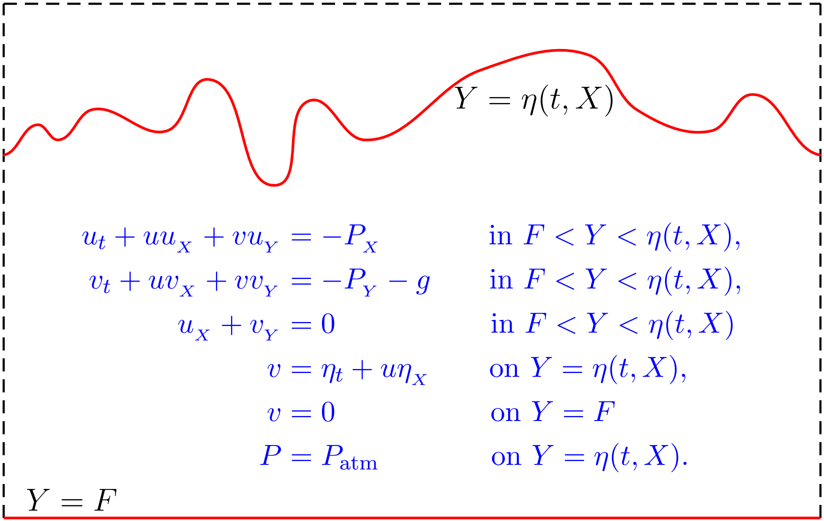
On the existence of extreme waves and the Stokes conjecture with vorticity

Eugen Varvaruca
Imperial College London

THE EULER EQUATIONS FOR A HEAVY FLUID WITH A FREE SURFACE



$$
\begin{aligned}
u_t + u u_X + v u_Y &= -P_X && \text{in } F < Y < \eta(t, X),\\
v_t + u v_X + v v_Y &= -P_Y - g && \text{in } F < Y < \eta(t, X),\\
u_X + v_Y &= 0 && \text{in } F < Y < \eta(t, X)\\
v &= \eta_t + u \eta_X && \text{on } Y = \eta(t, X),\\
v &= 0 && \text{on } Y = F\\
P &= P_{\text{atm}} && \text{on } Y = \eta(t, X).
\end{aligned}
$$

TRAVELLING WAVES

$\eta = \eta(X-ct), u = u(X-ct, Y), v = v(X-ct, Y), P = P(X-ct, Y)$, $(X-ct, Y) \mapsto (X, Y)$.

The stream function ψ:

$$\psi_X = -v, \quad \psi_Y = u - c.$$

Eliminating the pressure P:

$$(\Delta\psi)_X \psi_Y = (\Delta\psi)_Y \psi_X.$$

The vorticity $\omega := v_X - u_Y = -\Delta\psi$ satisfies:

$$-\Delta\psi = \gamma(\psi).$$

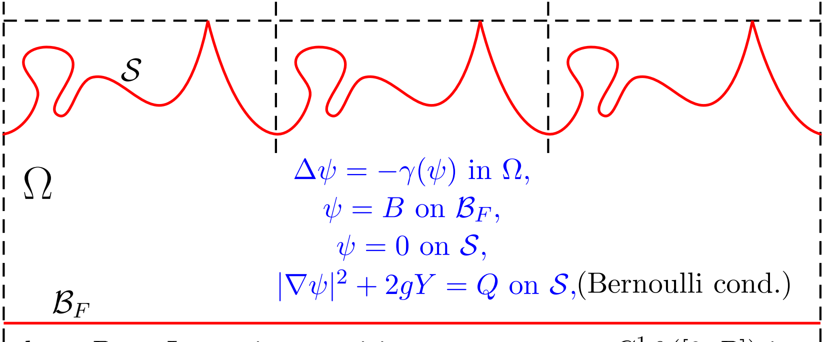
Recovering the pressure P:

$$P + \frac{1}{2}|\nabla\psi|^2 + gY + \Gamma(\psi) = \text{constant},$$

where $\Gamma(r) = \int_0^r \gamma(t)dt$.

