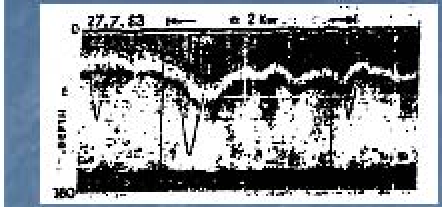
Modelling of “geniunely strong” internal waves

L. A. Ostrovsky

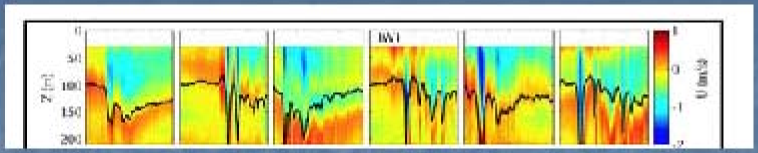
1. General outline
2. Hamiltonian evolution equation
3. Long waves: smooth stratification
4. Perturbation theory for strong kinks

“Internal Mach number:” *M* = *A*/*h*
(*A* is max. displacement, *h* is a characteristic vertical scale of stratification)

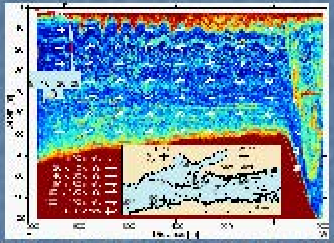
EXAMPLES OF STRONGLY NONLINEAR SOLITONS



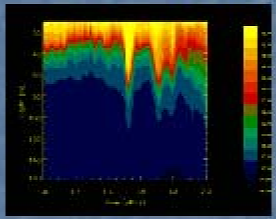
A soliton in Celtic Sea (white line).
From Pingree&Mardell (1986)



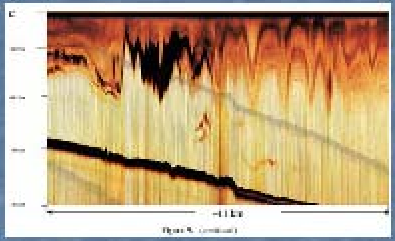
South China Sea, 2000, 6 semi-diurnal cycles, each for about 3 hrs
(Lien et al, 2005)



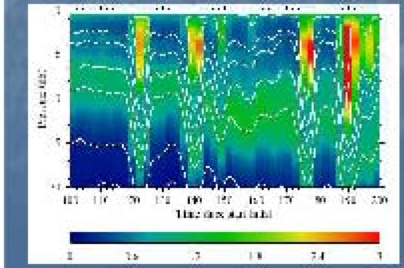
Knight Inlet, British Columbia (28 Feb 1995).
Echo-sounding image of solitary waves. [Farmer & Armi]



North-West UK shelf margin.
(Small et al,1995)

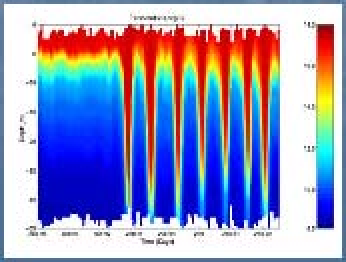


South China Sea, 2001
Orr & Mignerey (2003)



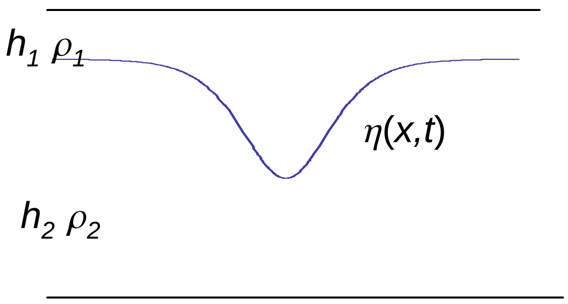
Mission Beach, CA, 1997

[Lennert-Cody&Franks]



COPE , Oregon Bay, 1995
(Stanton & Ostrovsky, 1998).

Weakly dispersive, two-layer models

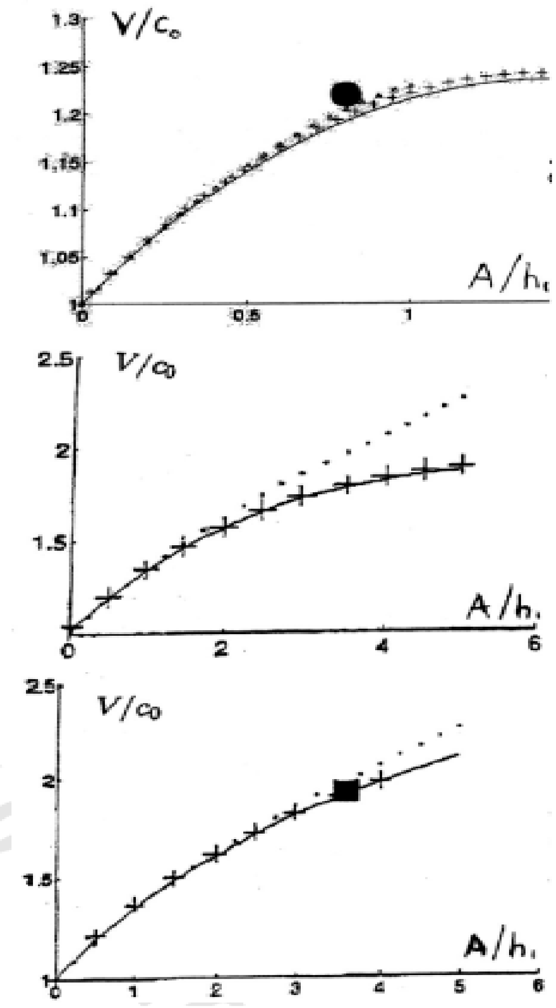


- KdV/BO - weak nonlinearity
- eKdV (Gardner) - weak/moderate nonlinearity
- Strong nonlinearity, weak dispersion – Miyata (1985, 1988), Choi&Camassa (1996, 1999)
- Evolution models (beta, e- and gamma-models) – Ostrovsky (1998), Ostrovsky&Grue (2003)

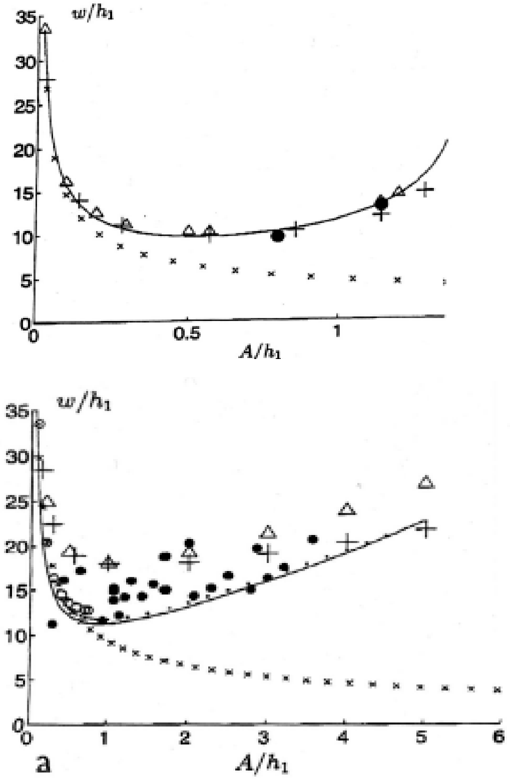
- Intrinsic contradiction of strongly nonlinear/weakly dispersive models is that, by definition, a soliton balances these two factors. Thus, only a comparison with fully nonlinear computations can confirm or disprove their applicability. In general, description of strongly nonlinear waves can not be fully consistently reduced to 1-D (2-D) equations. Still, in many cases these models work well enough.

