Smooth transition mixture GARCH models for forecasting financial risk measures

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Outline

- Introduction
- Value at Risk and Expected Shortfall
- Nonlinear volatility modelling
- Priors for identification and properness
- MCMC methods and simulation
- Value at Risk and Expected Shortfall application
- Conclusions and future work

Value at Risk and Expected Shortfall

Value-at-Risk

VaR is the α-level quantile of the l-period asset return *forecast* distribution

$$\alpha = Pr(r_t(l) < \text{VaR}_\alpha(l)|\Omega_t)$$

where $r_t(l) = \log(P_{t+l}) - \log(P_t)$

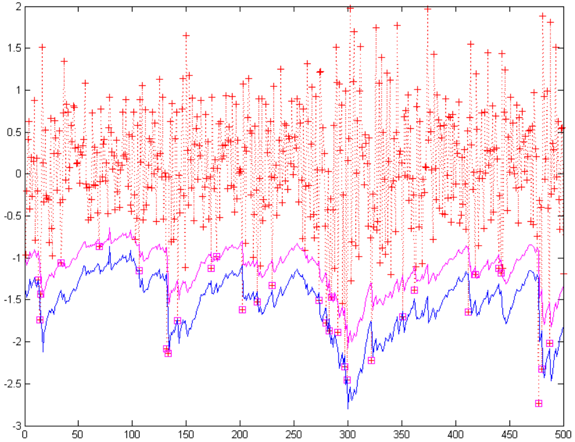
Expected shortfall: Conditional Value-at-Risk

The **conditional** l-period VaR for holding an asset is:

$$\text{ES}_\alpha(l) = E\left[r_t(l)|r_t(l) < \text{VaR}_\alpha(l)\right]$$

Dynamic VaR and ES for asset returns

Australian market index daily asset returns 2005-2007



Properties of (daily) asset returns

- (Close to) unpredictable in mean
- Dynamic, locally persistent conditional variance
- Fat-tailed, leptokurtic: subject to extreme observations
- Mildly negatively skewed (often)
- Asymmetric conditional variance (and mean)

Continuous Mixture GARCH Model

Mean

$$
\begin{aligned}
y_t &= \mu_t^{(1)} + F(z_{t-d};\gamma, c)\mu_t^{(2)} + a_t \\
\mu_t^{(l)} &= \phi_0^{(l)} + \sum_{i=1}^{p} \phi_i^{(l)} y_{t-i} + \sum_{i=1}^{r} \psi_i^{(l)} x_{t-i}
\end{aligned}
$$

Volatility

$$
\begin{aligned}
a_t &= \sqrt{h_t}\varepsilon_t, \qquad \varepsilon_t \sim t_\nu^*(0,1), \\
h_t &= h_t^{(1)} + F(z_{t-d};\gamma, c)h_t^{(2)} \\
h_t^{(l)} &= \alpha_0^{(l)} + \sum_{i=1}^{g} \alpha_i^{(l)} a_{t-i}^2 + \sum_{i=1}^{k} \beta_i^{(l)} h_{t-i}, \;\; l = 1, 2.
\end{aligned}
$$

Mixture probability functions

Sign Asymmetry

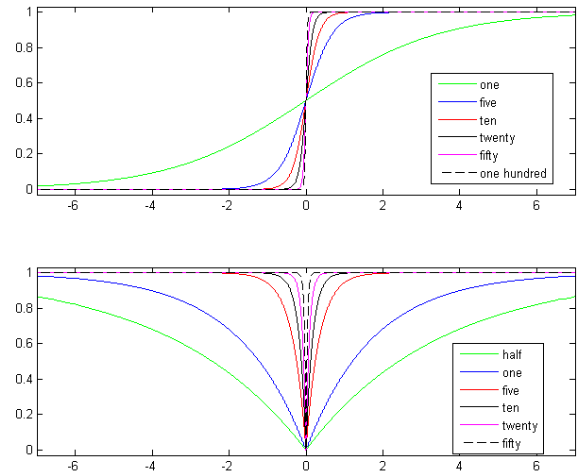
$$F(z_{t-d};\gamma,c) = \frac{1}{1+\exp\left\{-\gamma\left(\frac{z_{t-d}-c}{s_z}\right)\right\}},$$