THE PROBLEM

Find a $2L$-periodic curve $\mathcal{S}$, a line $\mathcal{B}_F = \{(X, F) : X \in \mathbb{R}\}$ in the (X, Y)-plane, and a $2L$-periodic function ψ in the domain Ω between $\mathcal{B}$ and $\mathcal{S}$, such that

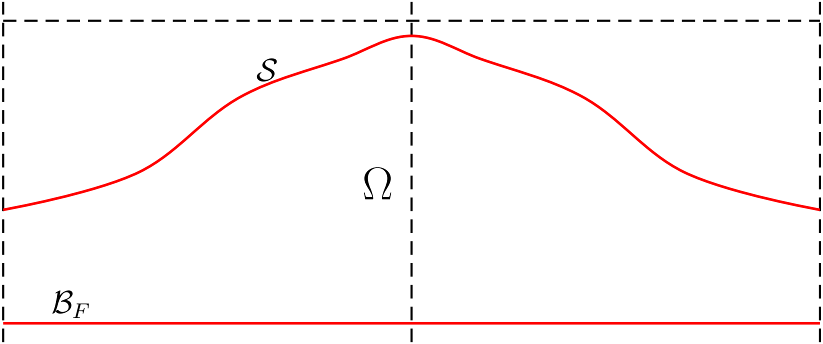


$$\Delta\psi = -\gamma(\psi) \text{ in } \Omega,$$
$$\psi = B \text{ on } \mathcal{B}_F,$$
$$\psi = 0 \text{ on } \mathcal{S},$$
$$|\nabla\psi|^2 + 2gY = Q \text{ on } \mathcal{S}, \text{(Bernoulli cond.)}$$

where B, g, L are given positive constants, $\gamma \in C^{1,\alpha}([0, B])$ is a given function ("vorticity function") and Q, F are parameters. (In fact, there is only one parameter, since one can always take $F = 0$ or $Q = 0$.)

Solutions of type (SMG)(symmetric monotone graphs):

$\mathcal{S} := \{(X, \eta(X)) : X \in \mathbb{R}\}$, η is $2L$-periodic, even, $\eta'(X) < 0$ on $(0, L)$, $\psi_Y < 0$ in Ω, $\psi_X < 0$ in $\Omega \cap \{(X,Y) : 0 < X < L\}$.



EXTREME WAVE: with stagnation points ($\nabla\psi = (0,0)$) on $\mathcal{S}$.

Trivial waves:

$F := 0$, $\mathcal{S} := \{(X,H) : X \in \mathbb{R}\}$,

$$\psi := \psi(Y) \quad \text{where} \quad \psi_{YY} = -\gamma(\psi).$$

There is a family of solutions ψ^λ, $\lambda \in (2\Gamma_{\max}, \infty)$, where $\Gamma_{\max} = \max_{r\in[0,B]} \Gamma(r)$, $\Gamma(r) = \int_0^r \gamma(t)dt$ for all $r \in [0,B]$, given implicitly by:

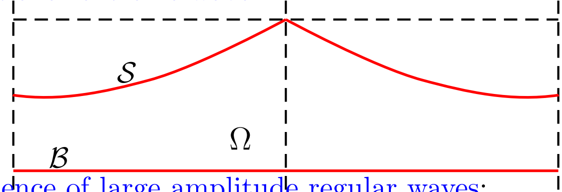
$$\int_{\psi^\lambda(Y)}^B (\lambda - 2\Gamma(r))^{-1/2} dr = Y \quad \text{for all } Y \in [0, H(\lambda)],$$

where $H(\lambda) := \int_0^B (\lambda - 2\Gamma(r))^{-1/2} dr$.
Whenever $\Gamma \leq 0 = \Gamma_{\max}$ and $\int_0^B (2\Gamma_{\max} - 2\Gamma(r))^{-1/2} dr < +\infty$, (e.g. when γ is a negative constant), the above formulae make sense for $\lambda = 0 = 2\Gamma_{\max}$ as well, and define a trivial extreme wave: $\psi_Y(X,H) = 0$ for all $X \in \mathbb{R}$, and of course also $\psi_X(X,H) = 0$ for all $X \in \mathbb{R}$.

