Topological Rings in Rigid Geometry

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Raynaud's theorem

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$$\left\{\begin{matrix}\text{coherent } V\text{-formal} \\ \text{schemes of finite type}\end{matrix}\right\} \Big/ \left(\begin{matrix}\text{admissible} \\ \text{blow-ups}\end{matrix}\right) \xrightarrow{\sim} \left\{\begin{matrix}\text{coherent rigid} \\ \text{spaces over } K\end{matrix}\right\}.$$

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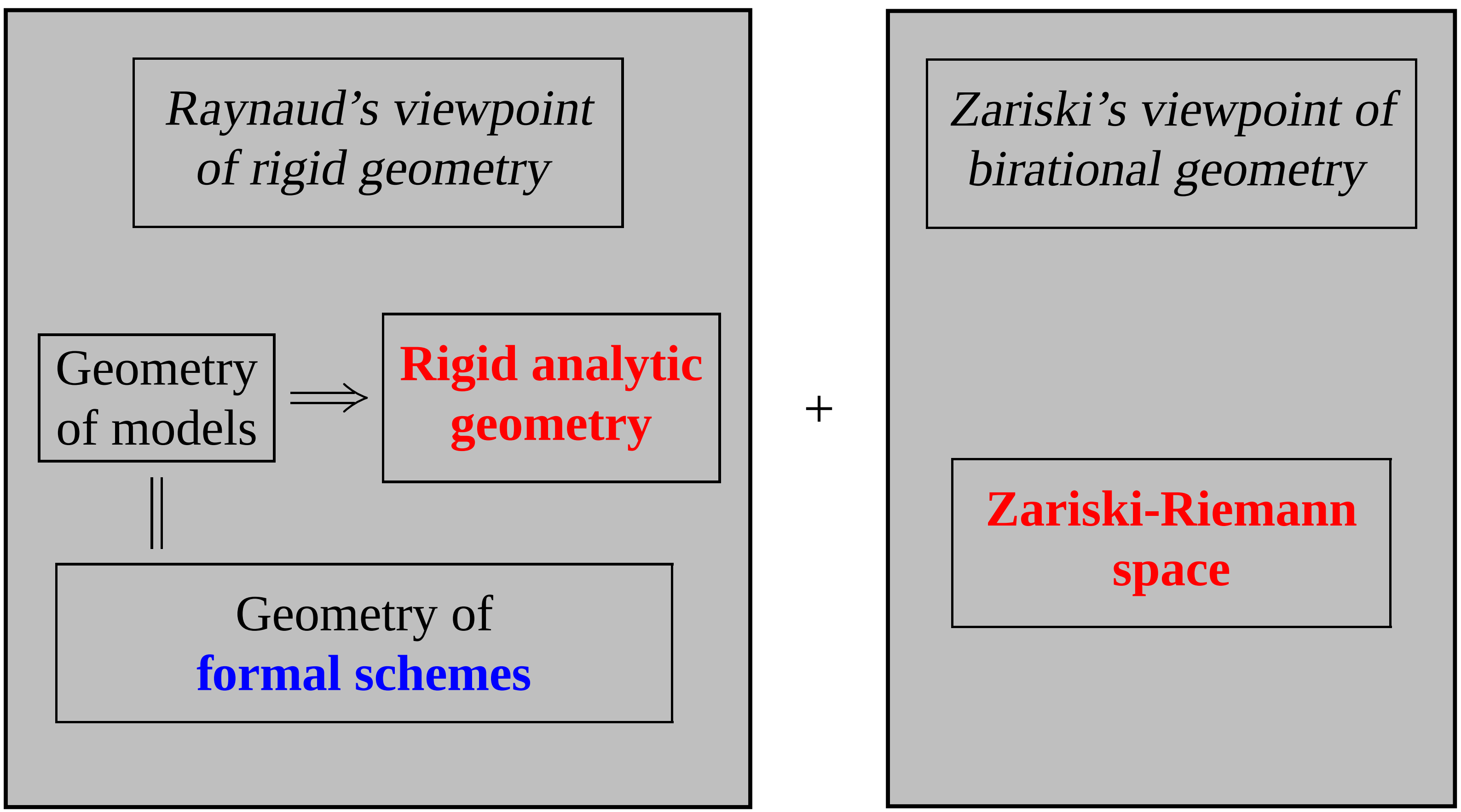
- Strong analogue with birational geometry.
- More general notion of rigid spaces; e.g.
  - rigid henselian spaces, rigid Zariskian spaces
  - absolute notion of rigid spaces...