Comparison of long-wave models with fully nonlinear computations (Ostrovsky&Grue, 2003)

SOLITON VELOCITY VS. AMPLITUDE



SOLITON WIDTH VS. AMPLITUDE



Solid- fully nonlinear; Triangles – CC model; Crosses – evolution (beta) model; Dots - Kdv

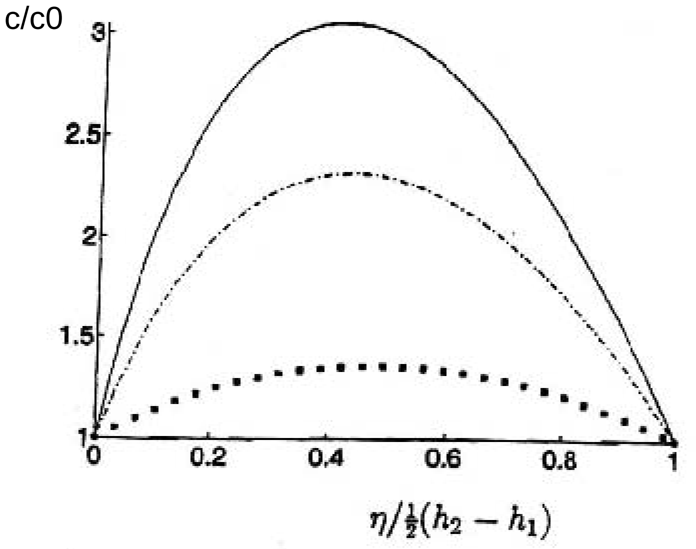
NON-DISPERSIVE (HYDROSTATIC-PRESSURE) MODEL

For a progressive (simple) wave with $u_{1,2} = u_{1,2}(\eta)$:

$$\frac{\partial \eta}{\partial t} + c_{\pm}(\eta) \frac{\partial \eta}{\partial x} = 0$$

$$c(\eta) = c(0) \left\{ 1 + 3 \frac{(h_1 - h_2)(h_1 - h_2 - 2\eta)}{(h_1 + h_2)^2} \left[\sqrt{\frac{(h_1 - \eta)(h_2 + \eta)}{h_1 h_2}} - \frac{h_2 - h_1 + 2\eta}{h_2 - h_1} \right] \right\}$$

Sandstrom&Quon, 1993
***In this form*: Slunyaev et al., 2001**



Simple wave velocity vs. local displacement for $h_2 / h_1 = 3.86$ (small squares), 12 (dash dots), and 20.4 (solid line).

7

I. Hamiltonian approach (H - model)

(with V. Irisov)

$$H = \alpha(\eta) - \frac{1}{2}\beta(\eta)\left(\frac{\partial \eta}{\partial x}\right)^2$$

$$\eta_t + \frac{\partial}{\partial x}\left(\frac{\delta H}{\delta \eta}\right) - \frac{\partial^2}{\partial x^2}\left(\frac{\delta H}{\delta \eta_x}\right) = 0$$

$$\eta_t + c(\eta)\eta_x + (\beta\eta_{xx})_x + \left(\frac{\beta_\eta}{2}\eta_x^2\right)_x = 0$$

$$c(\eta) = \alpha_{\eta\eta}$$

$$\beta(\eta) = \frac{1}{6}c(\eta)(h_1+\eta)(h_2-\eta)$$

Attenuation + sloping bottom

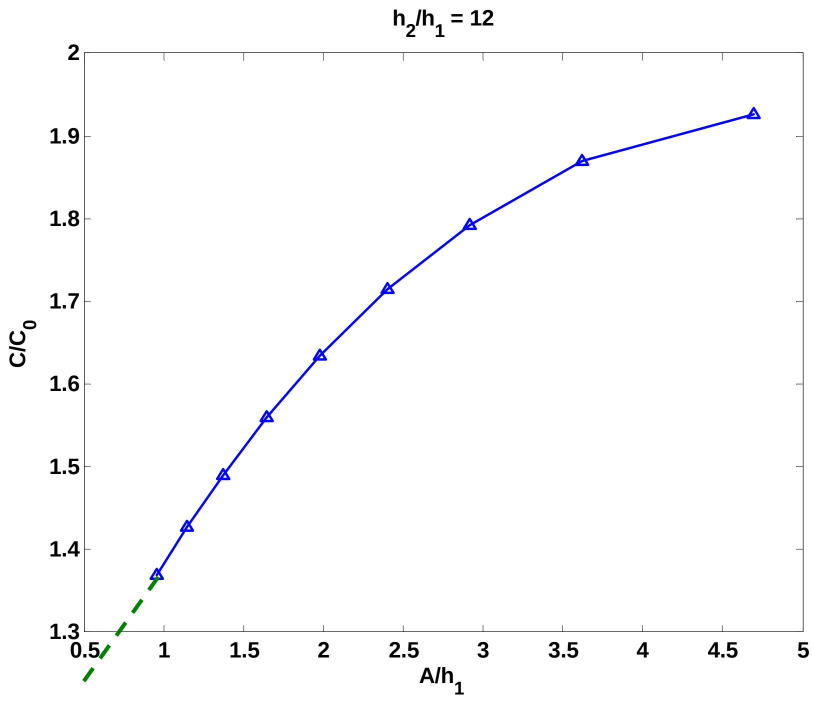
Turb. viscosity and bottom friction

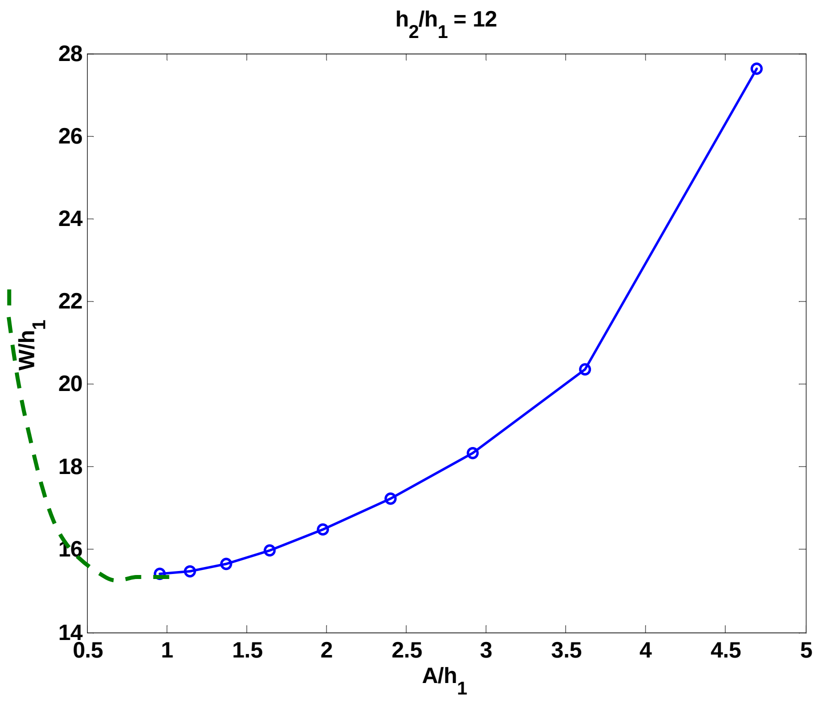
$$\eta_t + c(\eta)\eta_x + (\beta\eta_{xx})_x + \left(\frac{\beta_\eta}{2}\eta_x^2\right)_x - \nu_t\eta_{xx} + \frac{c\cdot u\cdot|u|}{C_z^2(h_2-\eta)\frac{\Delta\rho}{\rho}} = -\left(\alpha_{\eta h_2} + (\beta_{h_2}\eta_x)_x\right)\frac{\partial h_2}{\partial x}$$

$$C_z = \frac{(h_2-\eta)^{1/6}}{n} \qquad n \approx 0.025 \div 0.035$$

Normalized velocity and width of a soliton

(h2/h1 =12, close densities)





h2/h1 =12, close densities

Green – KdV part

SOUTH CHINA SEA-THEORY

Left: From Gardner equation (Orr & Mignerey, 2003), A_0=20 m
Right: From H – model , A_0=20 m