Irrotational flows: $\gamma \equiv 0$.

THE STOKES CONJECTURE Stokes (1880): the profile of any extreme wave must have corners with included angle of 120° at stagnation points.

An irrotational extreme wave



- Existence of large amplitude regular waves: Krasovskii(1961), Keady & Norbury(1977), Amick & Toland(1981).
- Existence of extreme waves: Toland(1978), McLeod(1979).
- Proof of the Stokes conjecture: Amick, Fraenkel & Toland(1982), Plotnikov(1982).

The theory is based on Nekrasov's integral equation for a function θ which gives the angle between the tangent to $\mathcal{S}$ and the horizontal.

Bernoulli free-boundary problems

(introduced by Shargorodsky & Toland (2004)):
$\gamma \equiv 0$, but the Bernoulli condition (with $Q = 0$)

$$|\nabla\psi| = (-2gY)^{1/2} \quad \text{on } \mathcal{S},$$

is replaced by

$$|\nabla\psi| = f(Y) \quad \text{on } \mathcal{S}.$$

where $f : (-\infty, 0] \to [0, \infty)$, $f(0) = 0$, $f(r) > 0$ for all $r < 0$.

V(2006): generalization of the Stokes conjecture with a much simpler proof than that for water waves of AF&T(1982)

- If $f(r) \sim c(-r)^{\alpha}$ as $r \nearrow 0$, then the free boundary must have a corner of size $\pi/(\alpha + 1)$.
- An explicit example of a singular free boundary which does not have a corner.

Flows with vorticity: $\gamma \neq 0$.

- Existence of large amplitude regular waves under very general assumptions on γ: Constantin & Strauss (2004), Hur (2006).

  The continuum of waves of type (SMG) found in C&S(2004) contains a sequence $\{(\mathcal{S}_j, \psi^j, Q_j)\}_{j\geq 1}$, with $F = 0$, for which $\max_{\overline{\Omega}_j} \psi^j_Y \to 0$ as $j \to \infty$. (Being of type (SMG), they satisfy $\psi^j_Y < 0$ everywhere in Ω_j.)
- Numerical evidence Simmen & Saffman(1985), Teles da Silva & Peregrine(1988), Vanden-Broeck(1994 -1996), Ko & Strauss(2007): existence of extreme waves for any negative constant vorticity and for small positive constant vorticity.
- Numerical evidence and formal speculation Milne-Thomson(1968), Delachenal(1973): extreme waves with vorticity have corners with angles of 120° at stagnation points (THE STOKES CONJECTURE).

The Dubreil-Jacotin equation:
instead of seeking $\psi = \psi(X, Y)$, seek $Y = Y(X, -\psi)$ or (upon relabeling) $h = h(q, p)$, where $(q, p) \in \mathbb{R} \times [-B, 0]$, such that:

$$\begin{gathered}(1 + h_q^2)h_{pp} - 2h_p h_q h_{pq} + h_p^2 h_{qq} = -\gamma(-p)h_p^3 \quad \text{in } \mathbb{R} \times [-B, 0], \\ 1 + h_q^2 + (2gh - Q)h_p^2 = 0 \quad \text{on } p = 0, \\ h = 0 \quad \text{on } p = -B.\end{gathered}$$

Note that $h_p = -\dfrac{1}{\psi_Y}$.