Birational approach

Idea: Enhancing "birational-geometry-like aspect" of rigid geometry by applying Zariski's classical approach to birational geometry.



Visualization: Zariski-Riemann space

- Rigid space = patching of coherent (= q-cpt & q-sep) objects.
- Coherent rigid spaces have coherent formal models (Raynaud's Theorem).

$\rightsquigarrow$ **Zariski-Riemann space**: for coherent $\mathscr{X}$ (in general, by patching),

$$\langle \mathscr{X} \rangle = \varprojlim \text{(all formal models)}.$$

– The limit is cofiltered. Admissible blow-ups of a fixed formal model comprise a cofinal portion of all formal models.

– For coherent $\mathscr{X}$, the topological space $\langle \mathscr{X} \rangle$ is coherent (= spectral), that is:

- sober;
- $\exists$ open basis consisting of quasi-compact open subsets;
- quasi-separated; i.e., U, V: quasi-compact $\Rightarrow U \cap V$: quasi-compact;
- quasi-compact.

N.B. Filtered projective limit of coherent topological spaces and quasi-compact maps is again coherent.

Visualization: Zariski-Riemann space

- **Two structure sheaves**
  - $\mathscr{O}^{\mathrm{int}}_{\mathscr{X}}$ (integral structure sheaf): given by the limit,
  - $\mathscr{O}_{\mathscr{X}}$ ((rigid) structure sheaf) := $\mathscr{O}^{\mathrm{int}}_{\mathscr{X}}[\frac{1}{a}]$.

  $\rightsquigarrow$ **Zariski-Riemann triple** $(\langle\mathscr{X}\rangle, \mathscr{O}^{\mathrm{int}}_{\mathscr{X}}, \mathscr{O}_{\mathscr{X}})$

  - At least in classical situation, $(\langle\mathscr{X}\rangle, \mathscr{O}^{\mathrm{int}}_{\mathscr{X}}, \mathscr{O}_{\mathscr{X}})$ coincides with the associated adic space. If $\mathscr{X} = (\mathrm{Spf}\, A)^{\mathrm{rig}}$, then

$$\Gamma(\langle\mathscr{X}\rangle, \mathscr{O}_{\mathscr{X}}) = A[\tfrac{1}{a}], \quad \Gamma(\langle\mathscr{X}\rangle, \mathscr{O}^{\mathrm{int}}_{\mathscr{X}}) = A^{\mathrm{int}}.$$

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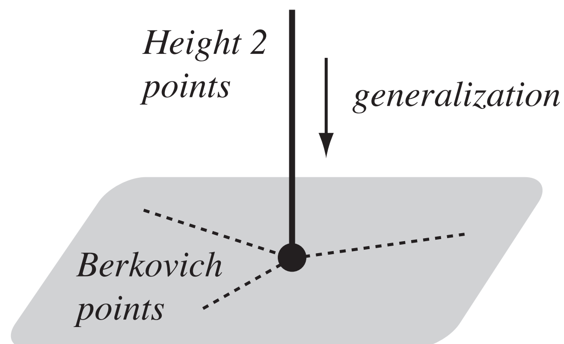
- **Separation map** $\mathrm{sep}_{\mathscr{X}} : \langle \mathscr{X} \rangle \longrightarrow [\mathscr{X}]$.
  - Defined by universal mapping property purely in the language of topological spaces; universal T_1 quotient.
  - $[\mathscr{X}]$ (the separated quotient) is homeomorphic to the underlying topological space of the associated Berkovich space of $\mathscr{X}$.