Size Asymmetry

$$F_t(z_{t-d};\gamma,c) = 1-\exp\left[-\gamma\, g\left(\frac{z_{t-d}-c}{s_z}\right)\right],\ \gamma>0,$$

Sign and Size Asymmetry

Logistic and exponential functions



Restrictions

Non-negative variances

$$0 < \alpha_0^{(1)} < b_1 \quad 0 < \alpha_i^{(1)} < b_2 \; 0 < \beta_i^{(1)} < b_3$$
$$0 < \alpha_0^{(1)} + \alpha_0^{(2)} < b_1 \qquad 0 < \textstyle\sum_i \alpha_i^{(1)} + \alpha_i^{(2)} < b_2$$
$$0 < \textstyle\sum_j \beta_j^{(1)} + \beta_j^{(2)} < b_3.$$

Covariance-stationarity

$$\sum_i \alpha_i^{(1)} + \beta_i^{(1)} \quad < \quad b_4$$
$$\sum_{i=1}^{g} (\alpha_i^{(1)} + 0.5\alpha_i^{(2)}) + \sum_{j=1}^{k} (\beta_i^{(1)} + 0.5\beta_i^{(2)}) \quad < \quad 1.$$

Mixture prior for identification

George and McCulloch, 1993

$$\phi_j^{(i)} | \delta_j^{(i)} \sim (1 - \delta_j^{(i)}) N(0, k^2 \tau_j^{(i)2}) + \delta_j^{(i)} N(0, \tau_j^{(i)2}),$$
$$j = 1, \ldots, p + r$$

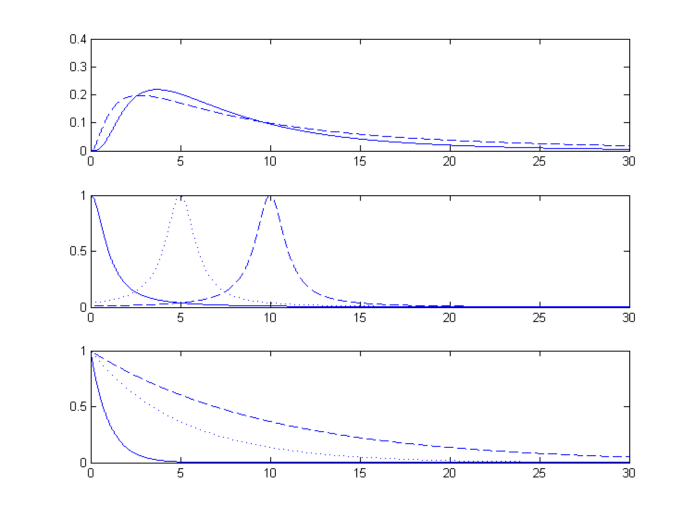
$$\delta_j^{(i)} | \gamma = \begin{cases} 1 & \text{if } i = 1 \text{ or } \gamma > \xi \ , \\ 0 & \text{if } i = 2 \text{ and } \gamma \leq \xi. \end{cases}$$

$$\boldsymbol{\theta}_i | \boldsymbol{\delta}_i \sim N_{p+r}(\mathbf{0}, \mathbf{D}_i \mathbf{V}_i \mathbf{D}_i),$$

- k chosen small to shrink mean parameters to 0
- ξ chosen so that $F(\min(z_{t-d})) \approx 0$

Proper prior on mixing

Gerlach and Chen (2008), Chen, Gerlach, Choy and Lin (2010)



Adaptive MH MCMC sampling

- Student-t proposal tuned random walk Metropolis

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- Observed acceptance rates $\in (5, 40)\%$
- Gelman's R very close to 1 from diverse positions

