An asymptotic description of internal waves

Joint work with J. Bona and J.-C. Saut,
Asymptotic models for internal waves, J. Math. Pures Appl. **89** (2008), 538-566

David Lannes

Ecole Normale Supérieure

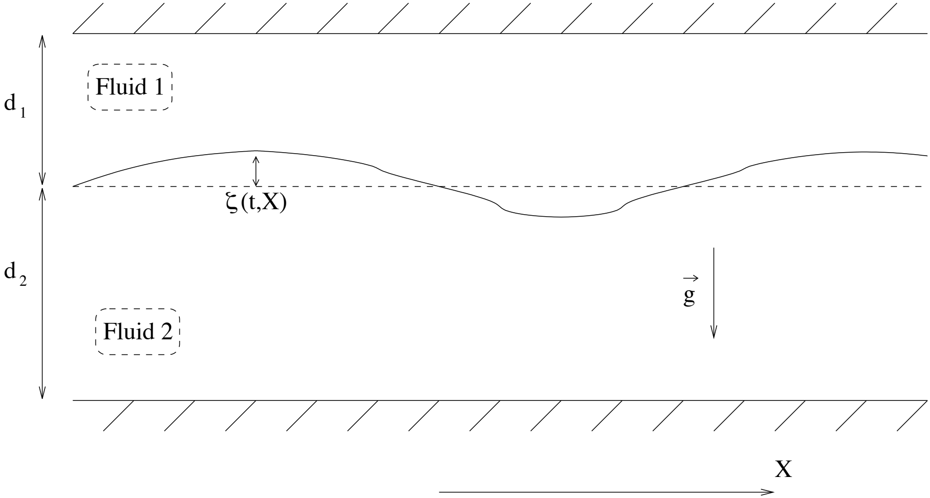
Edinburgh 2008

Loch Ness monster

> … *"a possibility worth looking into depends on the acoustic-reflective properties of internal waves or regions of shear instability, in particular the steep-fronted internal surges, associated in Loch Ness with short internal waves"*

(MORTIMER, C. H. *The Loch Ness monster-limnology or paralimnology?*)

Notations



The internal waves equations

Fluid 1

The internal waves equations

Fluid 1

- $v_1 = \nabla_{X,z}\Phi_1$

The internal waves equations

Fluid 1

- $v_1 = \nabla_{X,z}\Phi_1$
- $\Delta_{X,z}\Phi_1 = 0$

The internal waves equations

Fluid 1

- $v_1 = \nabla_{X,z}\Phi_1$
- $\Delta_{X,z}\Phi_1 = 0$
- $\partial_z \Phi_1 {}_{|_{top}} = 0$

The internal waves equations

Fluid 1

- $v_1 = \nabla_{X,z}\Phi_1$
- $\Delta_{X,z}\Phi_1 = 0$
- $\partial_z \Phi_1 {}_{|_{top}} = 0$
- $\partial_t \Phi_1 + \frac{1}{2}|\nabla_{X,z}\Phi_1|^2 = -\frac{P}{\rho_1} - gz.$

The internal waves equations

Fluid 1

- $v_1 = \nabla_{X,z}\Phi_1$
- $\Delta_{X,z}\Phi_1 = 0$
- $\partial_z\Phi_1\,_{|_{top}} = 0$
- $\partial_t\Phi_1 + \frac{1}{2}|\nabla_{X,z}\Phi_1|^2 = -\frac{P}{\rho_1} - gz.$

Fluid 2

- $v_2 = \nabla_{X,z}\Phi_2$
- $\Delta_{X,z}\Phi_2 = 0$
- $\partial_z\Phi_2\,_{|_{bottom}} = 0$
- $\partial_t\Phi_2 + \frac{1}{2}|\nabla_{X,z}\Phi_2|^2 = -\frac{P}{\rho_2} - gz.$

The internal waves equations

Fluid 1

- $v_1 = \nabla_{X,z}\Phi_1$
- $\Delta_{X,z}\Phi_1 = 0$
- $\partial_z\Phi_1\,|_{top} = 0$
- $\partial_t\Phi_1+\frac{1}{2}|\nabla_{X,z}\Phi_1|^2 = -\frac{P}{\rho_1}-gz.$

Fluid 2

- $v_2 = \nabla_{X,z}\Phi_2$
- $\Delta_{X,z}\Phi_2 = 0$
- $\partial_z\Phi_2\,|_{bottom} = 0$
- $\partial_t\Phi_2+\frac{1}{2}|\nabla_{X,z}\Phi_2|^2 = -\frac{P}{\rho_2}-gz.$

