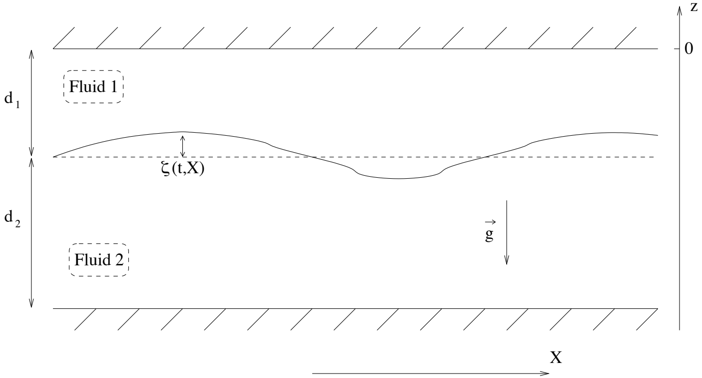
The Cauchy problem for a non local model of large amplitude internal waves

Jean-Claude Saut
Université Paris-Sud
(with Ph. Guyenne and D. Lannes)

Large Amplitude Internal waves
ICMS Edinburgh
December 1-4, 2008

The aim of this talk is to present a well-posedness result for the initial-value problem associated to a new system of large amplitude internal waves derived by J. Bona, D. Lannes and J.-C. S. : **BLS** Journal Math. Pures Appl. 2008, see David Lannes talk.


z
0
Fluid 1
d 1
ζ(t,X)
d 2
g
Fluid 2
X

$$\Delta_{X,z}\Phi_i = 0 \quad \text{in } \Omega_t^i \tag{1}$$

$$\partial_t\Phi_i + \frac{1}{2}|\nabla_{X,z}\Phi_i|^2 = -\frac{P}{\rho_i} - gz \quad \text{in } \Omega_t^i, \tag{2}$$

where Ω_t^i denotes the region occupied by fluid i at time t, $i = 1, 2$.
(Bernouilli)
The velocity must be horizontal at the two rigid surfaces
$\Gamma_1 := \{z = 0\}$ and $\Gamma_2 := \{z = -d_1 - d_2\}$,

$$\partial_z\Phi_i = 0 \quad \text{on} \quad \Gamma_i, \qquad (i = 1, 2). \tag{3}$$

The interface $\Gamma_t := \{z = -d_1 + \zeta(t, X)\}$ between the fluids is taken to be a bounding surface (no fluid particle crosses the interface). This condition, written for fluid i, is classically expressed by the relation $\partial_t \zeta = \sqrt{1 + |\nabla \zeta|^2} v_n^i$, where v_n^i denotes the upwards normal derivative of the velocity of fluid i at the surface. Since this equation must of course be independant of the fluid, it follows that the normal component of the velocity is continuous at the interface. Thus,

$$\partial_t \zeta = \sqrt{1 + |\nabla \zeta|^2} \partial_n \Phi_1 \quad \text{on} \quad \Gamma_t, \tag{4}$$

$$\partial_n \Phi_1 = \partial_n \Phi_2 \quad \text{on} \quad \Gamma_t, \tag{5}$$

with

$$\partial_n := \mathbf{n} \cdot \nabla_{X,z} \quad \text{and} \quad \mathbf{n} := \frac{1}{\sqrt{1 + |\nabla \zeta|^2}} (-\nabla \zeta, 1)^T$$

Finally,

$$P \text{ is continuous at the interface.} \tag{6}$$

The system (1)- (6) is the classical formulation of the two-layers system.

We reduce to a problem on a fixed domain (cf Zakharov 1968, Craig-Sulem-Sulem 1992 for the classical water waves problem) :

Introduce the trace of the potentials Φ_1 and Φ_2 at the interface,

$$\psi_i(t,X) := \Phi_i(t,X,-d_1+\zeta(t,X)), \qquad (i=1,2).$$

Define the Dirichlet-Neumann operator $G[\zeta]\cdot$ by

$$G[\zeta]\psi_1 = \sqrt{1+|\nabla\zeta|^2}(\partial_n\Phi_1)_{|_{z=-d_1+\zeta}},$$

which allows to determine $\partial_n\Phi_1$ from ψ_1 .

