Nearby cycles and composition with a two variable function

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Joint work with Gil Guibert and François Loeser.

Problem

Consider a field k of characteristic 0, a variety X and two functions $g_1 : X \longrightarrow \mathbb{A}^1_k$ and $g_2 : X \longrightarrow \mathbb{A}^1_k$.
Assume g_1 and g_2 depend on different sets of variables.

Express the nearby cycles of the composed map $f \circ \mathbf{g}$ at any point of $X_0(\mathbf{g}) := g_1^{-1}(0) \cap g_2^{-1}(0)$ in terms of

- *The nearby cycles of the two functions g_1 and g_2.*
- *Combinatorial data extracted from the function f.*

Remark: When g_1 and g_2 do not depend on different sets of variables, one has to consider the Alexander nearby cycles of the mapping $\mathbf{g}$ instead of the nearby cycles of g_1 and g_2.

Examples

- **Thom-Sebastiani formula**: $f(g_1, g_2) = g_1 + g_2$. For the motivic formula: Denef-Loeser, Looijenga.
- **Steenbrink Conjecture**: $f(g_1, g_2) = g_1 + (g_2)^N$ with N big enough. Here one do not even assume that g_1 and g_2 have different sets of variables.
  - With suitable assumptions on the function g_2 and on the singular locus of g_1: Steenbrink, Siersma (characteristic polynomial of the monodromy), M. Saito (Hodge Spectrum).
  - General motivic formula: Guibert-Loeser-Merle.
- **f is a r-variable, non degenerate polynomial (with respect to its Newton polyhedron)** and $g_1, \ldots, g_r$ have different sets of variables. Motivic formula given by G-L-M.
- **Composition of a polynomial f in $k[X_1, X_2]$** with a mapping $\mathbf{g} : X \longrightarrow \mathbb{A}^2_{\mathbb{C}}$.
  (Némethi (1991), Némethi-Steenbrink (1994-95)).

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Basic tools and constructions (Denef-Loeser)

Arc spaces

We fix an algebraically closed field k of characteristic 0. For a variety X over k, we denote by $\mathcal{L}(X)$ and $\mathcal{L}_n(X)$ the spaces of arcs, resp. arcs mod t^{n+1}.

There is a $\mathbb{G}_m$-action on $\mathcal{L}_n(X)$ (resp. $\mathcal{L}(X)$):
$(\lambda.\varphi)(t) := \varphi(\lambda t)$.

Grothendieck Rings

We denote by $\mathcal{M}_X$ the localization of the Grothendieck ring of varieties over X with respect to the class $\mathbb{L}$ of the relative line. We will consider as well the $\mathbb{G}_m$-equivariant variant $\mathcal{M}_{X\times\mathbb{G}_m^r}^{\mathbb{G}_m}$, generated by classes of objects $Y \to X \times \mathbb{G}_m^r$ endowed with a good $\mathbb{G}_m$-action.

If $h : Z \longrightarrow X$ is a morphism of varieties, fiber product induces a pullback morphism $h^* : \mathcal{M}_X \longrightarrow \mathcal{M}_Z$ and composition induces a push-forward morphism $h_! : \mathcal{M}_Z \longrightarrow \mathcal{M}_X$.

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Relations in the Grothendieck ring

- Additivity
- $Y \times \mathbb{A}_k^n \longrightarrow Y$ with a $\mathbb{G}_m$-action on $Y \times \mathbb{A}_k^n$ lifting an action σ on Y. Then the class $[Y \times \mathbb{A}_k^n]$ is equal to the class $[Y]\mathbb{L}^n$.
- The class of an object $[Y]$ does not change when passing to the action induced by a finite morphism $\mathbb{G}_m \longrightarrow \mathbb{G}_m$.