Interface

- $\partial_t\zeta - \sqrt{1+|\nabla\zeta|^2}\partial_n\Phi_j, \quad j=1,2$
- P is continuous

Interface equations

Interface ($\Psi_1 = \Phi_1\,|_{interface}$, $\Psi_2 = \Phi_2\,|_{interface}$)

- $\partial_t\zeta - \sqrt{1+|\nabla\zeta|^2}\partial_n\Phi_1\,|_{interface} = 0$
- $\partial_n\Phi_1\,|_{interface} = \partial_n\Phi_2\,|_{interface}$
- $\rho_1\Big(\partial_t\psi_1 + g\zeta + \frac{1}{2}|\nabla\psi_1|^2 - \frac{(\sqrt{1+|\nabla\zeta|^2}(\partial_n\Phi_1)+\nabla\zeta\cdot\nabla\psi_1)^2}{2(1+|\nabla\zeta|^2)}\Big) = -P.$
- $\rho_2\Big(\partial_t\psi_2 + g\zeta + \frac{1}{2}|\nabla\psi_2|^2 - \frac{(\sqrt{1+|\nabla\zeta|^2}(\partial_n\Phi_2)+\nabla\zeta\cdot\nabla\psi_2)^2}{2(1+|\nabla\zeta|^2)}\Big) = -P.$

Interface equations

Interface ($\Psi_1 = \Phi_1\,|_{interface}$, $\Psi_2 = \Phi_2\,|_{interface}$)

- $\partial_t\zeta - \sqrt{1+|\nabla\zeta|^2}\partial_n\Phi_1\,|_{interface} = 0$
- $\partial_n\Phi_1\,|_{interface} = \partial_n\Phi_2\,|_{interface}$
- $\rho_1\Big(\partial_t\psi_1 + g\zeta + \frac{1}{2}|\nabla\psi_1|^2 - \frac{(\sqrt{1+|\nabla\zeta|^2}(\partial_n\Phi_1)+\nabla\zeta\cdot\nabla\psi_1)^2}{2(1+|\nabla\zeta|^2)}\Big) = -P.$
- $\rho_2\Big(\partial_t\psi_2 + g\zeta + \frac{1}{2}|\nabla\psi_2|^2 - \frac{(\sqrt{1+|\nabla\zeta|^2}(\partial_n\Phi_2)+\nabla\zeta\cdot\nabla\psi_2)^2}{2(1+|\nabla\zeta|^2)}\Big) = -P.$

Fluid 1

$$\begin{cases} \Delta_{X,z}\Phi_1 = 0, \\ \partial_z\Phi_1\,|_{top} = 0, \\ \Phi_1\,|_{interface} = \Psi_1 \end{cases}$$

Fluid 2

$$\begin{cases} \Delta_{X,z}\Phi_2 = 0, \\ \partial_z\Phi_1\,|_{bottom} = 0, \\ \Phi_2\,|_{interface} = \Psi_2 \end{cases}$$

The Dirichlet-Neumann operator

Definition (Dirichlet-Neumann operator)

$$G[\zeta]: \quad \Psi_1 \quad \mapsto \quad G[\zeta]\Psi_1 = \sqrt{1+|\nabla\zeta|^2}\,\partial_n \Phi_1\,|_{interface}.$$

The Dirichlet-Neumann operator

Definition (Dirichlet-Neumann operator)

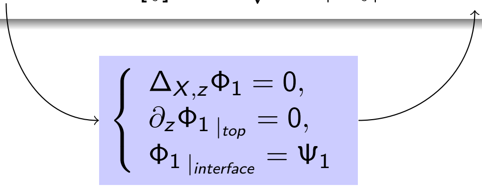
$$G[\zeta]: \quad \Psi_1 \quad \mapsto \quad G[\zeta]\Psi_1 = \sqrt{1+|\nabla\zeta|^2}\,\partial_n\Phi_1\,_{|_{interface}}.$$

$$\left\{\begin{array}{l} \Delta_{X,z}\Phi_1 = 0, \\ \partial_z\Phi_1\,_{|_{top}} = 0, \\ \Phi_1\,_{|_{interface}} = \Psi_1 \end{array}\right.$$

The Dirichlet-Neumann operator

Definition (Dirichlet-Neumann operator)

$$G[\zeta]: \quad \Psi_1 \quad \mapsto \quad G[\zeta]\Psi_1 = \sqrt{1+|\nabla\zeta|^2}\,\partial_n \Phi_1\,_{|interface}.$$



$$\begin{cases} \Delta_{X,z}\Phi_1 = 0, \\ \partial_z \Phi_1\,_{|top} = 0, \\ \Phi_1\,_{|interface} = \Psi_1 \end{cases}$$