Similarly, one remarks that ψ_2 is determined up to a constant by ψ_1 (solving a non-homogeneous Neumann problem). It follows that ψ_1 fully determines $\nabla\psi_2$ and we may thus define the "interface operator" $\mathbf{H}[\zeta]\cdot$ by

$$\mathbf{H}[\zeta]\psi_1 = \nabla\psi_2.$$

On can express (1)-(6) in terms of ψ_1, ψ_2 and ζ to get

$$\partial_t(\psi_2-\gamma\psi_1)+g(1-\gamma)\zeta+\frac{1}{2}\big(|\mathbf{H}[\zeta]\psi_1|^2-\gamma|\nabla\psi_1|^2\big)+\mathcal{N}(\zeta,\psi_1)=0.$$

where $\gamma=\rho_1/\rho_2$ and

$$\mathcal{N}(\zeta,\psi_1):=\frac{\gamma\big(G[\zeta]\psi_1+\nabla\zeta\cdot\nabla\psi_1\big)^2-\big(G[\zeta]\psi_1+\nabla\zeta\cdot\mathbf{H}[\zeta]\psi_1\big)^2}{2(1+|\nabla\zeta|^2)}.$$

Taking the gradient of this equation gives the set of equations

$$\begin{cases} \partial_t \zeta - G[\zeta]\psi_1 = 0, \\ \partial_t(\mathbf{H}[\zeta]\psi_1 - \gamma\nabla\psi_1) + g(1-\gamma)\nabla\zeta \\ \;+ \frac{1}{2}\nabla\big(|\mathbf{H}[\zeta]\psi_1|^2 - \gamma|\nabla\psi_1|^2\big) + \nabla\mathcal{N}(\zeta,\psi_1) = 0, \end{cases} \tag{7}$$

for ζ and ψ_1, which is the system of equations that will be used in the next sections to derive asymptotic models.
Setting $\rho_1 = 0$, and thus $\gamma = 0$, in the above equations, one recovers the usual surface water-wave equations written in terms of ζ and ψ as in the Zakharov formulation.

Denoting by **a** a typical amplitude of the deformation of the interface, and by λ a typical wavelenth, the following dimensionless indendent variables

$$\widetilde{X} := \frac{X}{\lambda}, \quad \widetilde{z} := \frac{z}{d_1}, \quad \widetilde{t} := \frac{t}{\lambda/\sqrt{gd_1},},$$

are introduced. Likewise, we define the dimensionless unknowns

$$\widetilde{\zeta} := \frac{\zeta}{a}, \quad \widetilde{\psi}_1 := \frac{\psi_1}{a\lambda\sqrt{g/d_1}},$$

as well as the dimensionless parameters

$$\gamma := \frac{\rho_1}{\rho_2}, \quad \delta := \frac{d_1}{d_2}, \quad \varepsilon := \frac{a}{d_1}, \quad \mu := \frac{d_1^2}{\lambda^2};$$

Though there is some redundancy here, it is convenient to introduce two parameters ε_2 and μ_2 defined as

$$\varepsilon_2 = \frac{a}{d_2} = \varepsilon\delta, \qquad \mu_2 = \frac{d_2^2}{\lambda^2} = \frac{\mu}{\delta^2}.$$

The parameters ε_2 and μ_2 correspond to ε and μ with d_2 rather than d_1 taken as the unit of length in the vertical direction.