ON THE EXISTENCE OF EXTREME WAVES

Theorem (V(2007))

Let $\{(\mathcal{S}_j, \psi^j, Q_j)\}_{j\geq 1}$ *be a sequence of regular waves of type* (SMG) *with* $F = 0$. *Suppose that*

$$\{Q_j\}_{j\geq 1} \textit{ is bounded above}.$$

Then $\{(\mathcal{S}_j, \psi^j, Q_j)\}_{j\geq 1}$ *'converges' along a subsequence to a 'weak solution'* $(\widetilde{\mathcal{S}}, \tilde{\psi}, \widetilde{Q})$ *of the water-wave problem.*
If, in addition,

$$\psi^j_Y(0, \eta_j(0)) \to 0 \textit{ as } j \to \infty,$$

then $(\widetilde{\mathcal{S}}, \tilde{\psi}, \widetilde{Q})$ *is an extreme wave.*

Theorem (Savin & V(2008))

Let $\{(\mathcal{S}_j, \psi^j, Q_j)\}_{j\geq 1}$ *be a sequence of regular waves of type* (SMG) *with* $F = 0$. *If either* (i) $\Gamma(r) \leq 0$ *for all* $r \in [0, B]$, *or* (ii) $\Gamma(r) \leq \Gamma(B)$ *for all* $r \in [0, B]$ *and* $\int_0^B (2\Gamma_{\max} - 2\Gamma(r))^{-1/2} dr < +\infty$, *then* $\{Q_j\}_{j\geq 1}$ *is bounded above.*

Theorem (Savin & V(2008))

Let $\{(\mathcal{S}_j, \psi^j, Q_j)\}_{j\geq 1}$ *be a sequence of regular waves of type* (SMG) *with* $F = 0$ *such that* $\max_{\overline{\Omega}_j} \psi^j_Y \to 0$ *as* $j \to \infty$. *If* $\gamma(r) \leq 0$ *for all* $r \in [0, B]$ *and* $\gamma'(r) \geq 0$ *for all* $r \in [0, \delta]$, *for some* $\delta \in [0, B]$, *then* $\psi^j_Y(0, \eta_j(0)) \to 0$ *as* $j \to \infty$.

Theorem (Savin & V(2008))

Let $\{(\mathcal{S}_j, \psi^j, Q_j)\}_{j\geq 1}$ *be a sequence of regular waves of type* (SMG) *with* $F = 0$. *Then* $\{\psi^j_Y(L, \eta_j(L))\}_{j\geq 1}$ *is bounded away from* 0.

In the following two definitions, it is required that $\mathcal{S}$ is locally rectifiable, $\psi \in \mathrm{Lip}(\overline{\Omega}) \cap C^2(\Omega)$, all conditions apart from (BC) are satisfied in the classical sense.

- $(\mathcal{S}, \psi, Q)$ is a *weak solution* if, for all $\zeta \in C_0^1(\mathcal{U}_F)$,

  $$\int_\Omega \nabla\psi\nabla\zeta \, d\mathcal{L}^2 = \int_\Omega \gamma(\psi)\zeta \, d\mathcal{L}^2 - \int_{\mathcal{S}} (Q - 2gY)^{1/2}\zeta \, d\mathcal{H}^1.$$

  where $\mathcal{U}_F := \{(X, Y) : Y > F\}$.
  (For classical solutions, $\psi = 0$ on $\mathcal{S}$ implies that $|\nabla\psi| = -\frac{\partial\psi}{\partial n}$ on $\mathcal{S}$.)
- $(\mathcal{S}, \psi, Q)$ is a *Hardy-space solution* if the partial derivatives of ψ have non-tangential boundary values $\mathcal{H}^1$-almost everywhere on $\mathcal{S}$ which satisfy (BC) $\mathcal{H}^1$-almost everywhere.

Theorem (V(2007))

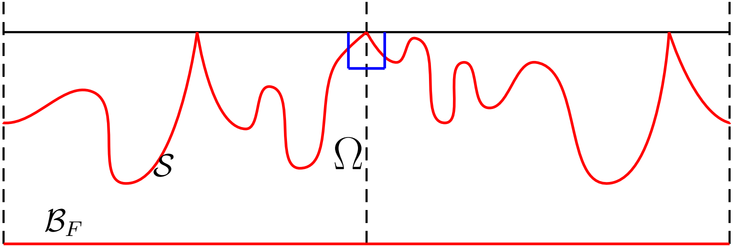
The Hardy-space solutions and the weak solutions coincide.

THE STOKES CONJECTURE

Let $(\mathcal{S}, \psi)$ be an extreme wave, with $Q = 0$, such that the origin ia a stagnation point.