The interface operator

Definition (Interface operator)

$$H[\zeta]: \quad \Psi_1 \quad \mapsto \quad H[\zeta]\Psi_1 = \nabla(\Phi_2\,|_{interface}).$$

The interface operator

Definition (Interface operator)

$$H[\zeta] : \quad \Psi_1 \quad \mapsto \quad H[\zeta]\Psi_1 = \nabla\left(\Phi_2{}_{|_{interface}}\right).$$

$$\partial_n \Phi_1{}_{|_{interface}} = \frac{G[\zeta]\Psi_1}{\sqrt{1+|\nabla\zeta|^2}}$$

The interface operator

Definition (Interface operator)

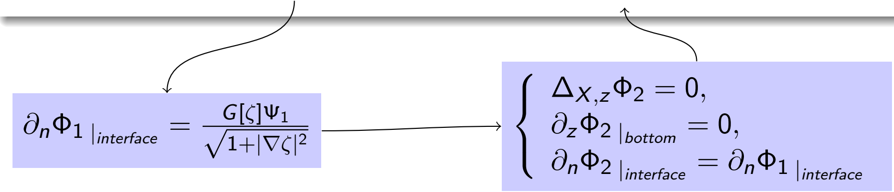
$$H[\zeta]: \quad \Psi_1 \quad \mapsto \quad H[\zeta]\Psi_1 = \nabla(\Phi_2\,|_{interface}).$$

$$\partial_n \Phi_1\,|_{interface} = \frac{G[\zeta]\Psi_1}{\sqrt{1+|\nabla\zeta|^2}} \longrightarrow \left\{ \begin{array}{l} \Delta_{X,z}\Phi_2 = 0, \\ \partial_z \Phi_2\,|_{bottom} = 0, \\ \partial_n \Phi_2\,|_{interface} = \partial_n \Phi_1\,|_{interface} \end{array} \right.$$

The interface operator

Definition (Interface operator)

$$H[\zeta]: \quad \Psi_1 \quad \mapsto \quad H[\zeta]\Psi_1 = \nabla(\Phi_2 \,|_{interface}).$$



$$\partial_n \Phi_1 \,|_{interface} = \frac{G[\zeta]\Psi_1}{\sqrt{1+|\nabla\zeta|^2}}$$

$$\begin{cases} \Delta_{X,z}\Phi_2 = 0, \\ \partial_z \Phi_2 \,|_{bottom} = 0, \\ \partial_n \Phi_2 \,|_{interface} = \partial_n \Phi_1 \,|_{interface} \end{cases}$$

The reduced internal waves equations

The reduced equations ($\gamma = \rho_1/\rho_2$)

$$\begin{cases} \partial_t \zeta - G[\zeta]\psi_1 = 0, \\ \partial_t(\mathbf{H}[\zeta]\psi_1 - \gamma \nabla \psi_1) + g(1-\gamma)\nabla\zeta \\ \qquad\qquad + \frac{1}{2}\nabla\big(|\mathbf{H}[\zeta]\psi_1|^2 - \gamma|\nabla\psi_1|^2\big) + \nabla\mathcal{N}(\zeta,\psi_1) = 0. \end{cases}$$

The reduced internal waves equations

$$\boxed{\partial_t\zeta - \sqrt{1+|\nabla\zeta|^2}\partial_n\Phi_1\,_{|_{interface}} = 0}$$

The reduced equations ($\gamma = \rho_1/\rho_2$)

$$\left\{\begin{array}{l}\partial_t\zeta - G[\zeta]\psi_1 = 0,\\ \partial_t(\mathbf{H}[\zeta]\psi_1 - \gamma\nabla\psi_1) + g(1-\gamma)\nabla\zeta\\ \qquad\qquad +\frac{1}{2}\nabla\big(|\mathbf{H}[\zeta]\psi_1|^2 - \gamma|\nabla\psi_1|^2\big) + \nabla\mathcal{N}(\zeta,\psi_1) = 0.\end{array}\right.$$