- **Dimensionless Dirichlet-Neumann operator $G^\mu[\varepsilon\zeta]\cdot$ associated to the non-dimensionalized upper fluid domain** :

$$\Omega_1 = \{(X,z) \in \mathbb{R}^{d+1}, -1+\varepsilon\zeta(X) < z < 0\},$$

assuming that there is a positive value H_1 such that

$$1-\varepsilon\zeta \geq H_1 \quad \text{on} \quad \mathbb{R}^d. \tag{8}$$

Define $G^\mu[\varepsilon\zeta]\psi_1 \in H^{1/2}(\mathbb{R}^d)$ by

$$G^\mu[\varepsilon\zeta]\psi_1 = -\mu\varepsilon\nabla\zeta\cdot\nabla\Phi_1\,_{|z=-1+\varepsilon\zeta} + \partial_z\Phi_1\,_{|z=-1+\varepsilon\zeta},$$

where, for $\zeta \in W^{2,\infty}(\mathbb{R}^d)$ be such that (8) is satisfied and for $\psi_1 \in H^{3/2}(\mathbb{R}^d)$, Φ_1 is the unique solution in $H^2(\Omega_1)$ of the boundary-value problem

$$\begin{cases} \mu\Delta\Phi_1 + \partial_z^2\Phi_1 = 0 \quad \text{in } \Omega_1, \\ \partial_z\Phi_1\,_{|z=0} = 0, \qquad \Phi_1\,_{|z=-1+\varepsilon\zeta(X)} = \psi_1, \end{cases} \tag{9}$$

Another way to approach G^μ is to define

$$G^\mu[\varepsilon\zeta]\psi_1 = \sqrt{1+\varepsilon^2|\nabla\zeta|^2}\partial_n\Phi_1\,_{|z=-1+\varepsilon\zeta}$$

where $\partial_n\Phi_1\,_{|z=-1+\varepsilon\zeta}$ stands for the upper conormal derivative associated to the elliptic operator $\mu\Delta\Phi_1 + \partial_z^2\Phi_1$.

- **We similarly define a dimensionless operator $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\cdot$ associated to the non-dimensionalized lower fluid domain**.

$$\Omega_2 = \{(X,z) \in \mathbb{R}^{d+1}, -1 - 1/\delta < z < -1 + \varepsilon\zeta(X)\},$$

where it is again assumed that

$$\exists H_2 > 0, \quad 1 + \varepsilon\delta\zeta \geq H_2 \quad \text{on} \quad \mathbb{R}^d. \tag{10}$$

Let $\zeta \in W^{2,\infty}(\mathbb{R}^d)$ be such that (8) and (10) are satisfied, and $\psi_1 \in H^{3/2}(\mathbb{R}^d)$. Let also Φ_2 be the unique solution (up to a constant) of the boundary-value problem

$$\begin{cases} \mu\Delta\Phi_2 + \partial_z^2\Phi_2 = 0 & \text{in } \Omega_2, \\ \partial_z\Phi_2\,_{|_{z=-1-1/\delta}} = 0, & \partial_n\Phi_2\,_{|_{z=-1+\varepsilon\zeta(X)}} = \frac{1}{(1+\varepsilon^2|\nabla\zeta|^2)^{1/2}} G^\mu[\varepsilon\zeta]\psi_1, \end{cases} \tag{11}$$

and define $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1 \in H^{1/2}(\mathbb{R}^d)$ by

$$\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1 = \nabla(\Phi_2\,_{|_{z=-1+\varepsilon\zeta}}).$$

The equations (7) can therefore be written in dimensionless variables as

$$\begin{cases} \partial_{\widetilde{t}}\widetilde{\zeta} - \frac{1}{\mu} G^{\mu}[\varepsilon\widetilde{\zeta}]\widetilde{\psi}_1 & = & 0 \\ \partial_{\widetilde{t}}\big(\mathbf{H}^{\mu,\delta}[\varepsilon\widetilde{\zeta}]\widetilde{\psi}_1 - \gamma\nabla\widetilde{\psi}_1\big) + (1-\gamma)\nabla\widetilde{\zeta} & & \\ \quad + \frac{\varepsilon}{2}\nabla\big(|\mathbf{H}^{\mu,\delta}[\varepsilon\widetilde{\zeta}]\widetilde{\psi}_1|^2 - \gamma|\nabla\widetilde{\psi}_1|^2\big) + \varepsilon\nabla\mathcal{N}^{\mu,\delta}(\varepsilon\widetilde{\zeta},\widetilde{\psi}_1) & = & 0, \end{cases} \tag{12}$$