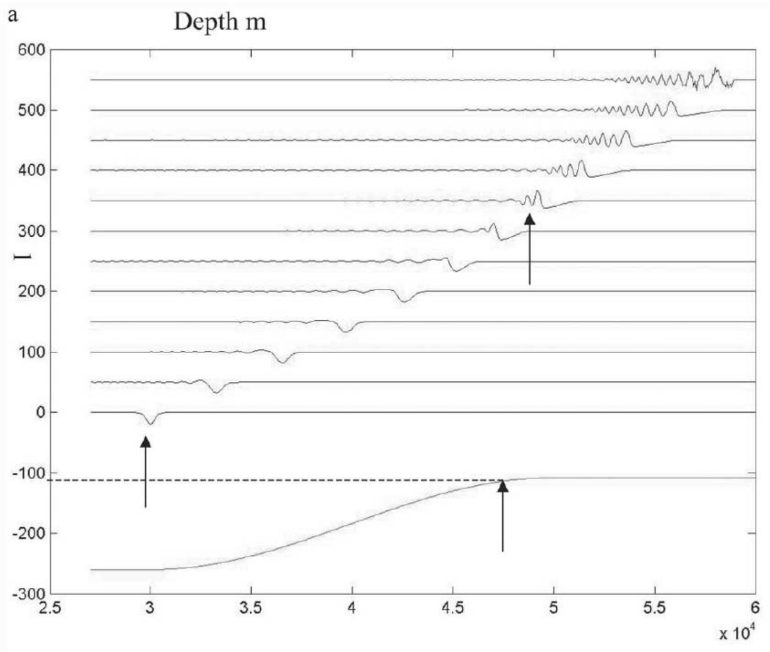
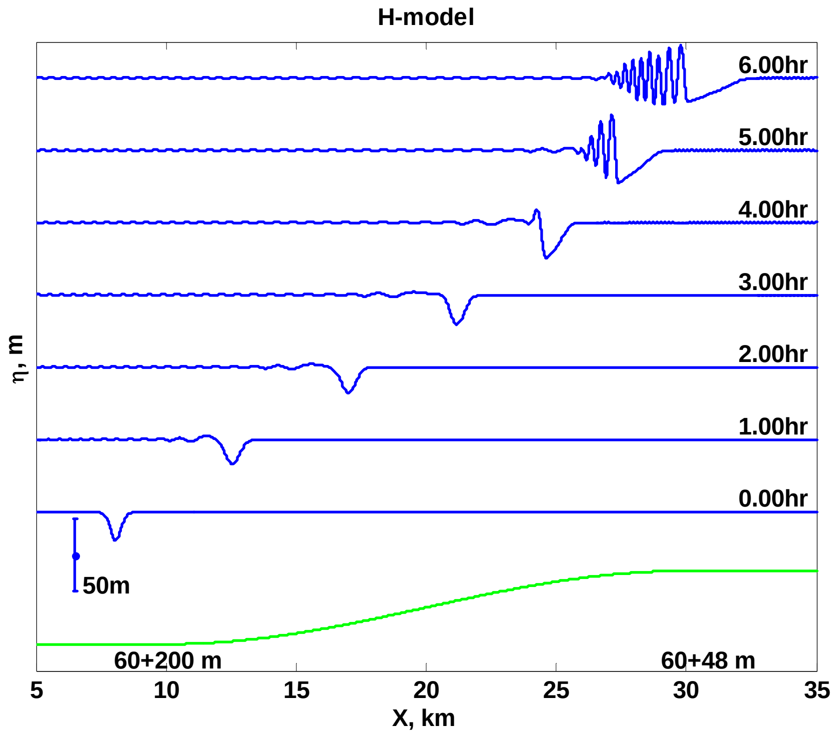


Figure 15. Plot of the calculated internal soliton waveform change as it propagates upslope. The initial wave amplitude was 20 m.



Formation of Limiting Soliton + Radiation

Left: From a fully nonlinear model (Vlasenko et al, 2005)
Right: From H - model

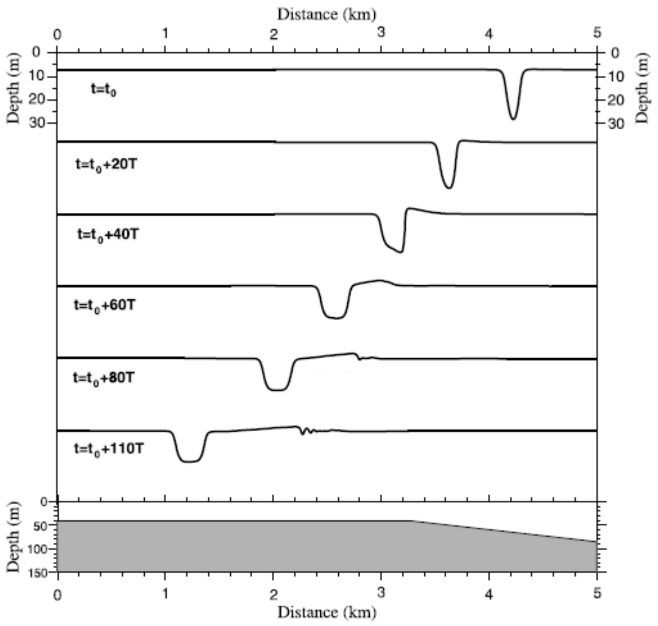
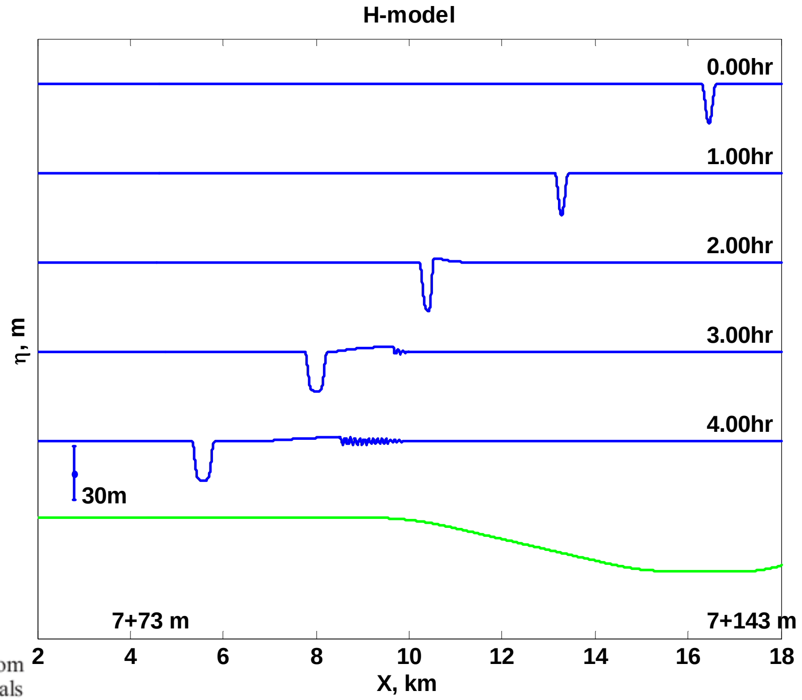
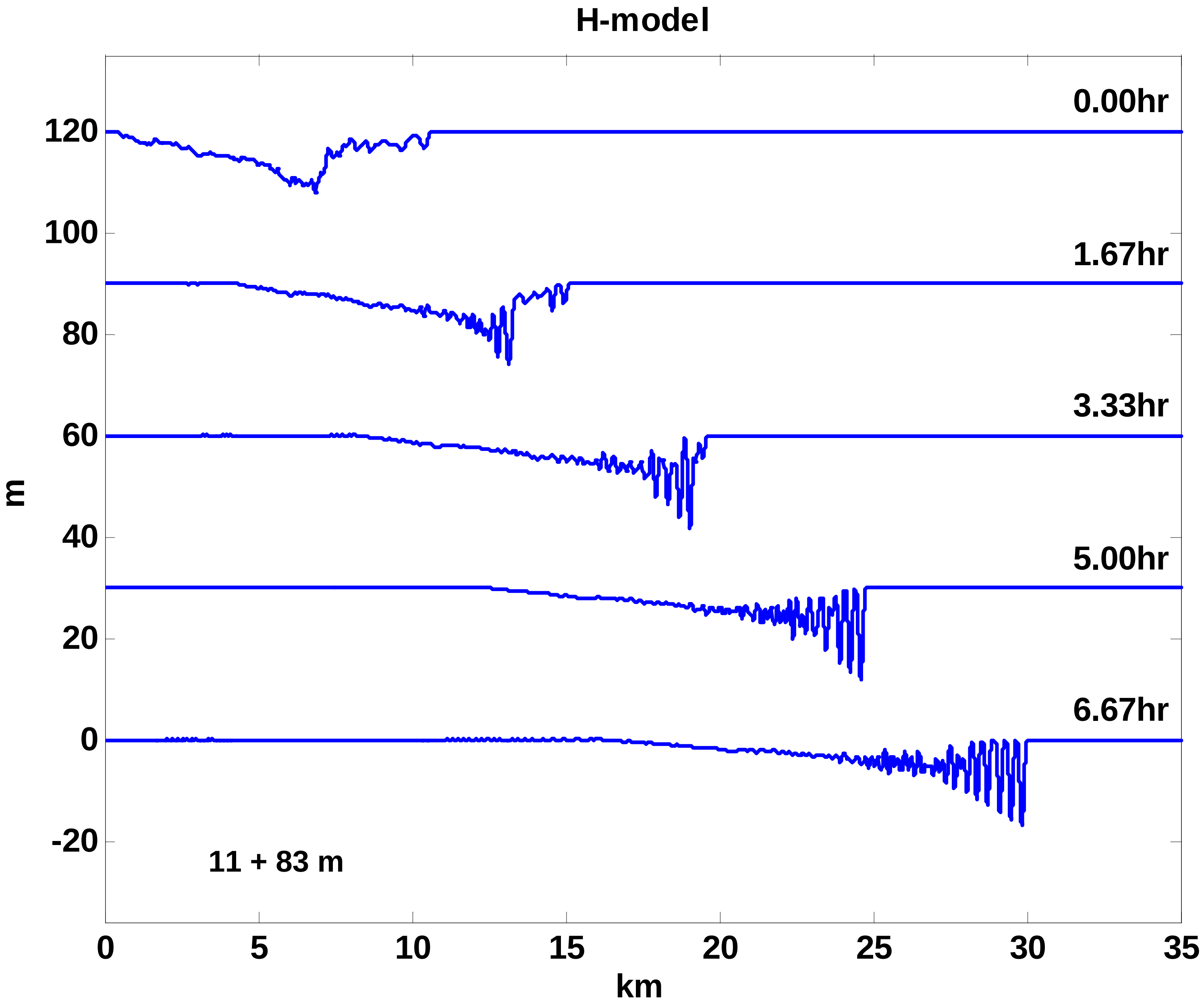