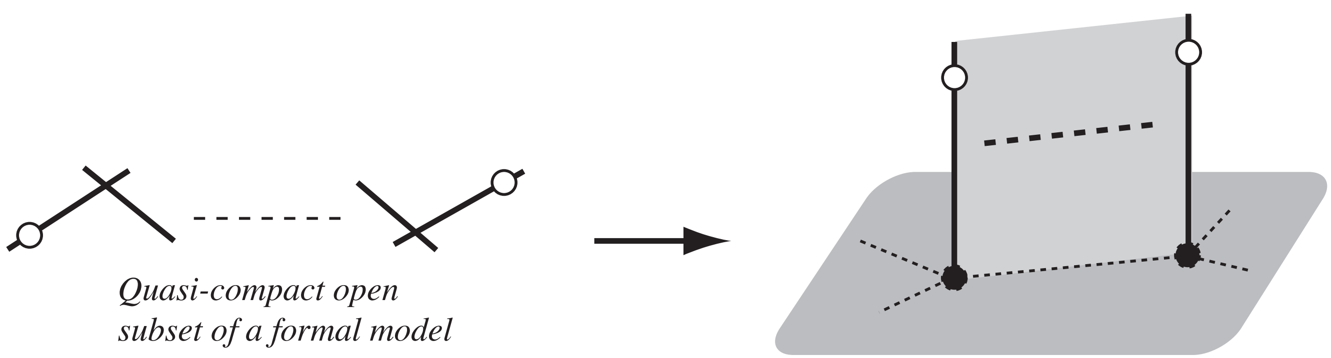
Visualization: Zariski-Riemann space

• Points of $\langle \mathscr{X} \rangle$ corresponds to valuations (similar to the classical situation).

Example. Analytic curve / cDVF



Annulus



Zariski's technique

A kind of inductive argument that has the following general structure:

$$\left.\begin{array}{c} \boxed{\text{local argument}} + \boxed{\text{birational patching}} \\ \boxed{\textbf{q-cpt'ness}\text{ of ZR space}} \end{array}\right\} \Longrightarrow \boxed{\textbf{desired result}}$$

E.g.

- Resolution of singularities (Zariski, Abhyankar)
- Embedding theorem (Nagata)
- (Formal) flattening theorem

Formal flattening theorem

V: a-adically complete valuation ring, $a \in \mathfrak{m}_V \setminus \{0\}$.

Formal flattening theorem (Bosch-Raynaud, Fujiwara). Let $f\colon X \to S$ be a morphism of finite type coherent formal schemes of finite type over V. Then the following conditions are equivalent:

(1) $f^{\mathrm{rig}}\colon X^{\mathrm{rig}} \to S^{\mathrm{rig}}$ is flat, that is, $\langle f^{\mathrm{rig}}\rangle\colon \langle X^{\mathrm{rig}}\rangle \to \langle S^{\mathrm{rig}}\rangle$ is flat as a mapping of local ringed spaces (with the rigid structure sheaf);

(2) there exists an admissible blow-up $S' \to S$ such that the strict transform $f'\colon X' \to S'$ is flat.

Absolute rigid space?

‘Points’ of rigid space = points of ZR-space.

$\rightsquigarrow$ Valuation rings (of arbitrary height) = **point-objects** in rigid geometry.

Cf. – Classical algebraic geometry: varieties over a field
– Scheme theory: fields = point objects, varieties = fiber objects

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$\rightsquigarrow$ Want to have a nice class of complete adic rings —

- containing Noetherian rings and a-adically complete valuation rings of arbitrary height;
- stable under topologically of finite type extension and under base change by topologically of finite type map;
- satisfying the Artin-Rees type condition;
- suitable for geometry, homological algebra, etc. that allows one to generalize theorems in [EGA, **III**] (especially, GFGA theorems).