where $\mathcal{N}^{\mu,\delta}$ is defined for all pairs (ζ,ψ) smooth enough as

$$\mathcal{N}^{\mu,\delta}(\zeta,\psi) := \mu\frac{\gamma\big(\frac{1}{\mu}G^{\mu}[\zeta]\psi + \nabla\zeta\cdot\nabla\psi\big)^2 - \big(\frac{1}{\mu}G^{\mu}[\zeta]\psi + \nabla\zeta\cdot\mathbf{H}^{\mu,\delta}[\zeta]\psi\big)^2}{2(1+\mu|\nabla\zeta|^2)}$$

The asymptotic models derived in **BLS** are $1+d$ dimensional systems coupling the surface elevation ζ to the variable $\mathbf{v}$ (which is linked to the second canonical variable in the hamiltonian formulation of Benjamin and Bridges (1996)) defined to be

$$\mathbf{v} := \mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1 - \gamma\nabla\psi_1. \tag{13}$$

For the surface water-wave problem ($\gamma = 0$ and $\delta = 1$), $\mathbf{v}$ is the horizontal velocity evaluated at the free surface). We will often refer to $\mathbf{v}$ as the velocity variable, though its precise interpretation will vary.

The Shallow water/Shallow water regime (SW/SW) :

It corresponds to $\mu \sim \mu_2 \ll 1$ and allows for large amplitude waves with respect to both the upper fluid ($\epsilon = 0(1)$) and the lower fluid ($\epsilon_2 = 0(1)$). The model is obtained by replacing the nonlocal operators $G^{\mu}[\zeta]$ and $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]$ by suitable asymptotic expansions. It reads in one dimension $d = 1$, setting $h_1 = 1 - \varepsilon\zeta$ and $h_2 = 1 + \varepsilon\delta\zeta$

$$\begin{cases} \partial_t \zeta + \partial_x \Big(\dfrac{h_1 h_2}{\delta h_1 + \gamma h_2} \mathbf{v} \Big) = 0, \\ \partial_t \mathbf{v} + (1-\gamma)\partial_x \zeta + \dfrac{\varepsilon}{2} \partial_x \Big(\dfrac{(\delta h_1)^2 - \gamma h_2^2}{(\delta h_1 + \gamma h_2)^2} \mathbf{v}^2 \Big) = 0, \end{cases} \tag{14}$$

where $h_1 = 1 - \varepsilon\zeta$ and $h_2 = 1 + \varepsilon\delta\zeta$. This system has been formally derived by Craig-Guyenne-Calish (2005) and has some similarities with a model derived by Camassa-Choi (1999). When $\gamma = 0$, $\delta = 1$, it reduces to the classical Saint-Venant system.

It has been proved in **BLS** that the internal wave equations are consistent with it.

The two-dimensional ($d = 2$) has been derived for the first time in **BLS** and exhibits **new nonlocal terms**. It writes

$$\begin{cases} \partial_t \zeta + \nabla \cdot \big(h_1 \mathfrak{R}[\zeta]\mathbf{v}\big) = 0, \\ \partial_t \mathbf{v} + (1-\gamma)\nabla\zeta + \frac{\varepsilon}{2}\nabla\Big(\big|\mathbf{v} - \gamma\mathfrak{R}[\zeta]\mathbf{v}\big|^2 - \gamma\big|\mathfrak{R}[\zeta]\mathbf{v}\big|^2\Big) = 0, \end{cases} \tag{15}$$

where again $h_1 = 1 - \varepsilon\zeta$, $h_2 = 1 + \varepsilon\delta\zeta$, and the operator $\mathfrak{R}[\zeta]$, which contains the nonlocal effects, is defined as follows.

