

# Non-archimedean geometry and complex singularities

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May 15, 2008/ ICMS, Edinburgh

# Outline

- 1 The motivic monodromy conjecture
  - The classical Milnor fibration
  - The motivic zeta function

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- 2 The analytic Milnor fiber
  - Points of the analytic Milnor fiber
  - Cohomology of the analytic Milnor fiber
  - MHS on the nearby cohomology

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  - The classical Milnor fibration
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- 2 The analytic Milnor fiber
  - Points of the analytic Milnor fiber
  - Cohomology of the analytic Milnor fiber
  - MHS on the nearby cohomology
- 3 Additional results

# Introduction

In this second talk, we'll explain how one can associate to a hypersurface singularity its so-called **analytic Milnor fiber**. It is a non-archimedean model for the classical topological Milnor fibration. The analytic Milnor fiber is an analytic variety over the field of Laurent series, and its geometric and arithmetic properties reflect the structure of the singularity.

We will show how this object serves as a bridge between certain arc spaces and the **motivic zeta function** on the one hand, and the cohomology of the classical Milnor fiber with its monodromy transformation on the other hand. Such a bridge is interesting in view of the motivic **monodromy conjecture**.

# The classical Milnor fibration

Let  $X$  be a complex manifold and  $f : X \rightarrow \mathbb{C}$  an analytic map.  
Denote by  $X_s$  the analytic space defined by  $f = 0$ , and fix a point  $x \in X_s$ .

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**Question:** What does the complex topology of  $X_s$  look like in a neighbourhood of  $x$ ? How is the germ  $(X_s, x)$  embedded in  $X$ ?

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**Question:** What does the complex topology of  $X_s$  look like in a neighbourhood of  $x$ ? How is the germ  $(X_s, x)$  embedded in  $X$ ?

**Answer:** If  $f$  is smooth at  $x$  this is easy:  $X_s$  is a complex submanifold of  $X$ . If  $f$  is not smooth at  $x$  the topology can be described by means of the **Milnor fibration**.

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Then for  $0 < \eta \ll \varepsilon \ll 1$  the map

$$f_x : f^{-1}(D^*) \cap B \rightarrow D^*$$

is a  $\mathcal{C}^\infty$  **locally trivial fibration**, called the **Milnor fibration** of  $f$  at  $x$ .

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Then for  $0 < \eta \ll \varepsilon \ll 1$  the map

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We denote by

$$\widetilde{D}^* = \{z \in \mathbb{C} \mid \Im(z) > -\log \eta\} \rightarrow D^* : z \mapsto \exp(iz)$$

the universal cover of  $D^*$  and we put

$$F_x = E \times_{D^*} \widetilde{D}^*$$

This is the universal fiber of the fibration  $f_x$ , and is called the **Milnor fiber** of  $f$  at  $x$ .

If  $f$  is smooth at  $x$ , then the fibration  $f_x$  is trivial. In general, the defect of triviality is measured by the **monodromy action** on the singular cohomology of  $F_x$ :

$$H_{\text{sing}}(F_x, \mathbb{Z})^{\circlearrowleft_{\pi_1(D^*)}}$$

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The action of the canonical generator  $z \mapsto z + 2\pi$  is called the **monodromy transformation**.

# The motivic zeta function

Let  $k$  be any field, let  $X$  be a smooth irreducible  $k$ -variety of dimension  $m$ , and let

$$f : X \rightarrow \mathbb{A}_k^1$$

be a morphism. We denote by  $X_s$  the special fiber of  $f$ , and we fix a closed point  $x$  of  $X_s$ .

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Recall that, for any  $k$ -algebra  $A$  and any  $d \geq 0$ ,

$$\mathcal{L}_d(X)(A) = X(A[t]/t^{d+1})$$

For  $\psi \in \mathcal{L}_d(X)$  we denote by  $\psi(0)$  the image of  $\psi$  under the truncation map  $\mathcal{L}_d(X) \rightarrow X$ .

For each  $d > 0$ , we consider the closed subset

$$\mathcal{X}_{d,1} = \{\psi \in \mathcal{L}_d(X) \mid f(\psi) \equiv t^d \pmod{t^{d+1}}, \psi(0) = x\}$$

of  $\mathcal{L}_d(X)$  and we endow it with its reduced induced structure.

