infty} |X^2+1| d\mu_{\mathcal{X}} = 1 - [\operatorname{Spec} \mathbb{F}_p[X]/(X^2+1)]\frac{1}{\mathbb{L}+1} \in \overline{\mathrm{K}_0}(\mathsf{Var}_{\mathbb{F}_p}).$$

This integral has the property that for every power of p, $\mathrm{C}_q \int_{\mathcal{X}_\infty} |X^2+1| d\mu_{\mathcal{X}} = \int_{\mathbf{W}(\mathbb{F}_q)} |X^2+1|_p dX$. So by computing the motivic integral we have simultaneously computed the corresponding integral over $\mathbf{W}(\mathbb{F}_q)$, for every power of p.

If $p \equiv 3 \pmod 4$ then -1 is a non-square in $\mathbb{F}_q$ and $\mathbb{F}_p[X]/(X^2+1) = \mathbb{F}_{p^2}$. So $\int_{\mathcal{X}_\infty} |X^2+1| d\mu_{\mathcal{X}} = 1 - [\operatorname{Spec} \mathbb{F}_{p^2}]\frac{1}{\mathbb{L}+1}$, showing, e.g., that $\int_{\mathbb{Z}_p} |X^2+1|_p dX = 1$ and $\int_{\mathbf{W}(\mathbb{F}_{p^2})} |X^2+1|_p dX = 1 - 2/(p^2+1)$.

2. Euler characteristics

The simplest solution to the problem that the counting homomorphism is not continuous with respect to the dimension filtration would perhaps be to just introduce the weakest topology on $\mathcal{M}_k$ such that it is. However, we want a definition that is independent of p, and works over any field k.

We will use the cohomology of varieties to define to define this topology. In fact, we will follow the lead of Ekedahl, [Eke07], who develops a topology for which taking the trace of Frobenius on the ℓ-adic cohomology is continuous, and then uses this to extend C_p.

It will turn out that, in the end, Ekedahl's topology is too strong for our purposes. However, we will still use it to define the topology that we use. So we begin by a discussion of the topology from [Eke07], and for this we will first need to discuss the Euler characteristic with values in a ring of Galois representations.

In this section, we let $\mathcal{G}$ be the absolute Galois group of k. For X a separated k-scheme of finite type we use $\mathrm{H}^i_c(X)$ to denote the cohomology of the extension of X to a separable closure of k with $\mathbb{Q}_\ell$-coefficients, with its natural $\mathcal{G}$-action.

2.1. **Euler characteristic taking values in a ring of Galois representation.** Let $\mathrm{K}_0(\mathsf{Rep}_{\mathcal{G}}\mathbb{Q}_\ell)$ be the Grothendieck ring of continuous $\mathcal{G}$-representations. Let X be a k-variety. The assignment

$$X \mapsto \sum_i (-1)^i [\mathrm{H}^i_c(X)]$$

defines a compactly supported Euler characteristic, $\chi_c\colon \mathrm{K}_0(\mathsf{Var}_k) \to \mathrm{K}_0(\mathsf{Rep}_{\mathcal{G}}\mathbb{Q}_\ell)$. Since $\chi_c(\mathbb{L}) = [\mathbb{Q}_\ell(-1)]$, the class of the dual of the cyclotomic representation, is invertible, it follows that χ_c factors through $\mathcal{M}_k$. It hence can be used to distinguish elements in $\mathcal{M}_k$ as well as in $\mathrm{K}_0(\mathsf{Var}_k)$. To illustrate this we consider the following example by Naumann, [Nau07]:

Example 2.1 (Naumann)**.** *Let L/k be a Galois extension of degree d. Then*

$$[\operatorname{Spec} L]^2 = [\operatorname{Spec} L^d] = d[\operatorname{Spec} L],$$

hence $[\operatorname{Spec} L]\big([\operatorname{Spec} L] - d\big) = 0$. Now, to every $\sigma \in \mathcal{G}$ there exists an invariant $\mathrm{C}_\sigma\colon \mathrm{K}_0(\mathsf{Rep}_{\mathcal{G}}\mathbb{Q}_\ell) \to \overline{\mathbb{Q}}$, induced by mapping V to the character of V evaluated in σ. Using this it is easy to show that $\chi_c[\operatorname{Spec} L] \notin \mathbb{Z}$, in particular it is not equal to zero or d. Hence $[\operatorname{Spec} L]$ is a zero divisor in both $\mathrm{K}_0(\mathsf{Var}_k)$ and $\mathcal{M}_k$.