Let $\gamma \in [0,1)$, $\varepsilon, \delta > 0$ and $\zeta \in L^\infty(\mathbb{R}^d)$ be such that

$$\left|\frac{1-\gamma}{\gamma+\delta}\varepsilon\delta\zeta\right|_\infty < 1.$$

The operator $\mathfrak{R}[\zeta]$ is then defined as

$$\mathfrak{R}[\zeta] : \begin{array}{ccl} L^2(\mathbb{R}^d)^d & \to & L^2(\mathbb{R}^d)^d \\ \mathbf{u} & \mapsto & \mathfrak{R}[\zeta]\mathbf{u} := \dfrac{1}{\gamma+\delta}\left(1 - \Pi\left(\dfrac{1-\gamma}{\gamma+\delta}\varepsilon\delta\zeta\Pi\cdot\right)\right)^{-1}\Pi\left(h_2\mathbf{u}\right), \end{array}$$

where $h_2 = 1 + \varepsilon\delta\zeta$, and $\Pi := \frac{\nabla\nabla^T}{\Delta}$ denotes the projection onto gradient vector fields.

The assumption $\left|\frac{1-\gamma}{\gamma+\delta}\varepsilon\delta\zeta\right|_\infty < 1$ allows one to define $(1-\Pi(\frac{1-\gamma}{\gamma+\delta}\varepsilon\delta\zeta\Pi\cdot))^{-1}$ by its Neumann series :

$$\mathfrak{R}[\zeta]\mathbf{u} = \frac{1}{\gamma+\delta}\sum_{n=0}^{\infty}\big(\Pi(\frac{1-\gamma}{\gamma+\delta}\varepsilon\delta\zeta\Pi\cdot)\big)^n\Pi(h_2\mathbf{u}). \qquad (16)$$

Note also that when $d=1$, one has $\Pi = 1$ and

$$\mathfrak{R}[\zeta]\mathbf{u} = \frac{h_2}{\delta h_1 + \gamma h_2}\mathbf{u},$$

so that both systems coincide as expected.

The nonlocal operator $\mathfrak{R}$ arises when one looks for an approximate solution :

$$\underline{\Phi}_{app} = \Phi^{(0)} + \mu_2 \Phi^{(1)},$$

of the linear elliptic problem which leads to the definition of $\mathbf{H}^{\mu,\delta}$. Matching the boundary condition at $z = -1$ leads to the restriction

$$\nabla \cdot (h_2 \nabla \Phi^{(0)}) = -\delta \nabla \cdot (h_1 \nabla \psi_1),$$

which implies that $\Pi(h_2 \nabla \Phi^{(0)}) = \Pi(-\delta h_1 \nabla \psi_1)$, where $\Pi = -\frac{\nabla \nabla^T}{|D|^2}$ is the orthogonal projector onto the gradient vector fields of $L^2(\mathbb{R}^d)^d$.

The first step for solving the Cauchy problem is to put the system under an equivalent "quasilinear" form. This is easy when $d = 1$:

$$\partial_t U + A(U)\partial_x U = 0, \qquad U = (\zeta, \mathbf{v})^T, \tag{17}$$

with

$$A(U) = \begin{pmatrix} a(U) & b(\zeta) \\ c(U) & d(U) \end{pmatrix};$$

and

$$a(U) = \epsilon \frac{(\delta h_1)^2 - \gamma h_2^2}{(\delta h_1 + \gamma h_2)^2}\mathbf{v}, \tag{18}$$

$$b(\zeta) = \frac{h_1 h_2}{\delta h_1 + \gamma h_2}, \tag{19}$$

$$c(U) = 1 - \gamma(1 + \varepsilon\delta \frac{(1+\delta)^2}{(\delta h_1 + \gamma h_2)^3}\mathbf{v}^2), \tag{20}$$

$$d(U) = \epsilon \frac{(\delta h_1)^2 - \gamma h_2^2}{(\delta h_1 + \gamma h_2)^2}\mathbf{v}. \tag{21}$$