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The  $k$ -scheme  $\mathcal{X}_{d,1}$  carries a natural  $\mu_d$ -action which is given by

$$\mu_d \times \mathcal{X}_{d,1} \rightarrow \mathcal{X}_{d,1} : (\xi, \psi) \mapsto \psi(\xi \cdot t)$$

### Definition (Denef-Loeser)

The local **motivic zeta function** of  $f$  at  $x$  is the generating series

$$Z_x(f; T) = \sum_{d>0} [\mathcal{X}_{d,1}] \mathbb{L}^{-md} T^d \in \mathcal{M}_x[[T]]$$

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*In fact,  $Z_x(f; T)$  is defined over a finer, equivariant Grothendieck ring  $\mathcal{M}_x^{\hat{\mu}}$  which keeps track of the  $\mu_d$ -action on the scheme  $\mathcal{X}_{d,1}$ .*

## Theorem (Denef-Loeser)

If  $k$  has characteristic zero, then the motivic zeta function  $Z_X(f; T)$  is rational over  $\mathcal{M}_X$ . More precisely, it belongs to the subring

$$\mathcal{M}_X \left[ \frac{\mathbb{L}^a T^b}{1 - \mathbb{L}^a T^b} \right]_{(a,b) \in \mathbb{Z}_{<0} \times \mathbb{N}^*} \subset \mathcal{M}_X[[T]]$$

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The **proof** uses an explicit computation on a resolution of singularities for the morphism  $f$ .

## Conjecture (Motivic monodromy conjecture)

If  $k$  is a subfield of  $\mathbb{C}$ , then there exists a finite subset  $S$  of  $\mathbb{Z}_{<0} \times \mathbb{N}^*$  such that

$$Z_x(f; T) \in \mathcal{M}_x \left[ \frac{\mathbb{L}^a T^b}{1 - \mathbb{L}^a T^b} \right]_{(a,b) \in S}$$

and such that for each couple  $(a, b)$  in  $S$ ,  $\exp(2\pi i a/b)$  is an *eigenvalue of the monodromy transformation* of

$$(f \times_k \mathbb{C})^{an} : (X \times_k \mathbb{C})^{an} \rightarrow \mathbb{C}$$

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Introducing the set  $S$  is an artificial way to speak about the “poles” of  $Z_x(f; T = \mathbb{L}^{-s})$  (the notion of pole is not well-defined since it is not known whether  $\mathcal{M}_x$  is a domain).

This conjecture was motivated by Igusa's  *$p$ -adic monodromy conjecture*, whose most basic formulation says the following: if  $f$  is a polynomial in  $\mathbb{Z}[x_1, \dots, x_m]$ , then for almost all prime numbers  $p$ , and for each pole  $\alpha$  of the  *$p$ -adic zeta function*

$$Z_p(f; s) = \int_{\mathbb{Z}_p^m} |f|_p^s dx$$

the value  $\exp(2\pi i \Re(\alpha))$  is an eigenvalue of the monodromy transformation of  $(f \times_{\mathbb{Z}} \mathbb{C})^{an} : \mathbb{C}^m \rightarrow \mathbb{C}$  at some complex point of the hypersurface  $f = 0$ .

Igusa's conjecture expresses a connection between the **complex singularities** of  $f$  and the number of **solutions of the congruences**

$$f \equiv 0 \pmod{p^d}$$

for  $d \geq 0$ . The motivic monodromy conjecture **implies** the  $p$ -adic one, since the motivic zeta function “specializes” to the  $p$ -adic one for almost all  $p$ .

# The analytic Milnor fiber

Now we will construct an object which establishes a bridge between arcs and monodromy: the **analytic Milnor fiber**.

Let  $k$  be an algebraically closed field of characteristic zero (*this condition is not essential*), and put  $R = k[[u]]$  and  $K = k((u))$ . Let  $X$  be **any**  $k$ -variety,

$$f : X \rightarrow \mathbb{A}_k^1 = \text{Spec } k[u]$$

a morphism, and  $x$  a closed point of the special fiber  $X_s$ .

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a morphism, and  $x$  a closed point of the special fiber  $X_s$ . By completing  $f$  at  $x$ , we get a **special formal  $R$ -scheme**

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## Definition (N-Sebag)

We denote the generic fiber of  $\widehat{f}_x$  by  $\mathcal{F}_x$ , and call it the **analytic Milnor fiber** of  $f$  at  $x$ .

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This is a rigid variety over the non-archimedean field  $K = k((u))$ , and it is smooth if  $X - X_s$  is smooth.