Remark 2.2. *This example constructs zero divisors when k is not separably closed. On the other hand, the classical example of zero divisors is by Poonen, and holds over any field of characteristic zero, including the algebraically closed ones. However, the zero-divisors of Poonen may not remain non-zero in $\mathcal{M}_k$. For a discussion of the status of the problem of zero divisors, see* [Nic08]*, Section* 2.5*. For a discussion about the effects of inverting $\mathbb{L}$, see* [DL04].

2.2. **Euler characteristic taking values in a ring of mixed Galois representations.** We use Coh_k to denote the category of mixed Galois $\mathbb{Q}_\ell$-representations. In case k is finitely generated, this is the full subcategory of $\mathsf{Rep}_{\mathcal{G}}\mathbb{Q}_\ell$ consisting of those representations that have a weight filtration. We then have an injection $\mathrm{K}_0(\mathsf{Coh}_k) \hookrightarrow \mathrm{K}_0(\mathsf{Rep}_{\mathcal{G}}\mathbb{Q}_\ell)$, the image being generated by modules of pure weight. (When $k = \mathbb{F}_q$ this means that all the absolute values of the trace of the geometric Frobenius are in $\overline{\mathbb{Q}}$ and have absolute value a power of q.) The image of χ_c is contained in this subring, i.e., we have $\chi_c\colon \mathcal{M}_k \to \mathrm{K}_0(\mathsf{Coh}_k)$.

For an arbitrary field k we use a construction of a category of mixed Galois $\mathbb{Q}_\ell$-representations Coh_k due to Ekedahl, [Eke07], Section 2. It has the property that if $\{k_\alpha\}$ is the collection of finitely generated subfields of k then we have an isomorphism $\lim_{\to\alpha} \mathrm{K}_0(\mathsf{Coh}_{k_\alpha}) \to \mathrm{K}_0(\mathsf{Coh}_k)$. If X is a k-scheme of finite type then it is defined over some finitely generated subfield k_{α_0}. The Euler characteristic of X can therefore be defined in $\mathrm{K}_0(\mathsf{Coh}_{k_{\alpha_0}})$, we define $\chi_c(X)$ to be the image of this in $\mathrm{K}_0(\mathsf{Coh}_k)$. This is well defined, so we have a ring homomorphism $\chi_c\colon \mathcal{M}_k \to \mathrm{K}_0(\mathsf{Coh}_k)$ for any field k.

The reason for us to work in $\mathrm{K}_0(\mathsf{Coh}_k)$ rather than in $\mathrm{K}_0(\mathsf{Rep}_\mathcal{G}\mathbb{Q}_\ell)$ is that we have some control of the structure of $\mathrm{K}_0(\mathsf{Coh}_k)$: If V is any mixed representation then there is a Jordan-Hölder sequence $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$. Here V_{i+1}/V_i is simple, hence of pure weight. Since $[V] = \sum_i [V_{i+1}/V_i]$, and since this sum is independent of the chosen sequence, it follows that any $v \in \mathrm{K}_0(\mathsf{Coh}_k)$ may be written uniquely as

$$v = \sum_n v_n, \text{ where } v_n = \sum_i c_{ni}[V_{ni}], \tag{2.3}$$

where the V_{ni} are irreducible, pairwise non-isomorphic representations of pure weight n. Since this also respects the multiplication, it follows that $\mathrm{K}_0(\mathsf{Coh}_k)$ is graded by weight. A important fact is that the image of $\mathrm{K}_0(\mathsf{Var}_k)$ under χ_c is contained in the part of non-negative degree.

Example 2.4. *This gives an easy way to see that $\mathbb{L}$ is not invertible in $\mathrm{K}_0(\mathsf{Var}_k)$. For if $x\mathbb{L} = 1$ then $\chi_c(x) = [\mathbb{Q}_\ell(1)]$ in $\mathrm{K}_0(\mathsf{Coh}_k)$. Since the right hand side is pure of weight -2, this is impossible.*

More generally, it is showed in [Eke07]*, Proposition* 3.11*, that if p and q are integer polynomials, q non-zero, such that $p(\mathbb{L})$ divides $q(\mathbb{L})$ in $\mathrm{K}_0(\mathsf{Var}_k)$, then p divides q in $\mathbb{Z}[T]$.*

Before we continue, we illustrate with two examples the usefullness of this grading. The first example is a result of Nicaise, (part of Proposition 3.13 in [Nic08]). Our proof is essentially the same, we just use a different setting.