The previous system is strictly hyperbolic provided that

$$\begin{cases} \inf_{\mathbb{R}}(1-\varepsilon\zeta)>0, \\ \inf_{\mathbb{R}}(1+\varepsilon\delta\zeta)>0, \\ \inf_{\mathbb{R}}\left[1-\gamma(1+\varepsilon\delta\dfrac{(1+\delta)^2}{(\delta+\gamma-\varepsilon\delta(1-\gamma)\zeta)^3}\mathbf{v}^2)\right]>0. \end{cases} \tag{22}$$

Theorem

Let $\varepsilon, \delta > 0$ and $\gamma \in (0, 1)$. Let also $t_0 > 1/2$, $s \geq t_0 + 1$ and $U^0 = (\zeta^0, \mathbf{v}^0)^T \in H^s(\mathbb{R})^2$ be such that (22) is satisfied. Then there exists $T_{max} > 0$ and a unique maximal solution $U = (\zeta, \mathbf{v})^T \in C([0, T_{max}); H^s(\mathbb{R})^2)$ to with initial condition U^0.

- **Construction of the symmetrizer.**

$$S(U) = \begin{pmatrix} b(\zeta)^{-1} & 0 \\ 0 & c(U)^{-1} \end{pmatrix},$$

where $b(\zeta)$ and $c(U)$ are the off-diagonal terms in $A(U)$. The matrix $S(U)$ is a symmetrizer in the sense that :

1. The matrix $S(U)$ is symmetric and $|S(U)\cdot|_2$ is uniformly equivalent to $|\cdot|_2$: for all $V \in L^2(\mathbb{R})^{1+1}$,

$$|V|_2^2 \leq \mathfrak{c}_2(U)\,(S(U)V, V) \quad \text{and} \quad (S(U)V, V) \leq \mathfrak{c}_2(U)|V|_2^2. \tag{23}$$

2. The matrix $S(U)A(U)$ is symmetric.

In dimension 2 it is not so easy to put the system under a "quasilinear " form because of the presence of the nonlocal term $\mathfrak{R}[\zeta]\mathbf{v}$. Nevertheless we find the equivalent form

$$\partial_t U + A^j[U]\partial_j U = 0, \qquad U = (\zeta, \mathbf{v})^T, \tag{24}$$

where

$$A^j[U] = \begin{pmatrix} a^j(U) & \mathbf{b}^j(U)^T \\ \mathbf{c}^j[U] & D^j[U] \end{pmatrix}, \qquad (j = 1, 2),$$

and

$$a^j(U) = \varepsilon(\mathbf{v} - \gamma\mathfrak{R}[\zeta]\mathbf{v})_j - \varepsilon\gamma(\mathfrak{S}[\zeta]\mathbf{v})_j \frac{h_2}{\delta h_1 + \gamma h_2}, \tag{25}$$

$$\mathbf{b}^j(U) = \frac{h_1 h_2}{\delta h_1 + \gamma h_2}\mathbf{e}^j, \tag{26}$$

$$\mathbf{c}^j[U]\bullet = \mathbf{e}^j - \gamma\Big[\mathbf{e}^j + \varepsilon\delta(\mathfrak{S}[\zeta]\mathbf{v})_j\mathfrak{R}[\zeta]\Big(\frac{\mathfrak{S}[\zeta]\mathbf{v}}{h_2}\bullet\Big)\Big], \tag{27}$$

$$D^j[U]\bullet = \varepsilon(\mathbf{v} - \gamma\mathfrak{R}[\zeta]\mathbf{v})_j \mathrm{Id}_{2\times 2} - \varepsilon\gamma(\mathfrak{S}[\zeta]\mathbf{v})_j\mathfrak{R}[\zeta]\bullet, \tag{28}$$

$\mathfrak{S}[\zeta]\mathbf{v} = \mathbf{v} + (1-\gamma)\mathfrak{R}[\zeta]\mathbf{v}$