## Topological intuition:

- $\mathfrak{X}_x = \mathrm{Spf} \hat{\mathcal{O}}_{X,x}$  is the infinitesimal neighbourhood of  $x$  in  $X$ , corresponding to a small ball  $B$  around  $x$  in  $X$

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*By the dictionary “finite covers of  $D^*$ ” vs. “finite extensions of  $K$ ”, the universal cover  $\widetilde{D}^*$  corresponds to the algebraic closure  $K^a$ .*

The analytic Milnor fiber contains a wealth of information on the singularity of  $f$  at  $x$ :

### Theorem (de Jong)

*If  $X$  is normal, then the analytic Milnor fiber is a complete invariant of the formal germ  $(f, x)$ . More precisely, the  $R$ -algebra  $\widehat{\mathcal{O}}_{X,x}$  is canonically isomorphic to the  $R$ -algebra*

$$\{f \in \mathcal{O}_{\mathcal{F}_x}(\mathcal{F}_x) \mid |f(x)| \leq 1 \text{ for all } x \in \mathcal{F}_x\}$$

## Points of the analytic Milnor fiber

Inside a fixed algebraic closure  $K^a$ , the field  $K = k((u))$  has a **unique extension  $K(d)$  of degree  $d$**  for each  $d > 0$ . It is obtained by joining a  $d$ -th root  $\sqrt[d]{u}$  of  $u$  to  $K$ . The Galois group  $G(K(d)/K)$  is canonically isomorphic to  $\mu_d(k)$ , and acts on  $K(d)$  by multiplication on  $\sqrt[d]{u}$ .

A  $d$ -th root  $\sqrt[d]{u}$  having been chosen, there exists a **canonical bijection**

$$\mathcal{F}_x(K(d)) = \{\psi \in \mathcal{L}(X)(k) = X(k[[t]]) \mid f(\psi) = t^d, \psi(0) = x\}$$

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for every  $d > 0$ . The **Galois action** of  $G(K(d)/K) = \mu_d(k)$  is given by  $\xi.\psi = \psi(\xi.t)$ .

We see that (if  $X$  is smooth) the set  $\mathcal{F}_X(K(d))$  with the Galois action of  $\mu_d(k)$  is very similar to the space  $\mathcal{X}_{d,1}$  with its  $\mu_d$ -action used in the definition of Denef and Loeser's motivic zeta function. We'll now see how the motivic zeta function  $Z_X(f; T)$  can be recovered from  $\mathcal{F}_X$  in a natural way.

## Proposition (N)

If  $\mathfrak{Y}$  is a regular special formal  $R$ -scheme of pure relative dimension  $n$ , then the natural morphism of  $\mathcal{O}_{\mathfrak{Y}_\eta}$ -modules

$$\Omega_{\mathfrak{Y}_\eta/K}^n = (\Omega_{\mathfrak{Y}/R}^n)_{\text{rig}} \xrightarrow{du \wedge (\cdot)} (\Omega_{\mathfrak{Y}/k}^{n+1})_{\text{rig}}$$

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## Definition (N)

If  $\phi$  is an element of  $\Omega_{\mathfrak{Y}/k}^{n+1}(\mathfrak{Y})$  we denote by  $\phi/du$  its inverse image in  $\Omega_{\mathfrak{Y}_\eta/K}^n(\mathfrak{Y}_\eta)$ , and we call  $\phi/du$  the **Gelfand-Leray form** associated to  $\phi$ .

## Definition (N)

Let  $\phi$  be any non-vanishing element in  $\Omega_{\mathcal{X}_x/k}^m(\mathcal{X}_x)$ . We define the **motivic Weil generating series**  $S(\mathcal{F}_x; T)$  of  $\mathcal{F}_x$  by

$$S(\mathcal{F}_x; T) = \sum_{d>0} \left( \int_{\mathcal{F}_x \times_K K(d)} \left| \frac{\phi}{du} \otimes_K K(d) \right| \right) T^d \in \mathcal{M}_k[[T]]$$

This definition clearly does not depend on the choice of  $\phi$ . Note that  $\phi/du$  is gauge since  $\phi$  has no zeroes.