Arc Spaces associated to a function

Let X be a smooth variety over k of pure dimension d and $f : X \to \mathbb{A}^1_k$. We set $X_0(f)$ for the zero locus of f, and consider, for $n \geq 1$, the constructible subset of $\mathcal{L}_n(X)$:

$$\mathcal{X}_n(f) := \left\{ \varphi \in \mathcal{L}_n(X) \;\middle|\; \mathrm{ord}_t f(\varphi) = n \right\}.$$

- The truncation map $\varphi \longmapsto \varphi(0)$ induces a map from $\mathcal{X}_n(f)$ to $X_0(f)$.
- $\mathcal{X}_n(f)$ is invariant by the $\mathbb{G}_m$-action on $\mathcal{L}_n(X)$.
- f induces a morphism $f_n : \mathcal{X}_n(f) \to \mathbb{G}_m$ homogeneous of weight n with respect to the $\mathbb{G}_m$-action on $\mathcal{X}_n(f)$.

Hence we can consider the class $[\mathcal{X}_n(f)]$ of $\mathcal{X}_n(f)$ in $\mathcal{M}^{\mathbb{G}_m}_{X_0(f)\times\mathbb{G}_m}$.

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Motivic Zeta Function

We now consider the motivic zeta function

$$Z_f(T) := \sum_{n\geq 1} [\mathcal{X}_n(f)]\, \mathbb{L}^{-nd}\, T^n$$

in $\mathcal{M}^{\mathbb{G}_m}_{X_0(f)\times\mathbb{G}_m}[[T]]$. Note that $Z_f = 0$ if $f = 0$ on X.

Rationality

Denef and Loeser showed that $Z_f(T)$ is a **rational series** by giving a formula for $Z_f(T)$ in terms of a log-resolution of $X_0(f)$.

$$Z_f(T) = \sum_{\emptyset \neq I \subset A} [U_I] \prod_{i \in I} \frac{T^{N_i(f)}\mathbb{L}^{-\nu_i}}{1 - T^{N_i(f)}\mathbb{L}^{-\nu_i}}$$

Z_f is a finite linear combination of geometric sums, hence they consider $\lim_{T \mapsto \infty} Z_f(T)$ in $\mathcal{M}^{\mathbb{G}_m}_{X_0(f) \times \mathbb{G}_m}$ and define the **Motivic Milnor fiber** of f as

$$\mathcal{S}_f := -\lim_{T \mapsto \infty} Z_f(T).$$

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Properties

- If f is a smooth function on X, then $S_f = [\mathrm{Id} : X_0(f) \times \mathbb{G}_m \longrightarrow X_0(f) \times \mathbb{G}_m]$.
- $\mathcal{S}_f := -\sum_{\emptyset \neq I \subset A} (-1)^{|I|} [U_I]$
- $\mathcal{S}_f = h_! \mathcal{S}_{f \circ h}$.

Motivic vanishing cycles

If we set

$$\mathcal{S}_f^{\phi} := (-1)^{d-1}(\mathcal{S}_f - [\mathbb{G}_m \times X_0(f)])$$

then $\mathcal{S}_f^{\phi}$ will be called the Motivic vanishing cycles of f and the virtual variety $i_x^*\mathcal{S}_f^{\phi}$ is the **Motivic vanishing cycles** of f at a point x of $X_0(f)$.

When $k = \mathbb{C}$, any class in $\mathcal{M}_{X_0(f)\times\mathbb{G}_m}^{\mathbb{G}_m}$ has an image in the Grothendieck group of monodromic Hodge structures by the canonical realization. Denef-Loeser proved that the image of $i_x^*\mathcal{S}_f^{\phi}$ is equal to the class of the Steenbrink Mixed Hodge Structure on the fiber at x of the sheaf $\phi_f(\mathbb{C}_X)$. From this one can recover A'Campo type formulae.

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Extensions

The virtual variety $\mathcal{S}_f$ *we have just defined is the value at* X *of an* $\mathcal{M}$*-linear group morphism*

$$\mathcal{S}_f : \mathcal{M}_X \longrightarrow \mathcal{M}^{\mathbb{G}_m}_{X_0(f)\times\mathbb{G}_m}.$$

(G-L-M, F. Bittner).

When $k = \mathbb{C}$, as before, $i_x^* \mathcal{S}_f([j : U \longrightarrow X])$ has a realization as the class of $\psi_{f,x}(j_! \mathbb{C}_U)$ in the suitable Grothendieck group.