Simulation Study: 100 data sets, $n = 2000$

		γ=1		γ=2		γ=4	
	real	Med	Std	Med	Std	Med	Std
$\phi_0^{(1)}$	-0.10	-0.061	0.104	-0.081	0.106	-0.095	0.085
$\phi_1^{(1)}$	-0.20	-0.143	0.057	-0.197	0.058	-0.198	0.049
$\psi_1^{(1)}$	0.40	0.427	0.064	0.410	0.067	0.399	0.061
$\phi_0^{(2)}$	0.10	0.024	0.194	0.072	0.185	0.097	0.134
$\phi_1^{(2)}$	0.30	0.207	0.096	0.298	0.091	0.305	0.069
$\psi_1^{(2)}$	0.15	0.129	0.077	0.147	0.069	0.153	0.063
ν	7.00	7.153	1.179	7.323	1.268	7.150	1.188
γ		1.666	4.692	2.232	3.956	4.123	5.310
c	-0.20	-0.067	0.308	-0.167	0.234	-0.186	0.154
		γ=5		γ=7		γ=10	
$\phi_0^{(1)}$	-0.10	-0.098	0.077	-0.098	0.074	-0.106	0.071
$\phi_1^{(1)}$	-0.20	-0.197	0.046	-0.208	0.044	-0.204	0.041
$\psi_1^{(1)}$	0.40	0.400	0.059	0.400	0.057	0.393	0.057
$\phi_0^{(2)}$	0.10	0.100	0.114	0.101	0.107	0.118	0.098
$\phi_1^{(2)}$	0.30	0.299	0.062	0.315	0.059	0.310	0.053
$\psi_1^{(2)}$	0.15	0.148	0.062	0.150	0.061	0.146	0.062
ν	7.00	7.381	1.26	7.029	1.130	7.120	1.171
γ		5.473	7.091	6.376	7.637	9.065	10.953
c	-0.20	-0.214	0.126	-0.206	0.113	-0.212	0.090

Empirical Study

Data:

- Six international daily market indices:
  — Australia (AORD), UK (FTSE100), Italy (MIBTel)
  — Canada (TSE300), Japan (Nikkei) and Taiwan (TWSI)
- **Sample period**: $\approx$ 2800 trading days from January, 1994
- **Forecast period**: March, 2005 to March-April, 2007

Methods/models:

- Short (25 days) and Medium (100 days) sample percentiles
- Symmetric CAViaR model: Engle and Manganelli (2004)
- Risk-Metrics (JP Morgan), GARCH-t, GJR-GARCH-t, DT-GARCH-t with S&P500 threshold.
- Proposed LST-GARCH-t model with S&P500 threshold.

Parameter Estimates, 1994-2005

	Australia		UK		Italy	
	Est	std	Est	std	Est	std
$\psi_1^{(1)}$	0.32	(0.03)	0.34	(0.04)	0.25	(0.04)
$\psi_1^{(2)}$	-0.035	(0.03)	-0.10	(0.05)	-0.11	(0.06)
$\alpha_1^{(1)}$	0.08	(0.02)	0.08	(0.02)	0.09	(0.03)
$\beta_1^{(1)}$	1.04	(0.04)	1.06	(0.04)	1.06	(0.04)
$\beta_1^{(2)}$	-0.22	(0.08)	-0.25	(0.09)	-0.33	(0.09)
γ	1.40	(0.5)	2.22	(2.1)	1.19	(0.5)
c	0.25	(0.18)	-0.01	(0.13)	0.35	(0.19)
ν	10.6	(1.78)	25.8	(11.7)	9.8	(1.7)
Post prob.	**(0.95, 0.97)**		**(1.0,1.0)**		**(0.96,0.999)**	
	Canada		Japan		Taiwan	
$\psi_1^{(1)}$	0.074	(0.03)	0.38	(0.04)	0.26	(0.05)
$\psi_1^{(2)}$	0.004	(0.05)	0.04	(0.07)	0.07	(0.06)
$\alpha_1^{(1)}$	0.08	(0.02)	0.09	(0.03)	0.08	(0.03)
$\beta_1^{(1)}$	**1.09**	(0.05)	0.94	(0.04)	0.95	(0.03)
$\beta_1^{(2)}$	-0.39	(0.10)	-0.05	(0.06)	-0.06	(0.05)
γ	1.37	(0.6)	5.8	(6.1)	7.2	(6.3)
c	0.22	(0.24)	-0.41	(0.24)	-0.48	(0.16)
ν	6.9	(0.8)	8.6	(1.2)	5.6	(0.6)
Post prob.	**(0.995, 1.0)**		**(0.90,0.94)**		**(0.96, 0.98)**	

VaR Forecast assessment

Violation rates:

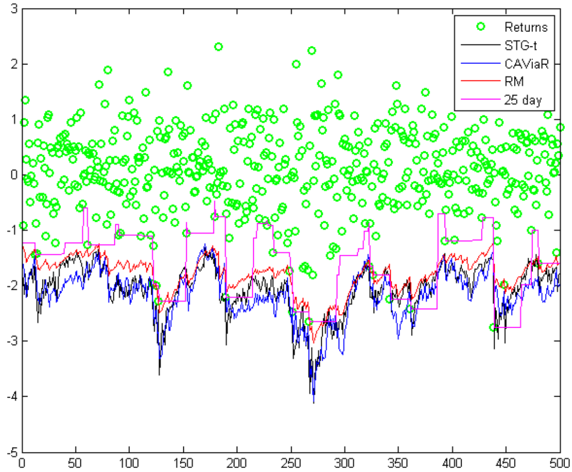
$$\text{VRate} = \frac{1}{m} \sum_{t=n+1}^{n+m} I(y_t < \text{VaR}_t)$$

- VRate $\approx \alpha \equiv \hat{\alpha}/\alpha \cong 1$
- Compare and rank across models
- PLUS: 3 popular tests: unconditional, conditional coverage (UC, CC) and DQ test

Ratio of $\hat{\alpha}/\alpha$ at $\alpha = 0.01, 0.05$

$\alpha = 0.01$		Aust.	UK	Italy	Canada	Japan	Taiwan
	25 day	4.40	4.60	3.80	4.40	4.20	4.40
	100 day	2.20	2.00	2.20	1.80	1.80	1.80
	CAViaR	1.60	1.20	0.80	1.20	1.20	1.20
	RM	2.60	2.60	2.80	2.80	2.20	2.20
	G-t	1.40	1.20	1.80	1.40	1.20	1.60
	GJR-t	1.20	1.40	1.80	1.20	1.40	1.40
	DTG-t	1.20	1.00	1.80	1.40	1.20	1.60
	STG-t	1.00	0.80	1.80	1.40	1.20	1.60
$\alpha = 0.05$	25 day	1.52	1.60	1.36	1.32	1.24	1.48
	100 day	1.28	1.20	1.24	1.36	1.20	1.28
	CAViaR	1.20	1.00	0.72	1.00	0.84	0.64
	RM	1.24	1.12	1.12	1.20	1.04	1.04
	G-t	1.24	1.16	1.00	1.24	0.96	0.72
	GJR-t	1.28	1.16	1.00	1.20	1.00	0.60
	DTG-t	1.20	1.04	0.96	1.20	1.20	0.72
	STG-t	1.16	0.96	0.88	1.20	1.16	0.72

1% VaR forecasts: Canada



Summary: Ratio of $\hat{\alpha}/\alpha$ and Rank of $\hat{\alpha}/\alpha$

$\alpha = 0.01$		Mean	Median	Std	1st	Mean	Std
	25 day	4.30	4.40	3.63	0	8.00	7.67
	100 day	1.97	1.90	1.08	0	6.00	5.48
	CAViaR	1.20	1.20	0.33	4	2.42	2.22
	RM	2.53	2.60	1.70	0	7.00	6.57
	G-t	1.43	1.40	0.53	1	3.58	2.89
	GJR-t	1.40	1.40	0.49	1	3.25	2.89
	DTG-t	1.37	1.30	0.50	2	2.92	2.40
	STG-t	1.30	1.30	0.50	2	2.83	2.35
$\alpha = 0.05$	25 day	1.42	1.42	0.48	0	7.83	7.50
	100 day	1.26	1.26	0.29	0	6.08	5.95
	CAViaR	0.90	0.92	0.23	2	3.75	4.08
	RM	1.13	1.12	0.16	1	3.50	3.08
	G-t	1.05	1.08	0.21	1	3.75	3.54
	GJR-t	1.04	1.08	0.25	2	3.75	3.68
	DTG-t	1.05	1.12	0.20	0	3.50	3.13
	STG-t	1.01	1.06	0.19	1	3.17	2.74

VaR Model rejections at $\alpha = 0.01, 0.05$

$\alpha = 0.01$	UC	Ind	CC	DQ	Total (out of 6)	
	25 day	6	1	6	6	6
	100 day	4	1	1	6	6
	CAViaR	0	0	0	5	5
	RM	6	0	5	6	6
	G-t	0	0	0	5	5
	GJR-t	0	0	0	4	4
	DTG-t	0	0	0	3	3
	STG-t	0	0	0	3	3
$\alpha = 0.05$	25 day	3	0	1	6	6
	100 day	0	1	1	5	5
	CAViaR	1	0	0	0	1
	RM	0	0	0	1	1
	G-t	0	0	0	0	0
	GJR-t	1	0	0	0	1
	DTG-t	0	0	0	1	1
	STG-t	0	0	0	1	1