Example 2.5. *Let X and Y be k-varieties. If $[X] = [Y]$ in $\mathcal{M}_k$ then X and Y have the same dimension n, and the same number of geometrical components of dimension n.*

We use following results of Deligne: $\mathrm{H}^i_c(X)$ is of mixed weight $\leq i$, and if $i > 2\dim X$ then $\mathrm{H}^i_c(X) = 0$. Furthermore, if $\dim X = n$ then $\mathrm{H}^{2n}_c(X)$ is of pure weight $2n$, and its dimension equals the number of geometric components of X of dimension n.

Now, let n be the dimension of X and m the dimension of Y. We have that

$$\chi_c(X) = [\mathrm{H}^{2n}_c(X)] + \big(\textit{terms of weight} < 2n\big).$$

Since the weight $2n$-part of this is a non-zero, non-virtual representation, it follows that the same must hold for the degree $2n$-part of $\chi_c(Y)$, hence that $m = n$, and also that these parts have the same dimension.

The following example is a variation of the preceding one, which we will need in defining our topology.

Example 2.6. *Let X_i be non-empty k-varieties and let n_i be non-negative integers. Let $x := \sum_{i=1}^N n_i[X_i]$. If $x = 0$ then $n_i = 0$ for every i. Moreover, if $x \in \mathrm{F}^m \mathrm{K}_0(\mathsf{Var}_k)$ then $[X_i] \in \mathrm{F}^m \mathrm{K}_0(\mathsf{Var}_k)$ for every i. (We use F to denote the dimension filtration.)*

The first part is a special case of the preceding example. For the second part, let m' be the maximal dimension of the X_i and suppose that $m' > m$. Since the dimension of x is m, the weight m' part of $\chi_c(x)$ is 0. On the other hand

$$\chi_c(x) = \sum_{\dim X_i = m'} n_i[\mathrm{H}^{2m'}_c(X_i)] + \big(\textit{terms of weight} < 2m'\big),$$

and the weight m'-part of this is non-zero, a contradiction.

Remark 2.7. *The reason why we use Coh_k, rather than for instance $\mathsf{WRep}_\mathcal{G}\mathbb{Q}_\ell$, to denote the category of mixed Galois $\mathbb{Q}_\ell$-representations is that, in case $k = \mathbb{C}$, we could equally well use it to denote the category of mixed polarisable rational Hodge structures. Everything in this section would then hold also with this alternative definition of Coh_k.*

3. Topologies

3.1. **The dimension filtration and its drawbacks.** The grading of $\mathrm{K}_0(\mathsf{Coh}_k)$ imposes a filtration, the *weight filtration*: For $v \in \mathrm{K}_0(\mathsf{Coh}_k)$, let v_n be its component of degree n, as in (2.3). Define the filtration in the negative direction as $\mathrm{F}^{\leq N}\,\mathrm{K}_0(\mathsf{Coh}_k) = \{v = \sum_{n \leq N} v_n : \text{finite sums}\} \subset \mathrm{K}_0(\mathsf{Coh}_k)$. If we complete with respect to the weight filtration, we get $\widehat{\mathrm{K}_0(\mathsf{Coh}_k)}$, a ring that behaves nicely in that elements can be represented uniquely as sums, infinite in the negative direction:

$$\widehat{\mathrm{K}_0(\mathsf{Coh}_k)} = \{\sum_{n \leq N} v_n : \text{infinite sums}\}.$$

The Euler characteristic is continuous with respect to this topology an the dimension filtration. For suppose that $\dim x_i = d$, where $x_i \in \mathcal{M}_k$. Then, using the above mentioned results of Deligne, one shows that $\chi_c(x_i) \in \mathrm{F}^{\leq 2d}\,\mathrm{K}_0(\mathsf{Coh}_k)$. Hence, if $x_i \to 0$, then $\chi_c(x_i) \to 0$. Therefore χ_c extends by continuity to a continuous homomorphism $\chi_c \colon \widehat{\mathcal{M}_k} \to \widehat{\mathrm{K}_0(\mathsf{Coh}_k)}$.

