Computing with the Lie correspondence

Scott H. Murray

University of Sydney

July 30, 2009

Linear algebraic groups

Definition (Linear algebraic group)

A subgroup G of $\mathrm{GL}_n(\mathbb{F})$ defined by polynomial equations

Linear algebraic groups

Definition (Linear algebraic group)

A subgroup G of $\mathrm{GL}_n(\mathbb{F})$ defined by polynomial equations

Definition (Lie algebra)

The tangent space at 1 is a Lie algebra $L(G)$

Linear algebraic groups

Definition (Linear algebraic group)

A subgroup G of $\mathrm{GL}_n(\mathbb{F})$ defined by polynomial equations

Definition (Lie algebra)

The tangent space at 1 is a Lie algebra $L(G)$

Example

$\mathrm{sl}_n(\mathbb{F})$ is the Lie algebra of $\mathrm{SL}_n(\mathbb{F})$
$\mathrm{g}_2(\mathbb{R})$ is the Lie algebra of $\mathrm{Aut}(\mathbb{O})$

Lie correspondence

Over $\mathbb{C}$:

$$\text{connected linear algebraic groups} \longleftrightarrow \text{Lie algebras}$$

Lie correspondence

Over $\mathbb{C}$:

$$\text{connected linear algebraic groups} \longleftrightarrow \text{Lie algebras}$$

This breaks down for number fields

Lie correspondence

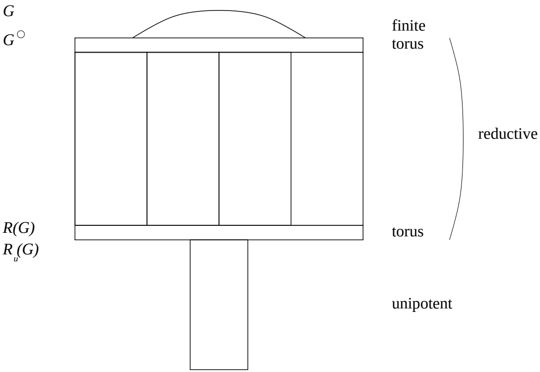
Over $\mathbb{C}$:

$$\text{connected linear algebraic groups} \longleftrightarrow \text{Lie algebras}$$

This breaks down for number fields

It breaks down more in characteristic p

Structure



Classification of simple Lie algebraic groups

Dynkin diagram, diagram automorphism, isogeny

Type A: $\mathrm{SL}_n(\mathbb{F})$, $\mathrm{PGL}_n(\mathbb{F})$, $\mathrm{SU}_n(\mathbb{F})$, etc
Type C: symplectic groups $\mathrm{Sp}_{2n}(\mathbb{F})$
Types B and D: orthogonal
Types E, F, G: exceptional

Almost reductive Lie algebras

The Lie algebra of a connected reductive linear algebraic group

Almost reductive Lie algebras

The Lie algebra of a connected reductive linear algebraic group

- $\mathrm{sl}_n(\mathbb{F})$ is almost reductive
  $\mathbb{F}$ has characteristic $p|n \implies Z(\mathrm{sl}_n(\mathbb{F})) = kI_n$

Almost reductive Lie algebras

The Lie algebra of a connected reductive linear algebraic group

- $\mathrm{sl}_n(\mathbb{F})$ is almost reductive
  $\mathbb{F}$ has characteristic $p|n \implies Z(\mathrm{sl}_n(\mathbb{F})) = kI_n$
- $\mathrm{pgl}_n(\mathbb{F})$ has derived subalgebra of index 1

Almost reductive Lie algebras

The Lie algebra of a connected reductive linear algebraic group

- $\mathfrak{sl}_n(\mathbb{F})$ is almost reductive
  $\mathbb{F}$ has characteristic $p|n \implies Z(\mathfrak{sl}_n(\mathbb{F})) = kI_n$
- $\mathfrak{pgl}_n(\mathbb{F})$ has derived subalgebra of index 1
- $\mathbb{F}$ has characteristic $3 \implies \mathfrak{g}_2(\mathbb{F}) = \dfrac{\mathfrak{sl}_n(\mathbb{F})/kI_n}{\mathfrak{sl}_n(\mathbb{F})/kI_n}$

Recognition of algebraic Lie algebras

We can find the soluble radical.

For almost reductive algebras we have:

- Statistical recognition for $p > 2$
- Name recognition
- Constructive recognition for $p > 3$

Conjugating semisimple elements

G simple linear algebraic group over $\mathbb{F}$
T_0 standard maximal torus
Extend to the algebraic closure $\overline{\mathbb{F}}$

$s \in G(\overline{\mathbb{F}})$ semisimple

Wish to find $\quad x \in G(\overline{\mathbb{F}}) \quad$ s.t. $\quad s^x \in T_0$

Algorithm

- $L = L(G)$, $\quad M = C_L(s)$ is almost reductive
- H a Cartan subalgebra of M [de Graaf, Ivanyos, Rónyai]
- find Chevalley bases of L w.r.t. H and H_0
- we now have $a \in \mathrm{Aut}(L)$ s.t. $H^a = H_0$
- decompose $a = xb$ s.t. $x \in G(\overline{\mathbb{F}})$ and $H_0^b = H_0$
- now $s^x \in T_0$

Rational conjugation

Rational tori:
T_w for w a class representatives in the Weyl group W

For $s \in G(\mathbb{F})$ semisimple
wish to find $\quad x \in G(\mathbb{F})$ and $w \in W \quad$ s.t. $\quad s^x \in T_w$

Rational conjugation

Rational tori:
T_w for w a class representatives in the Weyl group W

For $s \in G(\mathbb{F})$ semisimple
wish to find $x \in G(\mathbb{F})$ and $w \in W$ s.t. $s^x \in T_w$

We can:

- Use the Lang's Theorem algorithm [Cohen, M] or
- Generalise Chevalley bases

Conjugation

Semisimple part:

- conjugate into a torus
- orbits of W on the torus [Carter]

Conjugation

Semisimple part:

- conjugate into a torus
- orbits of W on the torus [Carter]

Unipotent part:
Need to find a torus stabilising a given “flag”

Sylow subgroups

S a Sylow r-subgroup of G

If $(r, p) = 1$ then S consists of semisimple elements

Sylow subgroups

S a Sylow r-subgroup of G

If $(r, p) = 1$ then S consists of semisimple elements

If S abelian $p > 3$,
algorithm for conjugation into a torus generalises

In general we conjugate into the normaliser of torus
$N_G(T_0)/T_0$ is the Weyl group

Stabiliser of a space of forms

U a subspace of $M_n(\mathbb{F})$
$G = N_{\mathrm{GL}_n(\mathbb{F})}(U)$ is algebraic
$L = L(G)$ is the normaliser in $\mathrm{gl}_n(\mathbb{F})$

For very large p we can compute G° from L