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- A: I-adically adhesive $\Longrightarrow$ A is Noetherian outside I.
- A: I-adically adhesive $\Longrightarrow$ any finite A-algebra B is IB-adically adhesive.

Example. V: valuation ring, $a \in \mathfrak{m}_V \setminus \{0\}$.

$$V \text{ is } a\text{-adically adhesive} \Longleftrightarrow V \text{ is } a\text{-adically separated.}$$

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$$V \text{ is } a\text{-adically adhesive} \iff V \text{ is } a\text{-adically separated.}$$

Definition. A is said to be ***I-adically universally adhesive***

$\overset{\text{def}}{\iff}$ $A[X_1, \ldots, X_n]$ is I-adically adhesive for any n.

First properties

• **(Artin-Rees type property)** A: I-adically universally adhesive
$\Longrightarrow$ **(AR)**: for any finitely generated A-module M, any A-submodule $N \subseteq M$, and any $n \geq 0$, there exists $m \geq 0$ such that

$$N \cap I^m M \subseteq I^n N.$$

(In particular, the subspace topology on N coincides with the I-adic topology.)

$\Longrightarrow A \to \widehat{A}$ is flat.

A technically important consequence: if A is I-adically complete and I-adically universally adhesive, and if $f_0, \ldots, f_r \in A$ such that $(f_0, \ldots, f_r) = A$, then

$$A \longrightarrow \prod_{i=0}^{r} A\langle\langle f_i^{-1} \rangle\rangle$$

is faithfully flat.

First properties

• **(Structure sheaf coherency)** A: I-adically universally adhesive, and I-torsion free
$\Longrightarrow$ any finitely presented A-algebra is a coherent ring; in particular, the structure sheaf $\mathscr{O}$ of $\operatorname{Spec} A$ is coherent as an $\mathscr{O}$-module.

• **(Flat descent)** $A \to B$ faithfully flat, and B is IB-adically adhesive (resp. I-adically universally adhesive)
$\Longrightarrow$ A is I-adically adhesive (resp. I-adically universally adhesive).

• **(Local criterion of flatness)** $A \to B$: adic map between I-adically universally adhesive rings, M: a finitely generated B-module. Suppose that B is I-adically Zariskian (that is, $1 + IB \subset B^{\times}$). Then the following conditions are equivalent:

(a) M is A-flat;
(b) $M_k = M/I^{k+1}M$ is flat over $A_k = A/I^{k+1}$ for any $k \geq 0$.

I-adically t.u.a. rings

Definition. A is said to be ***I-adically topologically universally adhesive***

$$\overset{\text{def}}{\Longleftrightarrow} \begin{cases} \bullet\ A \text{ is } I\text{-adically universally adhesive;} \\ \bullet\ \text{for any } n\text{, the } I\text{-adic completion } \widehat{A}\langle\langle X_1,\ldots,X_n\rangle\rangle \text{ of } A[X_1,\ldots,X_n] \text{ is} \\ \quad\text{again } I\text{-adically universally adhesive.} \end{cases}$$

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Example. Noetherian rings (with arbitrary ideal) are clearly t.u.a.

Theorem (Gabber).

Any a-adically complete valuation ring of arbitrary height is a-adically t.u.a.

I-adically t.u.a. rings

Theorem. Let A be a ring, and $I = (a)$ a principal ideal. Then the following conditions are equivalent:

(a) A is a-adically t.u.a.;
(b) $A\langle\langle X_1, \ldots, X_n\rangle\rangle$ is a-adically adhesive for any $n \geq 0$.

Proposition. Let A be a ring, and I a finitely generated ideal. Suppose:

(a) $\widehat{A}$ (the I-adic completion) is I-adically adhesive;
(b) $A \to \widehat{A}$ is flat;
(c) A is Noetherian outside I.