In [Eke07] the author wants to extend C_p to a completion of $\mathcal{M}_k$ in such a way that it still is compatible with taking the trace of Frobenius. In other words, when $k = \mathbb{F}_q$ we have, by the Lefschetz trace formula, a commutative diagram

$$\begin{array}{ccc} \mathcal{M}_k & \xrightarrow{\chi_c} & \mathrm{K}_0(\mathsf{Coh}_k) \\ & \searrow^{\mathrm{C}_q} & \downarrow{\scriptstyle \mathrm{Tr}} \\ & & \overline{\mathbb{Q}} \end{array}$$

We then first need to complete $\mathrm{K}_0(\mathsf{Coh}_k)$ in such a way that it is possible to extend Tr to the completion in case k is finite, and for this we need a stronger topology on $\mathrm{K}_0(\mathsf{Coh}_k)$ than the above mentioned. For define, for $v = \sum_n v_n$ as in (2.3),

$$\overline{\mathrm{w}}_n(v) := \sum_i |c_{ni}| \dim V_{ni}.$$

Then, if $k = \mathbb{F}_q$, the only possible extension of Tr to $\mathrm{K}_0(\mathsf{Coh}_k)$ would be to define $\mathrm{Tr}\, v = \sum_n \mathrm{Tr}\, v_n$. However, $\sum_n |\mathrm{Tr}\, v_n| = \sum_n q^{n/2}\, \overline{\mathrm{w}}_n(v)$. So in case $\overline{\mathrm{w}}_n$ grows to fast, $\mathrm{Tr}\, v$ is not convergent. In [Eke07] this is resolved in the following way:

3.2. **The topology of uniform polynomial growth.** Let $\{v_i\}_{i \in \mathbb{N}}$ be a sequence in $\mathrm{K}_0(\mathsf{Coh}_k)$. We say that the sequence is of *uniform polynomial growth* is there exist constants D and d such that for every i and n we have

$$\overline{\mathrm{w}}_n(v_i) \leq |n|^d + D. \tag{3.1}$$

We now define the sequence to be convergent if it is of uniform polynomial growth, and convergent with respect to the weight filtration. This topology does not come from a filtration anymore, neither from a metric. However, we may define the completion as the ring of all Cauchy sequences, meaning that $v_i - v_j \to 0$ as $i, j \to \infty$, modulo those converging to zero. It has the properties of a completion: $\mathrm{K}_0(\mathsf{Coh}_k)$ is dense in it, and every Cauchy sequence is convergent. Denote this completion by $\overline{\mathrm{K}_0}^{\mathrm{pol}}(\mathsf{Coh}_k)$. It has a very simple description as

$$\overline{\mathrm{K}_0}^{\mathrm{pol}}(\mathsf{Coh}_k) = \{v = \sum_{n \leq N} v_n : v \text{ of uniform polynomial growth}\}.$$

If $k = \mathbb{F}_q$ we may now define Tr, the trace of Frobenius, on $\overline{\mathrm{K}_0}^{\mathrm{pol}}(\mathsf{Coh}_k)$. For we have

$$|\mathrm{Tr}\, v| \leq \sum_{n \leq N} |\mathrm{Tr}\, v_n| \leq \sum_{n \leq N} q^{n/2}\, \overline{\mathrm{w}}_n(v) \leq \sum_{n \leq N} q^{n/2}(|n|^d + D)$$

which is convergent. Similarly one proves that Tr is continuous.

So this has the property that we want. However, with this stronger topology, χ_c is no longer continuous. We therefore need a stronger topology on $\mathcal{M}_k$ as well. This definition also uses the weight concept. Basically, one defines, for X a k-variety, $\overline{\mathrm{w}}_n(X) := \sum_i \overline{\mathrm{w}}_n(\mathrm{H}^i_c(X))$. (To make this well defined on $\mathcal{M}_k$

there are some technical details to take care of, however, this is the basic idea.) Note that $\overline{\mathrm{w}}_n(X) \geq \overline{\mathrm{w}}_n(\chi_c(X))$. (An alternative choice would be to just define $\overline{\mathrm{w}}_n(X) := \overline{\mathrm{w}}_n(\chi_c(X))$, however, this would make the definition less intrinsic; $\overline{\mathrm{w}}_n(X)$ could be small due to cancellations in the cohomology that do not correspond to cancellations in $\mathcal{M}_k$.)