Suppose that $\mathcal{S} = \{(X, \eta(X)) : X \in \mathbb{R}\}$, where $\eta : \mathbb{R} \to \mathbb{R}$ is continuous, of bounded variation, $\eta(0) = 0$ and η is nondecreasing on $[-A, 0]$ and nonincreasing on $[0, A]$ for some $A \in (0, L]$.

Suppose also that $\psi_Y < 0$ in Ω.



Theorem (V(2007), Weiss & V(2008))

Let $(\mathcal{S}, \psi)$ be an extreme wave satisfying the above assumptions, and also

$$P + (\max_{r\in[0,B]} \gamma^+(r))\psi \geq 0 \quad \textit{in } \Omega,$$

where P is the pressure in the fluid.
Then

$$\textit{either } \lim_{X\to 0\pm} \frac{\eta(X)}{X} = \mp\frac{1}{\sqrt{3}} \textit{ or } \lim_{X\to 0\pm} \frac{\eta(X)}{X} = 0.$$

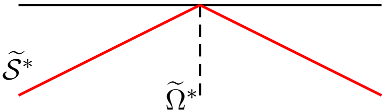
Moreover, if $\gamma(r) \geq 0$ for all $r \in [0,\delta]$, for some $\delta \in (0,B]$, then

$$\lim_{X\to 0\pm} \frac{\eta(X)}{X} = \mp\frac{1}{\sqrt{3}}.$$

One can prove that, for any regular wave,

$$P + (\max_{r\in[0,B]} \gamma^+(r))\psi \geq 0 \quad \text{in } \Omega.$$

Motivation for the Stokes conjecture (at least in the irrotational case): the Stokes corner flow.



Let $\widetilde{\mathcal{S}}^* := \{(X,\eta^*(X)) : X \in \mathbb{R}\}$, where

$$\eta^*(X) := -\frac{1}{\sqrt{3}}|X| \quad \text{for all } X \in \mathbb{R}.$$

Let $\widetilde{\Omega}^*$ be the domain below $\widetilde{\mathcal{S}}^*$, and let the function $\tilde{\psi}^*$ in $\widetilde{\Omega}^*$ be given, for all $(X,Y) \in \widetilde{\Omega}^*$, by

$$\tilde{\psi}^*(X,Y) := \frac{2}{3}g^{1/2}\,\mathrm{Im}\left(i(iZ)^{3/2}\right) \quad \text{where } Z = X + iY.$$

Then:

$$\Delta\tilde{\psi}^* = 0 \quad \text{in } \widetilde{\Omega}^*,$$
$$\tilde{\psi}^* = 0 \quad \text{on } \widetilde{\mathcal{S}}^*,$$
$$|\nabla\tilde{\psi}^*|^2 + 2gY = 0 \quad \text{everywhere on } \widetilde{\mathcal{S}}^*.$$

The blow-up sequence. Let $\varepsilon_j \searrow 0$ and, for any $j \geq 1$, let

$$\Omega_j := \frac{1}{\varepsilon_j}\Omega,$$

and $\psi^j : \Omega_j \to \mathbb{R}$ be given by

$$\psi^j(X,Y) := \frac{1}{\varepsilon_j^{3/2}}\psi(\varepsilon_j X, \varepsilon_j Y) \quad \text{for all } (X,Y) \in \Omega_j.$$

The boundary of the domain Ω_j consists of the curve $\mathcal{S}_j := \varepsilon_j^{-1}\mathcal{S}$ and the line $\mathcal{B}_j := \varepsilon_j^{-1}\mathcal{B}$. The function ψ^j satisfies

$$\begin{aligned} &\psi^j = 0 \quad \text{on } \mathcal{S}_j,\\ &|\nabla\psi^j|^2 + 2gY = 0 \quad \text{on } \mathcal{S}_j,\\ &\Delta\psi^j = -\gamma_j(\psi^j) \quad \text{in } \Omega_j, \end{aligned}$$

where $\gamma_j : [0, B\varepsilon_j^{-3/2}] \to \mathbb{R}$ is given by

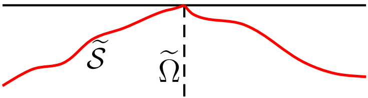
$$\gamma_j(r) = \varepsilon_j^{1/2}\gamma(\varepsilon_j^{3/2}r) \quad \text{for all } r \in [0, B\varepsilon_j^{-3/2}].$$