Theoretical Expected Shortfall

Distribution	ES_α
$N(0,1)$	$-\frac{1}{\alpha}\phi\left(\Phi^{-1}(\alpha)\right)$
t_ν^*	$-\frac{1}{\alpha}t_\nu\left(T_\nu^{-1}(\alpha)\right)\left(\frac{\nu+(T_\nu^{-1}(\nu))^2}{\nu-1}\right)\sqrt{\frac{\nu-2}{\nu}}$

		δ_α			
α	$N(0,1)$	t_2^*0	t_1^*0	t_7^*	t_5^*
0.01	0.0038	0.0037	0.0036	0.0035	0.0033
0.05	0.0196	0.0190	0.0184	0.0178	0.0171
α	$\Phi\left(ES_\alpha\right)$	$T_\nu\left(ES_\alpha\sqrt{\frac{\nu}{\nu-2}}\right)$			
Exp. no. (0.01)	1.92	1.86	1.80	1.75	1.67
Exp. no. (0.05)	9.79	9.50	9.20	8.92	8.55

Ratio of $\hat{\delta}_{alpha}/\delta_\alpha$ at $\alpha = 0.01, 0.05$

$\alpha = 0.01$	True (%)	Aust	UK	Italy	Canada	Japan	Taiwan
RM	0.38	3.68	2.63	5.26	3.16	4.21	4.74
G-t	0.355	1.69	2.15	1.683	1.73	1.69	1.76
GJR-t	0.356	2.23	1.61	1.680	1.73	1.125	1.16
DTG-t	0.357	1.65	1.059	1.675	1.72	1.124	2.944
STG-t	0.357	1.65	1.061	1.675	1.72	1.123	2.939
$\alpha = 0.05$	True (%)	Aust	UK	Italy	Canada	Japan	Taiwan
RM	1.96	1.84	1.63	1.74	1.84	1.53	1.63
G-t	1.81	1.87	1.37	1.65	1.692	0.993	1.03
GJR-t	1.82	1.53	0.95	1.75	1.690	1.10	1.02
DTG-t	1.824	1.30	1.25	1.42	1.681	0.989	0.921
STG-t	1.824	1.19	0.94	1.31	1.681	0.88	0.920

Forecast comparisons of $\hat{\delta}_{alpha}/\delta_\alpha$:

	Ratio of $\hat{\delta}_{alpha}/\delta_\alpha$			Rank of $\hat{\delta}_{alpha}/\delta_\alpha$		
$\alpha = 0.01$	Mean	Median	Std	1st	Mean	Std
RM	3.95	3.95	3.38	0	5.00	4.38
G-t	1.78	1.71	0.88	0	3.50	2.86
GJR-t	1.59	1.645	0.76	1	2.83	2.24
DTG-t	1.696	1.665	1.020	4	1.92	1.47
STG-t	1.695	1.665	1.018	4	1.75	1.07
$\alpha = 0.05$	Mean	Median	Std.	1st	Mean	Std
RM	1.70	1.68	0.78	0	4.83	4.22
G-t	1.43	1.51	0.60	1	3.17	2.69
GJR-t	1.34	1.31	0.52	2	2.50	2.16
DTG-t	1.26	1.27	0.40	1	2.25	1.50
STG-t	1.15	1.06	0.35	3	2.25	1.96

ES Model rejections at $\alpha = 0.01, 0.05$

$\alpha = 0.01$	UC	Ind	CC	DQ	Total (out of 6)
RM	4	0	4	6	6
G-t	0	0	0	2	2
GJR-t	0	0	0	1	1
DTG-t	1	0	0	1	1
STG-t	1	0	0	1	1
$\alpha = 0.05$	UC	Ind	CC	DQ	Total (out of 6)
RM	5	0	4	6	6
G-t	1	0	0	3	3
GJR-t	1	0	0	2	2
DTG-t	0	0	0	2	2
STG-t	0	0	0	1	1

Conclusions

- Continuous mixture of regression with volatility models extended.
- Mixture prior for identification and proper prior for mixing parameter.
- Proposed model ranked in top 2 for VaR and ES in all cases.
- And rejected the least number of times in each case.
- At least highly competitive and often better than alternatives on coverage rates for VaR and ES
- ... thanks for listening