## Theorem (N-Sebag)

*If  $X$  is smooth over  $k$ , then*

$$S(\mathcal{F}_x; T) = Z_x(f; \mathbb{L}T) \in \mathcal{M}_k[[T]]$$

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*It is possible to define an equivariant motivic integral of an invariant form on a rigid variety with finite group action. In this way, one defines a motivic Weil generating series which keeps track of the Galois action on  $\mathcal{F}_x \times_K K(d)$ , and the above comparison theorem extends to this setting.*

# Cohomology of the analytic Milnor fiber

## Theorem (Berkovich)

For each integer  $i \geq 0$ , there exists a canonical  $G(K^a/K)$ -equivariant isomorphism

$$H_{\text{ét}}^i(\mathcal{F}_x \widehat{\times}_K \widehat{K^a}, \mathbb{Q}_\ell) \cong R^i \psi_f(\mathbb{Q}_\ell)_x$$

## Corollary

If  $k = \mathbb{C}$  and  $X$  is smooth, then for each integer  $i \geq 0$ , there exists a canonical isomorphism

$$H_{\text{ét}}^i(\mathcal{F}_x \widehat{\times}_K \widehat{K^a}, \mathbb{Q}_\ell) \cong H_{\text{sing}}^i(F_x, \mathbb{Q}_\ell)$$

Under this isomorphism, the action of the canonical topological generator of  $G(K^a/K)$  on the left hand side corresponds to the monodromy transformation on the right hand side.

Hence,  $\mathcal{F}_x$  forms a bridge between **arc spaces** and **monodromy**, and the motivic zeta function can be recovered by measuring sets of rational points w.r.t. a Gelfand-Leray form.

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In this setting, our **trace formula** recovers an important result by Denef and Loeser: if we denote by  $M_x$  the monodromy transformation on the graded cohomology space  $H_{\text{sing}}(F_x, \mathbb{Q})$ , then

$$\chi_{\text{top}}(\mathcal{X}_{d,1}) = \text{Trace}((M_x)^d \mid H_{\text{sing}}(F_x, \mathbb{Q}))$$

for all  $d > 0$ .

More generally, the geometric and arithmetic properties reflect the nature of the singularity of  $f$  at  $x$ . Here's a toy example:

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### Proposition

*If  $X$  is smooth at  $x$ , the following are equivalent:*

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- 3 the analytic Milnor fiber  $\mathcal{F}_x$  of  $f$  at  $x$  has a  $K$ -rational point,

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- 1 the morphism  $f$  is smooth at  $x$ ,
- 2 the analytic Milnor fiber  $\mathcal{F}_x$  of  $f$  at  $x$  is isomorphic to an open unit polydisc,
- 3 the analytic Milnor fiber  $\mathcal{F}_x$  of  $f$  at  $x$  has a  $K$ -rational point,
- 4  $\text{Trace}(\varphi | H(\mathcal{F}_x \times_K \widehat{K^a}, \mathbb{Q}_\ell)) \neq 0$  for any topological generator  $\varphi$  of  $G(K^a/K)$ .

More generally, the geometric and arithmetic properties reflect the nature of the singularity of  $f$  at  $x$ . Here's a toy example:

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If  $k = \mathbb{C}$ , the last property can be rewritten as

$$\text{Trace}(M_x | H_{\text{sing}}(F_x, \mathbb{Q})) \neq 0$$

and then (1)  $\Leftrightarrow$  (4) is a classical result by A'Campo.

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*If  $k = \mathbb{C}$ , the singular cohomology of  $\mathcal{F}_x$  (as a Berkovich space) coincides with the weight zero part of the mixed Hodge structure on the cohomology of the topological Milnor fiber of  $f$  at  $x$ .*

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This is a local variant of an analogous result by Berkovich on the limit mixed Hodge structure of a proper family.

To conclude, let us cite some results which were omitted for lack of time. One can use the same expression as before to define the **motivic Weil generating series**  $S(\mathfrak{X}; T)$  of any regular special formal  $R$ -scheme  $\mathfrak{X}$ . It has similar rationality properties as the motivic zeta function.

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Taking a formal limit  $T \rightarrow \infty$  we defined the **motivic volume** of a smooth rigid  $K$ -variety  $X$ . This invariant should be seen as the “motivic incarnation” of  $X \times_K \widehat{K^a}$ .

If  $X$  is quasi-compact, this motivic volume coincides with the class of **Ayoub's motive of  $X$**  in an appropriate Grothendieck ring of  $k$ -motives. If  $X = \mathcal{F}_x$ , we recover Denef and Loeser's **motivic Milnor fiber**.

Our interpretation of the motivic zeta function  $Z_x(f; T)$  and the motivic Milnor fiber also applies to the global case, i.e. to Denef and Loeser's **motivic zeta function**  $Z(f; T) \in \mathcal{M}_{X_s}[[T]]$  and **motivic nearby cycles** (in the sense of Denef-Loeser or Ayoub): simply replace the completion of  $f$  at  $x$  by the  $t$ -adic completion of  $f$ .