The limiting problem: find a curve $\widetilde{\mathcal{S}} = \{(\tilde{u}(s), \tilde{v}(s)) : s \in \mathbb{R}\}$, where $s \mapsto (\tilde{u}(s), \tilde{v}(s))$ is injective on $\mathbb{R}$, $\tilde{u}(0) = 0$, $\tilde{v}(0) = 0$, $s \mapsto \tilde{u}(s)$ is nondecreasing on $\mathbb{R}$, $s \mapsto \tilde{v}(s)$ is nondecreasing on $(-\infty, 0]$ and nonincreasing on $[0, \infty)$, $\lim_{s \to \pm\infty} (|\tilde{u}(s)| + |\tilde{v}(s)|) = \infty$, and a function $\tilde{\psi}$ in the unbounded domain $\widetilde{\Omega}$ below $\widetilde{\mathcal{S}}$, such that

$$\Delta \tilde{\psi} = 0 \quad \text{in } \widetilde{\Omega},$$

$$\tilde{\psi} = 0 \quad \text{on } \widetilde{\mathcal{S}},$$

$$|\nabla \tilde{\psi}|^2 + 2gY = 0 \quad \mathcal{H}^1\text{-almost everywhere on } \widetilde{\mathcal{S}}.$$



This has a trivial solution $(\widetilde{\mathcal{S}}_0, \tilde{\psi}_0)$, where $\widetilde{\mathcal{S}}_0 = \{(X, 0) : X \in \mathbb{R}\}$ and $\tilde{\psi}_0 \equiv 0$ in $\mathbb{R}^2_-$.

The Stokes corner flow $(\widetilde{\mathcal{S}}^*, \tilde{\psi}^*)$ is a nontrivial solution.

Theorem (V(2007))

The only nontrivial symmetric solution $(\widetilde{\mathcal{S}}, \tilde{\psi})$ of the limiting problem is the Stokes corner flow $(\widetilde{\mathcal{S}}^*, \tilde{\psi}^*)$.

The blow-up limit problem can be described, in the symmetric case, by

$$\theta^*(x) = \mu \int_0^\infty \frac{1}{\pi} \log \left| \frac{x+y}{x-y} \right| \frac{\sin \theta^*(y)}{\int_0^y \sin \theta^*(u)\, du}\, dy, \;\; x \in (0, \infty),$$

$$0 \le \theta^*(x) \le \pi/2, \;\; \int_0^x \sin \theta^*(u)\, du > 0 \text{ for all } x \in (0, \infty),$$

with $\mu = 1/3$.

Theorem (V(2006))

For any $\mu \in (0,1)$, the only solution θ^* of the above integral equation is the constant function $\mu\pi/2$.

AF&T(1982): $\mu = 1/3$, $0 \le \theta^*(x) \le \pi/3$ for all $x \in (0, \infty)$.

Theorem (Weiss & V(2008))

Any nontrivial (not necessarily symmetric) solution $(\widetilde{\mathcal{S}}, \tilde{\psi})$ *of the limiting problem which arises as a blow-up limit of a solution of the original problem is necessarily homogeneous of degree* $3/2$, *and therefore is the Stokes corner flow* $(\widetilde{\mathcal{S}}^*, \tilde{\psi}^*)$.

The proof is based on a new monotonicity formula.

OPEN QUESTIONS

- the existence of waves with stagnation points in the interior or on the bottom of the flow domain;
- the existence of waves with overhanging profiles;
- the structure of the set of stagnation points for extreme waves with vorticity;
- the regularity of the wave profiles away from stagnation points;
- internal waves with vorticity?