For an open set $j : U \hookrightarrow X$, let $\mathcal{I}_F$ be any ideal defining the complement F of U in X. We consider, for $n \geq 1$, and γ a positive rational number, the variety

$$\mathcal{X}_n^{\gamma}(f, U) := \Big\{ \varphi \in \mathcal{L}_{\gamma n}(X) \;\Big|\; \mathrm{ord}_t f(\varphi) = n,\ \mathrm{ord}_t \varphi^*(\mathcal{I}_F) \leq \gamma n \Big\}.$$

The zeta function

$$Z_{f,U}^{\gamma}(T) := \sum_{n \geq 1} [\mathcal{X}_n^{\gamma}(f, U)]\, \mathbb{L}^{-nd}\, T^n$$

is a rational function. Its limit as T goes to infinity is independent of γ for γ large enough. For such a γ we set:

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Zeta function associated to r functions $g_1, \ldots, g_r$.

We set $X_0(\mathbf{g})$ for the zero locus of the ideal $(g_1, \ldots, g_r)$, and consider, for $\mathbf{n}$ in $\mathbb{N}^r_{>0}$, the variety

$$\mathcal{X}_{\mathbf{n}}(\mathbf{g}) := \left\{ \varphi \in \mathcal{L}_{\sigma(\mathbf{n})}(X) \;\middle|\; \mathrm{ord}_t g_i(\varphi) = n_i,\ 1 \leq i \leq r \right\}.$$

It defines a class in $\mathcal{M}^{\mathbb{G}_m}_{X_0(\mathbf{g}) \times \mathbb{G}_m^r}$. For any rational polyhedral open cone in $\mathbb{R}^r_{\geq 0}$ and any integral linear form ℓ, positive on $\overline{C} \setminus \{0\}$, the zeta function

$$Z^C_{\mathbf{g}}(T) := \sum_{\mathbf{n} \in C} [\mathcal{X}_{\mathbf{n}}(\mathbf{g})] \, \mathbb{L}^{-\sigma(\mathbf{n})d} \, T^{\ell(\mathbf{n})}$$

is rational and its limit, as T goes to infinity, is independent of ℓ. We denote it by $\mathcal{S}^C_{\mathbf{g}}$.

Convolution (Denef-Loeser, G-L-M)

Consider a r-variable quasi-homogeneous polynomial Q and a k-variety X_0. We define a **convolution operation**

$$\Psi_Q : \mathcal{M}^{\mathbb{G}_m}_{X_0 \times \mathbb{G}_m^r} \longrightarrow \mathcal{M}^{\mathbb{G}_m}_{X_0 \times \mathbb{G}_m}.$$

The morphism Ψ_Q is a group morphism and is $\mathcal{M}_{X_0}$-linear. For a variety $A \longrightarrow X_0 \times \mathbb{G}_m^r$ in $\mathrm{Var}^{\mathbb{G}_m}_{X_0 \times \mathbb{G}_m^r}$, the polynomial Q induces on A, by composition, a function we still denote by Q and we set:

$$\begin{aligned} &\Psi_Q([A \longrightarrow X_0 \times \mathbb{G}_m^r]) \\ &\qquad\qquad = -[Q : A \setminus Q^{-1}(0) \longrightarrow X_0 \times \mathbb{G}_m] + \mathcal{S}_Q(A). \end{aligned}$$

When $Q^{-1}(0)$ is a smooth subset of A and Q is of multiplicity e_i along the i-th component of $Q^{-1}(0)$, then

$$\begin{aligned}\Psi_Q([A \longrightarrow X_0 \times \mathbb{G}_m^r]) = -[Q : A \setminus Q^{-1}(0) \longrightarrow X_0 \times \mathbb{G}_m] \\ + \sum_i [\pi_{e_i} : Q^{-1}(0) \times \mathbb{G}_m \longrightarrow X_0 \times \mathbb{G}_m]\end{aligned}$$

where π_e is the product of the projection from $Q^{-1}(0)$ to X_0 by the morphism $\lambda \longmapsto \lambda^e$ from $\mathbb{G}_m$ to $\mathbb{G}_m$.