Then A is I-adically adhesive.

N.B. One can construct complete t.u.a. rings that have Berkovich' generalized affinoid rings (e.g. $K\langle\langle r^{-1}x\rangle\rangle$) as their generic fibers.

Adequate formal schemes

Definition. An adic formal scheme X of finite ideal type is said to be ***adequate*** if, for any finite type map $\mathrm{Spf}\, A \to X$ from an affine formal scheme, A is I-adically t.u.a., where I is an ideal of definition of A.

Example.

(1) Locally Noetherian formal schemes.
(2) Formal schemes of finite type over $\mathrm{Spf}\, V$, where V is an a-adically complete valuation ring (of arbitrary height).

Adequate formal schemes

Basic properties.

• Make sense to define ‘of finite presentation’ map between adequate formal schemes.

• If X is adequate, and $\mathscr{O}_X$ is $\mathscr{I}$-torsion free (where $\mathscr{I}$ is an ideal ideal of definition), then, for any $Y \to X$ of finite presentation, $\mathscr{O}_Y$ is coherent as $\mathscr{O}_Y$-module.

• Consistent notion of flatness; faithfully flat descent (of the so-called *adically quasi-coherent sheaves*).

Theorems

Theorem (GFGA Existence Theorem). Let B be I-adically complete t.u.a., I-adically topologically universally cohesive (e.g. I-torsion free), $f\colon X \to Y = \operatorname{Spec} B$ proper morphism of algebraic spaces of finite presentation; then

$$\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(X) \xrightarrow{\sim} \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\widehat{X})$$

exact equivalence of triangulated categories.

Theorem (Finiteness Theorem). Let $f\colon X \to Y$ be a proper of finite presentation between quasi-compact adequate formal schemes, and Y universally cohesive. Then, $\mathrm{R}f_*$ maps $\mathbf{D}^{*}_{\mathrm{coh}}(X)$ to $\mathbf{D}^{*}_{\mathrm{coh}}(Y)$, where $* =$“ ”$, +, -, \mathrm{b}$.

Cf. Ullrich, Math. Ann. 301

General rigid space

Definition of rigid spaces:

- **Coherent rigid space** = an object of the quotient category

$$\left\{\begin{matrix}\text{coherent adequate}\\ \text{formal schemes}\end{matrix}\right\}\Big/\left(\begin{matrix}\text{admissible}\\ \text{blow-ups}\end{matrix}\right).$$

- **(General) rigid space** = patching of coherent rigid spaces.

- **Visualization** = Zariski-Riemann triple $(\mathscr{X}, \mathscr{O}^{\mathrm{int}}_{\mathscr{X}}, \mathscr{O}_{\mathscr{X}})$.

GAGA

- B: I-adically t.u.a. ring, $D = V(I) \subseteq S = \operatorname{Spec} B$, $U = S \setminus D$.

$\rightsquigarrow$ **GAGA functor**

$$\left\{\begin{matrix}\text{locally of finite} \\ \text{type } U\text{-schemes}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{rigid spaces over} \\ \mathscr{S} = (\operatorname{Spf} B)^{\mathrm{rig}}\end{matrix}\right\}, \quad X \longmapsto X^{\mathrm{an}}$$

(Ingredient for the construction: Nagata's embedding theorem)

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GAGA Existence Theorem. Let $f\colon X \to U$ be a proper U-scheme. Then the comparison functor

$$\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(X) \longrightarrow \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(X^{\mathrm{an}})$$

is an exact equivalence of triangulated categories.

A theorem

Theorem. Let S be a coherent adequate formal scheme, and $X \to S$ a formal algebraic space of finite type. Then there exists an admissible blow-up $X' \to X$ such that X' is a formal scheme.

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- This theorem + Nagata's embedding theorem for algebraic spaces

$\rightsquigarrow$ GAGA functor for algebraic spaces (cf. Conrad-Temkin).