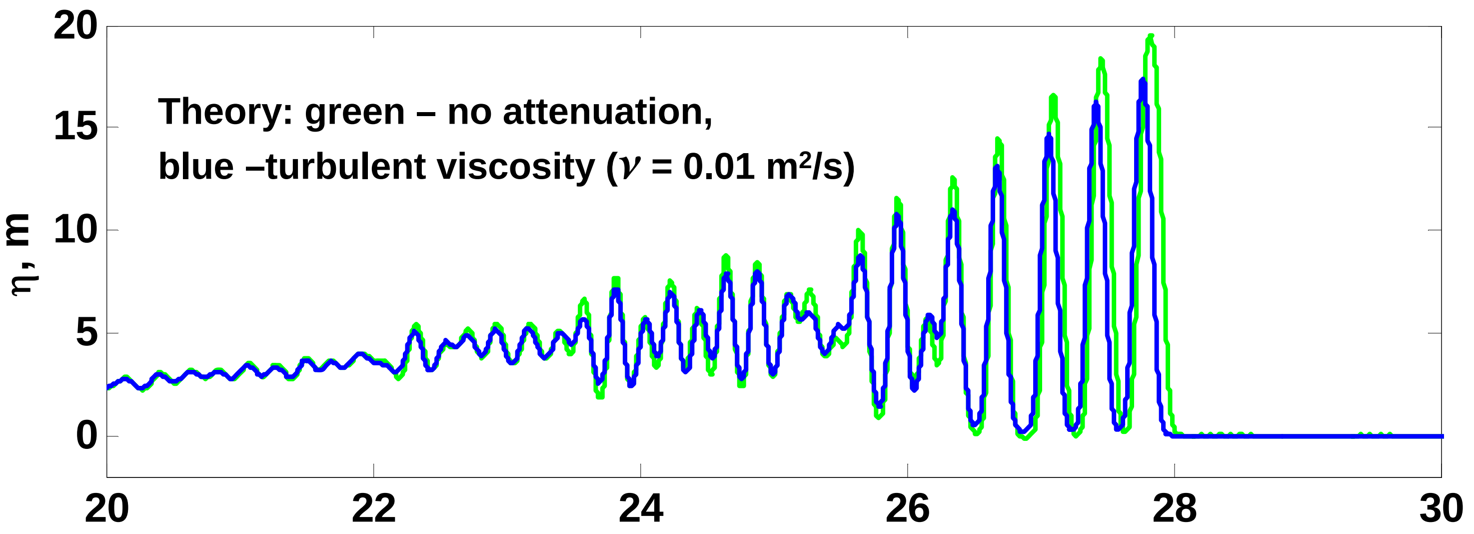
Figure 17. Example of nonadiabatic evolution of the ISW over the gently sloping ($\gamma = 1.47°$) bottom (only the isopycne corresponding to the middle of the density jump is shown). The timescale T equals 45 s. The initial amplitude of the incident ISW beyond the shelf at $H = 150$ m is 22.5 m.



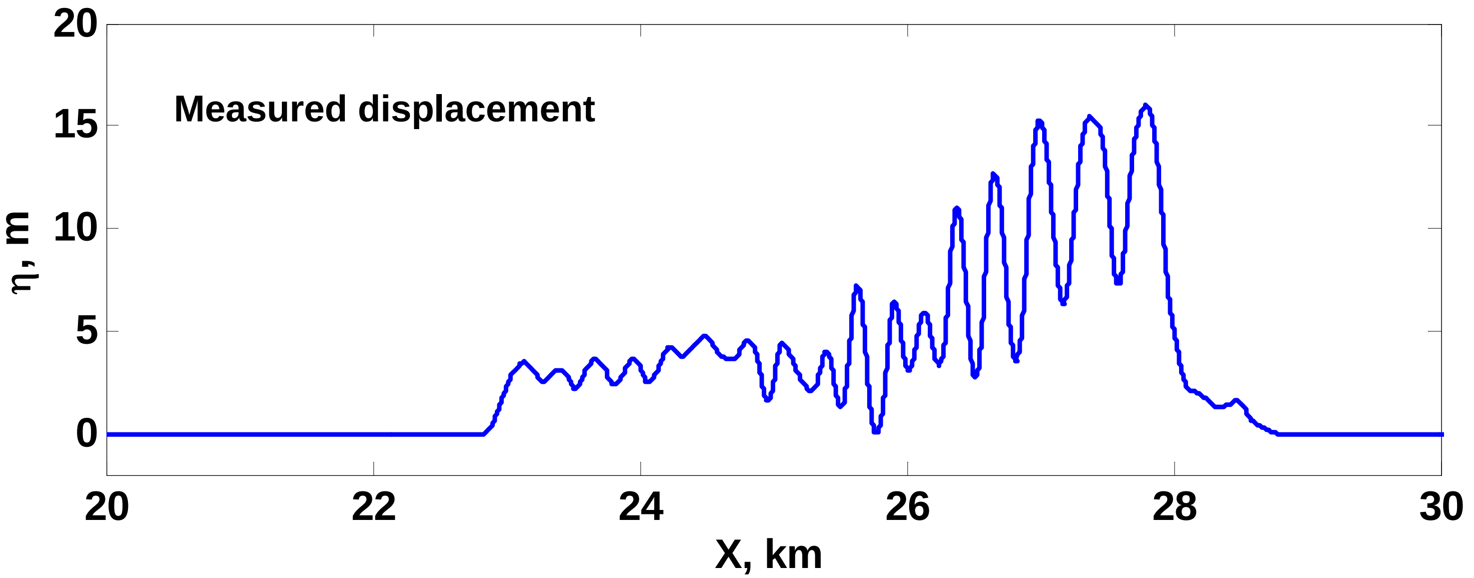
Evolution of the initial pulse taken from the Woods Hole data