- More precisely, let $T > 0$, $t_0 > 1$ and $s \geq t_0 + 1$. Let also $U = (\zeta, v) \in C([0, T]; H^s(\mathbb{R}^2)^3)$ be such that

$$\forall t \in [0, T], \qquad \frac{1-\gamma}{\gamma+\delta}\varepsilon\delta|\zeta(t,\cdot)|_\infty < 1 \quad \text{and} \quad \text{curl } \mathbf{v}(t,\cdot) = 0.$$

  Then U solves (15) if and only U solves (24).
- The system (24) is not *stricto sensu* a quasilinear system since $\mathbf{c}^j[U]$ (resp. $D^j[U]$) is not an $\mathbb{R}^2$-vector-valued (resp. 2×2-matrix-valued) function but a linear operator defined over the space of $\mathbb{R}^2$-vector-valued (resp. 2×2-matrix-valued) functions. However, these operator are of order zero and (24) can be handled roughly as a quasilinear system.

"Hyperbolicity" conditions

$$\begin{cases} 1 - \varepsilon|\zeta|_{H^{t_0}} > 0, \\ 1 - \varepsilon\delta|\zeta|_{H^{t_0}} > 0, \\ 1 - \gamma(1 + \varepsilon\delta \dfrac{(1+\delta)^2}{(\delta + \gamma - \varepsilon\delta(1-\gamma)|\zeta|_{H^{t_0}})^3} |\mathbf{v}|^2_{H^{t_0}}) > 0. \end{cases} \tag{29}$$

Theorem

Let $\varepsilon, \delta > 0$ and $\gamma \in [0, 1)$. Let also $t_0 > 1$, $s \geq t_0 + 1$ and $U^0 = (\zeta^0, \mathbf{v}^0)^T \in H^s(\mathbb{R}^2)^3$ be such that (29) is satisfied and $\operatorname{curl} \mathbf{v}^0 = 0$. *Then there exists $T_{max} > 0$ and a unique maximal solution $U = (\zeta, \mathbf{v})^T \in C([0, T_{max}); H^s(\mathbb{R}^2)^3)$ to (15) with initial condition U^0.*

- The existence proof on the transformed system is obtained via an energy method implemented on a regularized version of the system (truncation of large frequencies).
- Serious difficulties arise from the nonlocal terms for the construction of the symmetrizer (see below).
- In order to show that a solution of the transformed system yields a solution of the original one, we prove that a solution of the transformed system which is initially curl free remains curl free as long as it exists.

- **Construction of the symmetrizer.**

Let us look for $S[U]$ under the form

$$S[U] = \begin{pmatrix} s_1(U) & 0 \\ 0 & S_2[U] \end{pmatrix}, \tag{30}$$

with $s(\cdot) : H^s(\mathbb{R}^2)^3 \mapsto H^s(\mathbb{R}^2)$ and $S_2[U]$ a linear operator mapping $L^2(\mathbb{R}^2)^2$ into itself.
Defining $C[U]$ as

$$\forall \widetilde{\mathbf{v}} = (\widetilde{\mathbf{v}}_1, \widetilde{\mathbf{v}}_2)^T \in L^2(\mathbb{R}^2)^2, \qquad C[U]\widetilde{\mathbf{v}} = \mathbf{c}_1[U]\widetilde{\mathbf{v}}_1 + \mathbf{c}_2[U]\widetilde{\mathbf{v}}_2,$$

a straightforward generalization of the one dimensional case consists in taking $s_1(U) = b(U)^{-1}$ and $S_2[U] = C[U]^{-1}$; unfortunately, such a choice is not correct because the operator $C[U]$ is not self-adjoint. It turns out however that $C[U]$ is self-adjoint (up to a smoothing term) on the restriction of $L^2(\mathbb{R}^2)^2$ to gradient vector fields.

- Numerical simulations are in progress (in both one and two dimensions).
- Extension to the free surface case (Vincent Duchêne).
- Limit of a continuous stratification.