We now define a sequence in $\mathcal{M}_k$ to be of uniform polynomial growth in the same way as for sequences in $\mathrm{K}_0(\mathsf{Coh}_k)$, i.e., if it satisfy (3.1). Then, a sequence in $\mathcal{M}_k$ converges to 0 if it does so with respect to the dimension filtration, and is of uniform polynomial growth. Again we define $\overline{\mathrm{K}_0}^{\mathrm{pol}}(\mathsf{Var}_k)$ as the ring of Cauchy sequences, modulo those converging to zero. The dimension and weight functions extends to this ring, and it has the properties that every Cauchy sequence converges, and that the image of $\mathcal{M}_k$ is dense in it.

Now, for $k = \mathbb{F}_q$, we get a commutative diagram of continuous maps, where the extension of C_q is defined by continuity, or equivalently as just the composition $\mathrm{Tr} \circ \chi_c$.

$$\begin{array}{ccc} \overline{\mathrm{K}_0}^{\mathrm{pol}}(\mathsf{Var}_k) & \xrightarrow{\chi_c} & \overline{\mathrm{K}_0}^{\mathrm{pol}}(\mathsf{Coh}_k) \\ & \searrow^{\mathrm{C}_q} & \downarrow \mathrm{Tr} \\ & & \mathbb{C} \end{array}$$

3.3. **Our topology.** At first sight, the topology of uniform polynomial growth seems to be perfect for our purposes of motivic integration. The problem is that it is to strong. To define define a satisfying theory of motivic measures (in particular, to copy the existing theory) one needs to have some control of the measure of $U \subset V$, if one knows the measure of V. For example, suppose that $U_n \subset V_n$. Then it is reasonable to demand that if $\mu_{\mathcal{X}}(V_n) \to 0$ then $\mu_{\mathcal{X}}(U_n) \to 0$. However, we may have arbitrary complicated subvarieties of affine spaces, and for an affine space we have $\overline{\mathrm{w}}_n(\mathbb{A}_k^m) = 0$ if $n \neq 2m$ and $\overline{\mathrm{w}}_{2m}(\mathbb{A}_k^m) = 1$. Therefore there is no way to give an upper bound for $\overline{\mathrm{w}}_n(U)$ just by knowing $\overline{\mathrm{w}}_n(V)$ for $U \subset V$.

To overcome this problem we introduce a partial ordering on $\mathrm{K}_0(\mathsf{Var}_k)$ by saying that $x \leq y$ if there exists a variety X such that $x + [X] = y$. (That this really is a partial ordering is proved using Example 2.6.) One then extends this partial ordering to $\mathcal{M}_k$. We say that a sequence x_i in $\mathcal{M}_k$ is *strongly convergent* to zero, if it is convergent in the topology of uniform polynomial growth. We then define the sequence to be *convergent* to zero if there are two sequences a_n and b_n, strongly convergent to zero, such that $a_n \leq x_n \leq b_n$. Using Example 2.6 one shows that $\dim x_n \leq \max\{\dim a_n, \dim b_n\}$, hence that if a sequence is convergent, then it is convergent with respect to the dimension filtration. It is now easy to show that C_q is continuous when $k = \mathbb{F}_q$. We construct the completion of $\mathcal{M}_k$ with respect to this topology in the standard way, as a quotient of the ring of Cauchy sequences. We denote it $\overline{\mathrm{K}_0}(\mathsf{Var}_k)$.

This topology has one major drawback: The Euler characteristic is no longer continuous. However, it has the property that we are after: When $k = \mathbb{F}_q$, the counting homomorphism is continuous and hence extends to $\overline{\mathrm{K}_0}(\mathsf{Var}_k)$.

$$\begin{array}{cc} \overline{\mathrm{K}_0}(\mathsf{Var}_k) & \\ & \searrow^{\mathrm{C}_q} \\ & \mathbb{R} \end{array}$$

Using $\overline{\mathrm{K}_0}(\mathsf{Var}_k)$, instead of $\widehat{\mathcal{M}_k}$, we may now define the motivic measure theory in the standard way, everything is well defined also with respect to this stronger topology. Moreover, we may specialize to p-adic integration via point counting (in particular, we may use this theory to make sense of Example 1.1).

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