Comparison with the WHOI’06 experiment



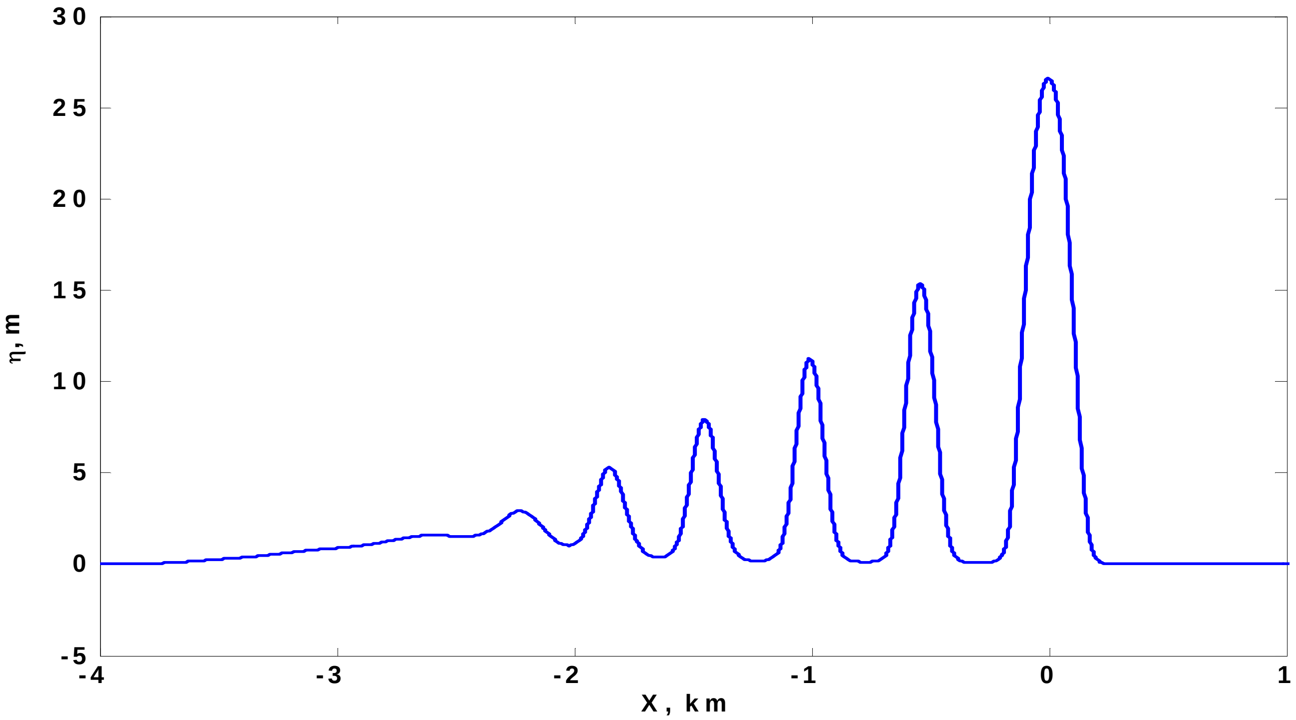
Theory: green – no attenuation,
blue –turbulent viscosity (ν = 0.01 m^2/s)



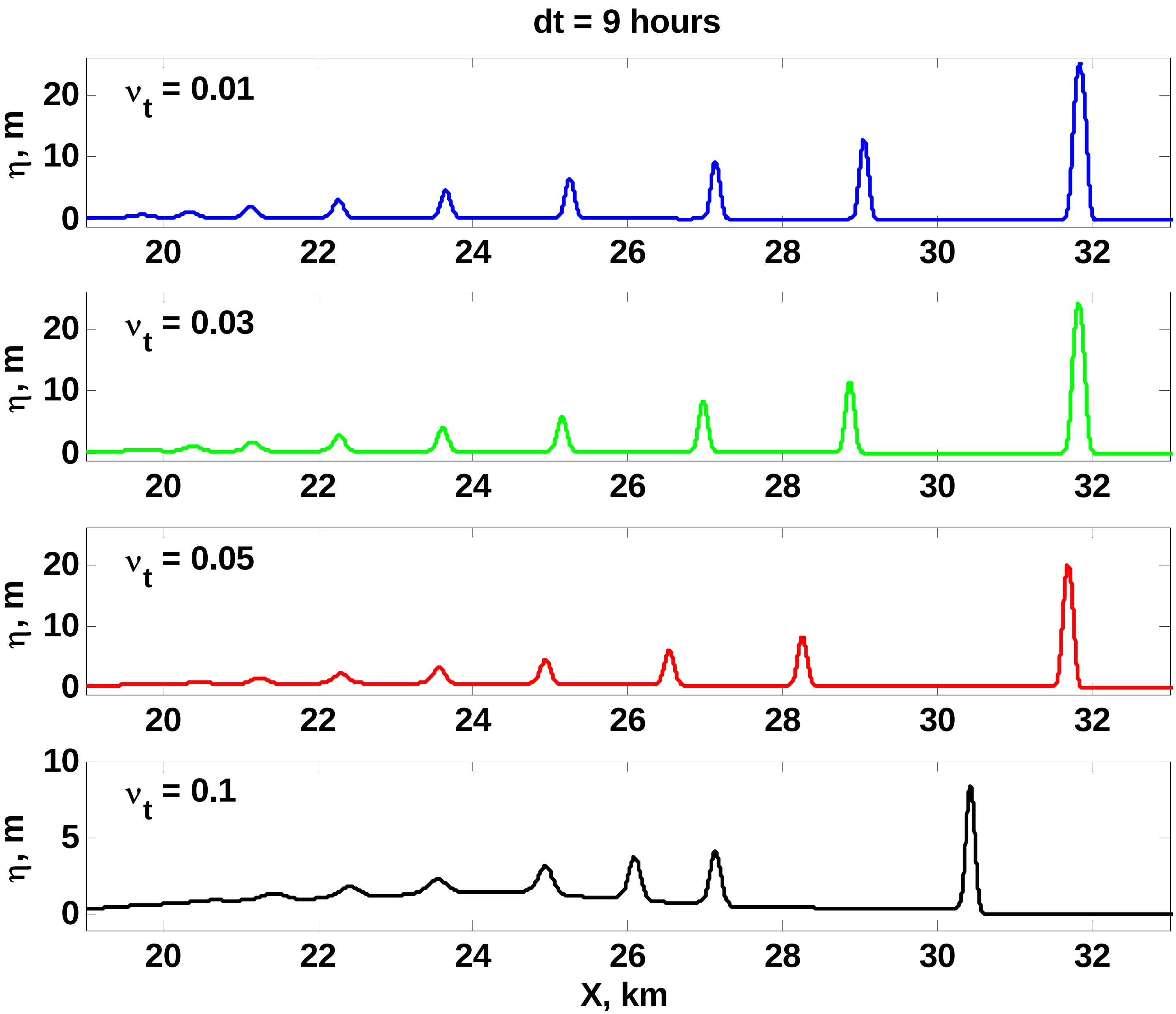
Measured displacement

Evolution of a soliton train with different turbulent viscosity coefficients

Initial condition



Calculation result at different attenuations
h1 = 11 m, h2 = 71 m



II. Strongly nonlinear, non-dispersive waves: Arbitrary stratification

$$\left(u_t + uu_x\right)_\beta = N^2(\beta)h_\beta,$$

$$h_{t\beta} + \left(uh_\beta\right)_x = 0$$

Generalized simple wave:

$$h = h(\xi,\beta), u = u(\xi,\beta), \quad \xi_t + c\xi_x = 0,$$

$$\xi = x - c(\xi) \qquad \xi_x = 1 - tc_\xi \xi_x = \frac{1}{1 + tc_\xi}.$$

After that, 2+1 eqs are reduced to 1+1:

$$\frac{\partial}{\partial\beta}\left[(u-c)u_{\xi}\right] = N^{2}(\beta)h_{\xi},$$

$$(u-c)\frac{\partial h_{\xi}}{\partial\beta} + u_{\xi}\frac{\partial h}{\partial\beta} = 0.$$