The definition of the convolution extends easily when replacing $\mathbb{G}_m^r$ by $\mathbb{A}_k^1 \times \mathbb{G}_m$. Thence, we consider the following convolution operation

$$\Psi_Q : \mathcal{M}_{X_0 \times \mathbb{A}_k^1 \times \mathbb{G}_m}^{\mathbb{G}_m} \longrightarrow \mathcal{M}_{X_0 \times \mathbb{G}_m}^{\mathbb{G}_m}.$$

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Spectrum

The convolution by the function Σ (usual convolution associated to the sum) induces the product on the spectra, namely:

$$\mathrm{Sp}(\Psi_\Sigma(A \boxtimes B)) = \mathrm{Sp}(A)\,\mathrm{Sp}(B),$$

where

$$\mathrm{Sp}(H) = \sum_{\substack{p,q \\ \alpha\in[0,1)}} \dim H_\alpha^{p,q}\; t^{\alpha+p}.$$

Examples

We denote by i the inclusion of $X_0(\mathbf{g}) = g_1^{-1}(0) \cap \ldots g_r^{-1}(0)$ in any larger set.

- **Thom-Sebastiani formula**:

$$i^* \mathcal{S}^{\phi}_{g_1 \oplus g_2} = \Psi_{\Sigma}(\mathcal{S}^{\phi}_{g_1} \boxtimes \mathcal{S}^{\phi}_{g_2})$$

- **f is a r-variable, non degenerate with respect to its Newton polyhedron polynomial** and $g_1, \ldots, g_r$ have different sets of variables.

$$i^* S_{f \circ \mathbf{g}}(U) = \sum_{\delta \in \Gamma} \Psi_{f_\delta}(\mathcal{S}_{\mathbf{g}}^{\sigma(\delta)}),$$

where Γ is the set of compact faces of the Newton polyhedron not contained in any coordinate hyperplane and U the open set in X where none of the g_i vanish.

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The tree associated to a polynomial $f(x, y)$ at the origin.

Eggers, Kuo-Lu,...

- To any Puiseux expansion $y(x)$ we associate an infinite metric rooted tree $\tau(y)$ with only one branch. Its vertices are the rational exponents with non zero coefficients and they are labeled with their coefficient (or rather a monic polynomial of degree 1). The distance to the root is defined as the height of a vertex, that is the value of the exponent.
- To a polynomial f in $k[x, y]$ and a point (the origin) we associate a tree $\tau(f)$. Consider the union $\cup|\tau(y)|$ for all Puiseux expansions y such that $f(x, y(x)) = 0$ and take the separated quotient by the natural action of roots of unity. This quotient is the underlying unlabeled tree $|\tau(f)|$. We label each vertex with (a convenient homogenization of) the product of the labels of its inverse image by the quotient map.

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The tree associated to a polynomial $f(x, y)$ at the origin.

- We say that a vertex v of $\tau(f)$ is a **rupture vertex** if its label is not a power of a degree one polynomial.
- The **augmented set of rupture vertices** is the set of the rupture vertices together with the first vertex v_0 of positive height.

The formula

Theorem

Assume g_1 and g_2 depend on different sets of variables. Then the following formula holds:

$$i^* \mathcal{S}_{f\circ\mathbf{g}} = \mathcal{S}_{(g_2)^{m_p}}(X_0(g_1)) - \sum_v \Psi_{Q_v}(A_v)$$

where the sum is runs over the augmented set of rupture vertices of the tree $\tau(f)$.

where m is the order of $f(0,y)$ and the A_v are elements of the group $\mathcal{M}^{\mathbb{G}_m}_{X_0(g)\times\mathbb{A}_1\times\mathbb{G}_m}$ defined by induction on the rupture vertices, starting with

$$A_{v_0} := \mathcal{S}_{g_1} \boxtimes \mathcal{S}_{g_2}.$$

For $\mathbf{g} = Id_{\mathbb{A}^2_k}$: G. Guibert.

Question

Compute the nearby cycles of the composed map $f \circ \mathbf{g}$ *of* f *with a mapping* $\mathbf{g} : X \longrightarrow \mathbb{A}^2_{\mathbb{C}}$*.*