Weakly nonlinear wave:

$$h = \beta + A(\xi)f(\beta),\ u = c_{0}A(\xi)f\quad(\beta)$$

$$f''_{\beta\beta} + \frac{N^{2}(\beta)}{c_{0}^{2}}f = 0,\ c(\xi) = c_{0}\left[1 + A(\xi)\frac{\int_{-H}^{0} f_{\beta}^{3}d\beta}{\int_{-H}^{0} f_{\beta}^{2}d\beta}\right]$$

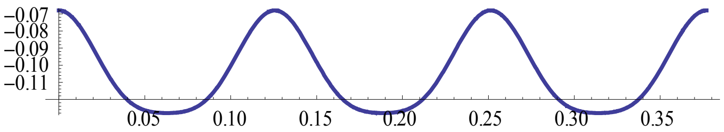
Strongly nonlinear wave
N = const

$$c(\xi) \propto 1 - 0.2\sin(k\xi)$$

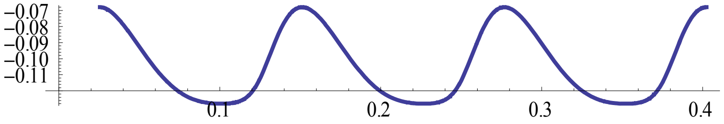
$$\xi(0,\beta) = \beta - 0.07\sin(\kappa\beta)$$

$$u(0,\beta) = -0.07\cos(\kappa\beta)$$

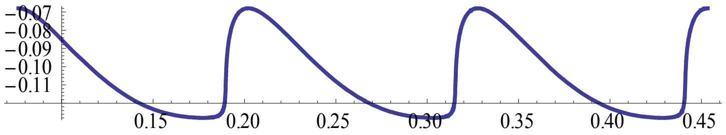
$$0 < \beta < 1$$



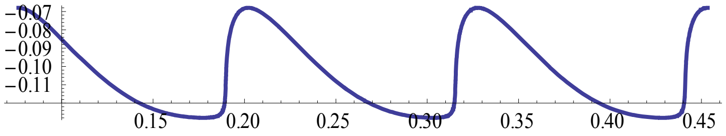
kx = 0



kx = 5

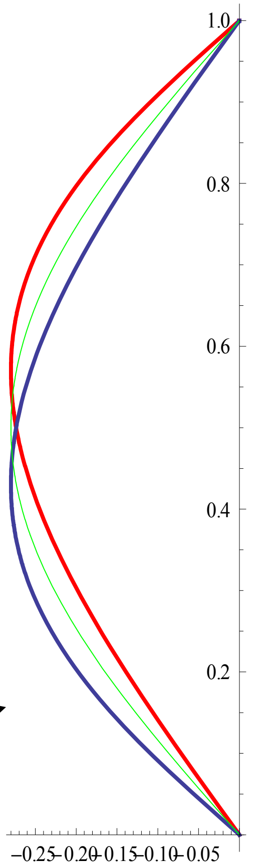


kx = 15



kx = 25

Vertical displacement profiles
Green – linear
Blue and red -
Nonlinear in different phases



III. INTERACTION OF KINKS AND COMPOUND SOLITONS IN THE CHOI-CAMASSA MODEL

(with K. Gorshkov and I. Soustova)

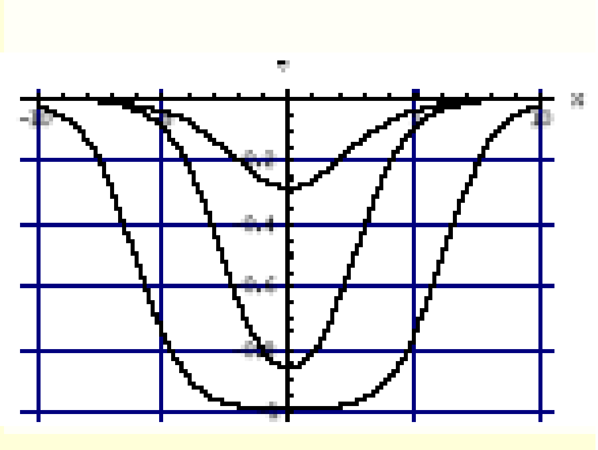
Direct perturbation theory for solitons: K. Gorshkov, L. Ostrovsky, Physica D, 1981
For kinks in Gardner eq.: K. Gorshkov, L. Ostrovsky, I. Soustova and V. Irisov, Phys. Rev. E, 2004

C-C equations for a two-layer model:

$$\eta_{1,2t} + \left(\eta_{1,2} u_{1,2}\right)_x = 0; \qquad \eta_{1,2} = h_{1,2} \pm \xi$$

$$\rho_{1,2}(u_{1,2t} + u_{1,2}u_{1,2x} + g\xi_x) = -p_x + \frac{1}{3\eta_{1,2}}\left(\eta_{1,2}^3\left(\frac{\partial}{\partial t} + u_{1,2}\frac{\partial}{\partial x}\right)^2 \xi\right)_x$$

Soliton



$$c^2 = \frac{c_0^2 (h_1 - a)(h_2 + a)}{h_1 h_2 - c_0^2 a / g}$$ -**Soliton velocity**

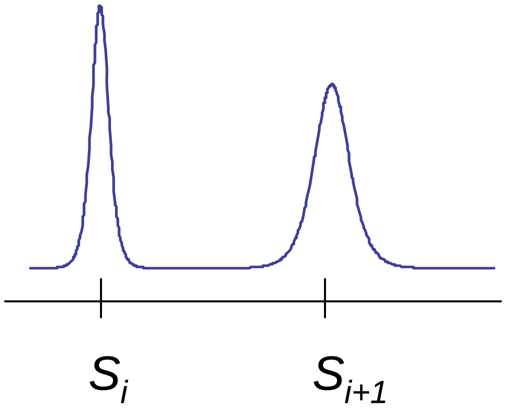
$$c_0^2 = g h_1 h_2 \frac{\rho_2 - \rho_1}{\rho_1 h_2 + \rho_2 h_1}$$ - **Linear velocity**

$$c_0 < c < c_m, \ 0 < |a| < |\zeta_m|$$

Limiting soliton:

$$c_m^2 = g\left(h_1 + h_2\right)\frac{1 - \sqrt{\rho_1 / \rho_2}}{1 + \sqrt{\rho_1 / \rho_2}}$$

$$\zeta_m = g h_1 h_2 \frac{h_1 - h_2 \sqrt{\rho_1 / \rho_2}}{1 + \sqrt{\rho_1 / \rho_2}}$$



Soliton interactions

(by exponential “tails”)

$$\varepsilon = O[(c_i - c_j)/c_{i,j}] << 1$$

$$\xi_i = \xi^0(x - S_i, c) + \sum_{n=1} \varepsilon^n \xi^n (x - S_i, \tau, \rho)$$

$$\boxed{m d^2 S_i / dt^2 = \alpha[e^{-\lambda_0 (S_i - S_{i-1})} - e^{-\lambda_0 (S_i - S_{i+1})}]}$$

← Toda lattice

*M = ∂P/ ∂**c** is effective mass (P is momentum).*

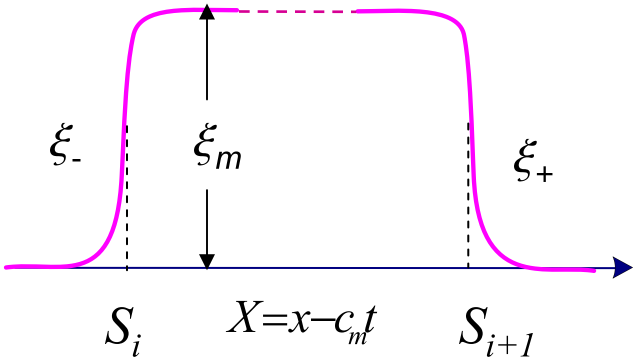
Phase shift:

$$\lambda_0 \Delta x_{i,j} = \pm \ln[2\lambda_0 m \left|(S_i - S_j)^2 / \alpha\right|]$$

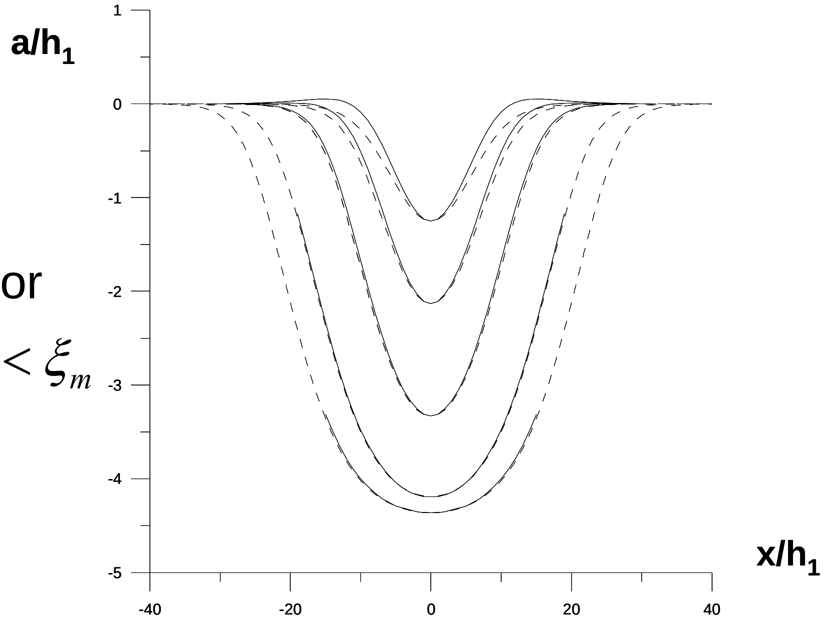
It works well at the “KdV level” when $\xi_{mi,j} \le 0.5$

Solitons close to limiting

$$\xi \approx \xi_+[x - c_m t - S_1(\tau,\rho)] + \xi_-[x - c_m t - S_2(\tau,\rho)] - \xi_m$$

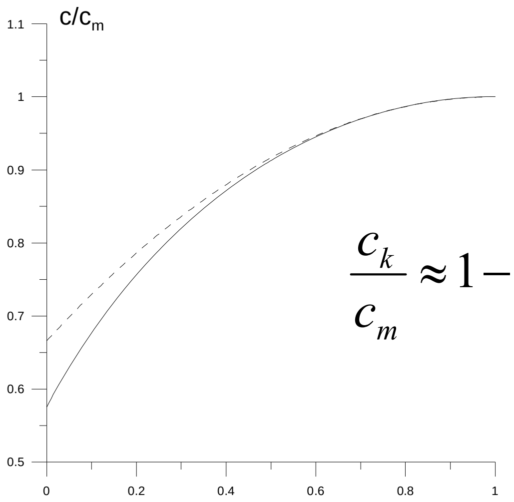


$$h_1 / h_2 = 1/10, \rho_1 / \rho_2 = 0.997$$



Good for

$$\xi_m / 2 < \xi < \xi_m$$

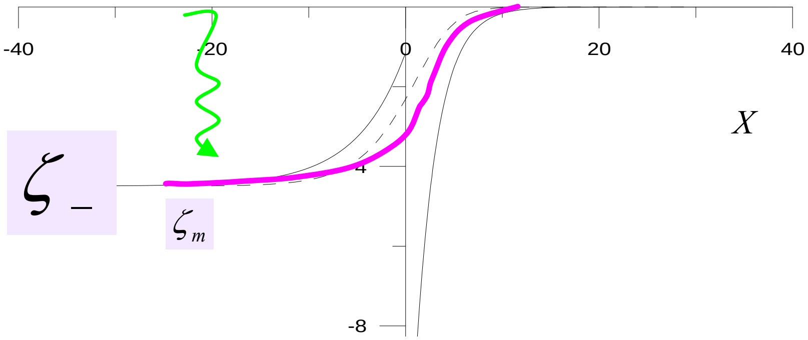


$$\frac{c_k}{c_m} \approx 1 - \frac{M_m(a-\xi_m)^2}{4c_m e_m^2}$$

—— Exact

- - - Two-kink

Kink



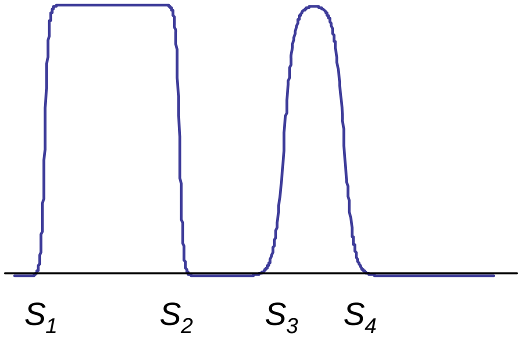
$$\left(\frac{\sqrt{a_*-\xi_\pm}+\sqrt{a_*}}{\sqrt{a_*-\xi_\pm}-\sqrt{a_*}}\right)\cdot\left(\frac{\sqrt{a_*-\xi_m}-\sqrt{a_*-\xi_\pm}}{\sqrt{a_*-\xi_m}+\sqrt{a_*-\xi_\pm}}\right)^{\frac{\sqrt{a_*-\xi_m}}{\sqrt{a_*}}}=Ae^{\mp\xi_m K_m(x-S_\pm)}$$

Asymptotics:

$$\xi_+=\begin{cases} e_0 e^{\Lambda_0(x-S_+)}\text{при }(x-S_+)\to-\infty \\ \xi_m+e_m e^{-\Lambda_m(x-S_+)}\text{при }(x-S_+)\to+\infty \end{cases}\qquad \xi_-=\begin{cases} \xi_m+e_m e^{\Lambda_m(x-S_-)}\ \text{при}(x-S_-)\to-\infty \\ e_0 e^{-\Lambda_0(x-S_-)}\ \text{при}(x-S_-)\to+\infty \end{cases}$$

$$\Lambda_0=(\xi_m K_m)/\sqrt{a_*},\ \Lambda_m=(\xi_m K_m)/\sqrt{a_*-\xi_m}$$

Interaction of kinks



$$\xi_N = \sum_{i=1}^{2N} \xi^0\left(x - S_i\right) - N\zeta_m$$

$$\xi_i\left(x,t\right) = \xi_i^0\left(X - S_i(\varepsilon x, \varepsilon t\right) + \sum_{n=1}^{\infty} \xi_i^{(n)}\left(X - S_i(\varepsilon x, \varepsilon t), \tau, \rho\right)$$

$$u_{i(1,2)}\left(x,t\right) = U_{i(1,2)}^0\left(X - S_i(\varepsilon x, \varepsilon t\right) + \sum_{n=1}^{\infty} U_{i(1,2)}^{(n)}\left(X - S_i(\varepsilon x, \varepsilon t), \tau, \rho\right)$$

$$\frac{dS_i}{dt} = I(\tau,\rho) - \begin{Bmatrix} M_0 e^{-\Lambda_0\left(S_i - S_{i-1}\right)} + M_m e^{-\Lambda_m\left(S_{i+1} - S_i\right)}, i - odd \\ M_m e^{-\Lambda_m\left(S_i - S_{i-1}\right)} + M_0 e^{-\Lambda_0\left(S_{i+1} - S_i\right)}, i - even \end{Bmatrix}$$

I(τ,ρ) is prescribed by the initial conditions

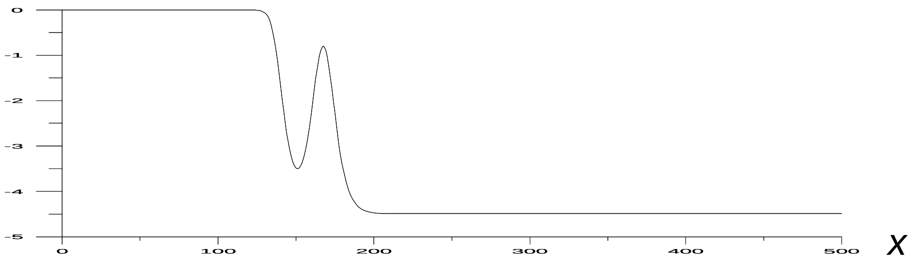
$$M_0 = \frac{2c_m m(0) e_0^2 \Lambda_0^2}{g\left(\rho_2 - \rho_1\right)\zeta_m^2}, \qquad M_m = \frac{2c_m m(\zeta_m) e_m^2 \Lambda_m^2}{g\left(\rho_2 - \rho_1\right)\zeta_m^2},$$

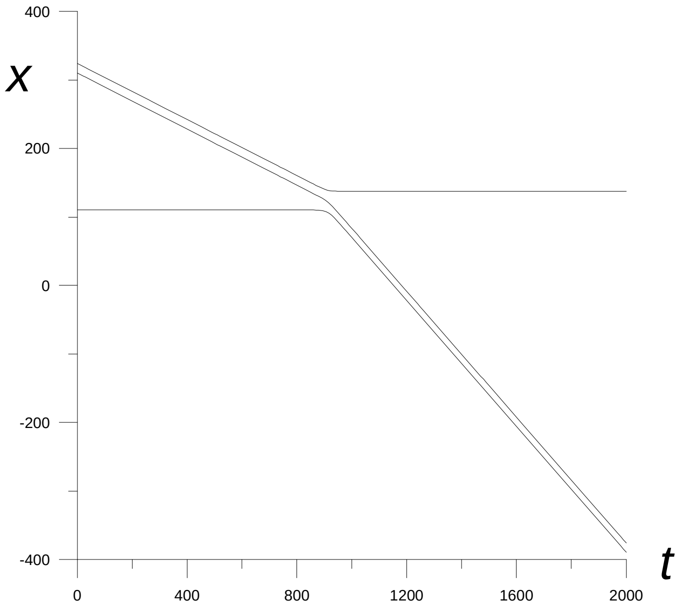
Example 1: Kink-soliton interaction

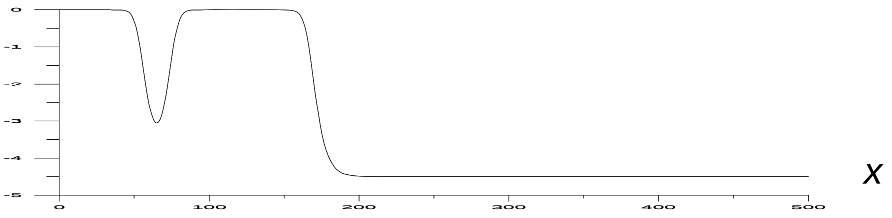
$$\frac{dS_2}{dt} = -M_0 e^{-\Lambda_0 (S_3 - S_2)} \longrightarrow \text{kink}$$

$$\frac{dS_3}{dt} = -M_0 e^{-\Lambda_0 (S_3 - S_2)} - M_m e^{-\Lambda_m (S_4 - S_3)}$$

$$\frac{dS_4}{dt} = -M_m e^{-\Lambda_m (S_4 - S_3)}$$

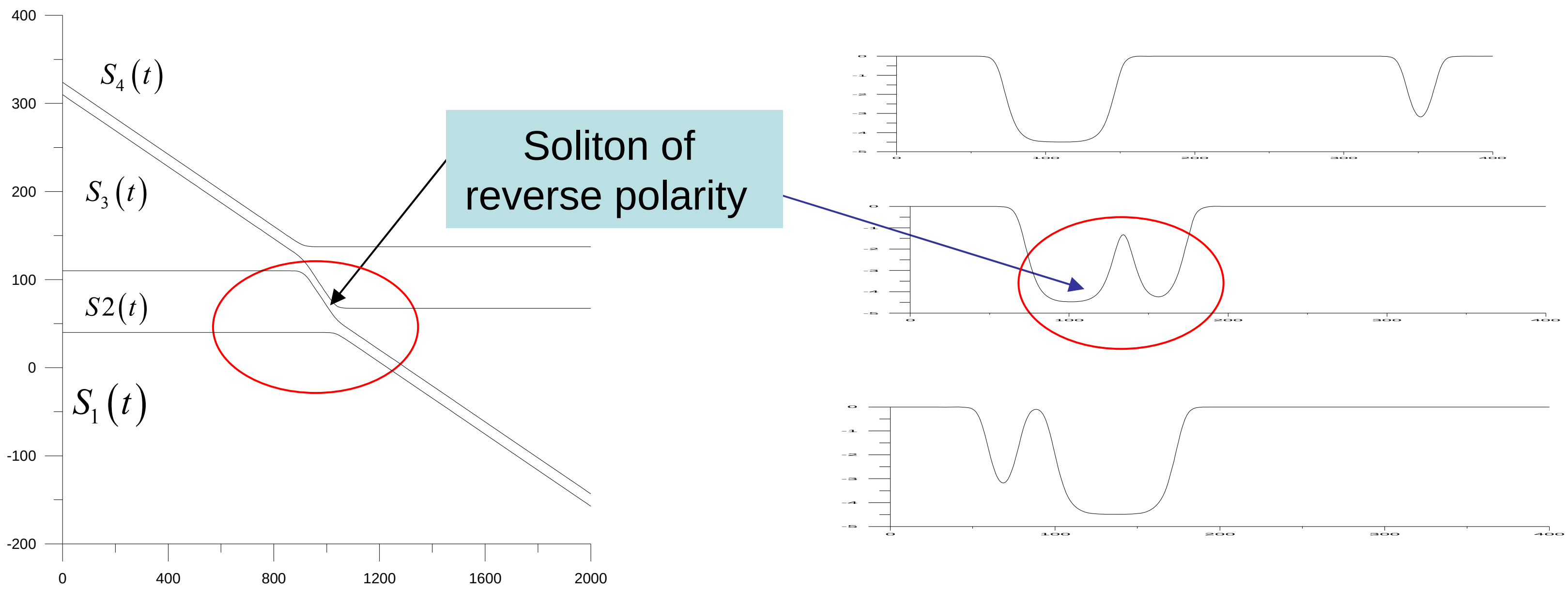






Example 2: Interaction of composite solitons

$$h_1 / h_2 = 1/10, \rho_1 / \rho_2 = 0.997$$



Soliton of reverse polarity

Soliton interaction occurs in three stages:

1. **Kink interaction giving rise to a reverse-polarity soliton (in this case, positive)**
2. **Uniform motion of the positive soliton inside of the bottom.**
3. **Kink rearrangement resulting in soliton divergence.**

Soliton interaction results in formation of a soliton phase shift due to its uniform motion with velocity different from its velocity at the asymptotic stage.

Conclusions

- Strongly nonlinear internal solitons (with a “Mach number” of order unity or larger) exist in many coastal areas.
- Strictly speaking, it is impossible to reduce description of strongly nonlinear, dispersive IW to 2-D (1-D) long-wave models. Still, strongly nonlinear-weakly dispersive models such as the Miyata-Choi-Camassa (MCC) model often (but not always) work well for a two-layer case.
- Hamiltonian approach with an exact long-wave velocity and semi-phenomenological nonlinear dispersion yields an evolution equation which provides a good description of numerical and observational results for strong internal wave trains on shelf.
- The description of strongly nonlinear, non-dispersive waves at smooth stratification can be reduced from a 2+1 to 1+1 problem by using the isopycnal variable and considering a progressive, deforming wave.
- For strong enough MCC solitons, their representation as pairs of interacting kinks allows to efficiently describe soliton interaction (including the phase shifts) using a perturbation theory for kinks.