The reduced internal waves equations

The reduced equations ($\gamma = \rho_1/\rho_2$)

$$\begin{cases} \partial_t \zeta - G[\zeta]\psi_1 = 0, \\ \partial_t(\mathbf{H}[\zeta]\psi_1 - \gamma\nabla\psi_1) + g(1-\gamma)\nabla\zeta \\ \qquad\qquad + \frac{1}{2}\nabla\big(|\mathbf{H}[\zeta]\psi_1|^2 - \gamma|\nabla\psi_1|^2\big) + \nabla\mathcal{N}(\zeta,\psi_1) = 0. \end{cases}$$

$$\rho_1\Big(\partial_t\psi_1 + g\zeta + \frac{1}{2}|\nabla\psi_1|^2 - \frac{(\sqrt{1+|\nabla\zeta|^2}(\partial_n\Phi_1)+\nabla\zeta\cdot\nabla\psi_1)^2}{2(1+|\nabla\zeta|^2)}\Big) = -P,$$

$$\rho_2\Big(\partial_t\psi_2 + g\zeta + \frac{1}{2}|\nabla\psi_2|^2 - \frac{(\sqrt{1+|\nabla\zeta|^2}(\partial_n\Phi_2)+\nabla\zeta\cdot\nabla\psi_2)^2}{2(1+|\nabla\zeta|^2)}\Big) = -P.$$

Non dimensionalization for one fluid

I. Identify the characteristic lengths:

Non dimensionalization for one fluid

I. Identify the characteristic lengths:

- Wavelength: λ

Non dimensionalization for one fluid

I. Identify the characteristic lengths:

- Wavelength: λ
- Amplitude: a

Non dimensionalization for one fluid

I. Identify the characteristic lengths:

- Wavelength: λ
- Amplitude: a
- Depth: d

Non dimensionalization for one fluid

I. Identify the characteristic lengths:

- Wavelength: λ
- Amplitude: a
- Depth: d

II. Nondimensionalization

- $\widetilde{\zeta} = \frac{\zeta}{a}$, $\widetilde{X} = \frac{X}{\lambda}$, etc.

Non dimensionalization for one fluid

I. Identify the characteristic lengths:

- Wavelength: λ
- Amplitude: a
- Depth: d

II. Nondimensionalization

- $\widetilde{\zeta} = \frac{\zeta}{a}$, $\widetilde{X} = \frac{X}{\lambda}$, etc.

III. Asymptotic analysis in terms of dimensionless parameters

- Amplitude/nonlinearity parameter $\varepsilon = \frac{a}{d}$

Non dimensionalization for one fluid

I. Identify the characteristic lengths:

- Wavelength: λ
- Amplitude: a
- Depth: d

II. Nondimensionalization

- $\widetilde{\zeta} = \frac{\zeta}{a}$, $\widetilde{X} = \frac{X}{\lambda}$, etc.

III. Asymptotic analysis in terms of dimensionless parameters

- Amplitude/nonlinearity parameter $\varepsilon = \frac{a}{d}$
- Shallowness/dispersive parameter $\mu = \frac{d^2}{\lambda^2}$.

Asymptotic regimes for one fluid

Definition

- A shallow water regime corresponds to $\mu \ll 1$
- A small amplitude regime corresponds to $\varepsilon \ll 1$.

Example

- The Nonlinear Shallow Water equations (Saint-Venant) is shallow water and large amplitude
- Boussinesq, KdV, KP are shallow water and small amplitude
- Full dispersion models are deep water and small amplitude

Ref: Alvarez-Samaniego and D. L., Large time existence for 3d water-waves and asymptotics. Invent. math., **171**:485–541, 2008.

Asymptotics for two fluids

I. Identify the characteristic lengths:

- Wavelength: λ
- Amplitude: a
- Depth: d_1 and d_2

Asymptotics for two fluids

I. Identify the characteristic lengths:

- Wavelength: λ
- Amplitude: a
- Depth: d_1 and d_2

II. Nondimensionalization

Asymptotics for two fluids

I. Identify the characteristic lengths:

- Wavelength: λ
- Amplitude: a
- Depth: d_1 and d_2

II. Nondimensionalization

III. Dimensionless parameters

- Amplitude $\varepsilon_1 = \frac{a}{d_1}$
- Shallowness $\mu_1 = \frac{d_1^2}{\lambda^2}$.
- Amplitude $\varepsilon_2 = \frac{a}{d_2}$
- Shallowness $\mu_2 = \frac{d_2^2}{\lambda^2}$.

Asymptotics for two fluids

I. Identify the characteristic lengths:

- Wavelength: λ
- Amplitude: a
- Depth: d_1 and d_2

II. Nondimensionalization

III. Dimensionless parameters

- Amplitude $\varepsilon_1 = \frac{a}{d_1}$
- Shallowness $\mu_1 = \frac{d_1^2}{\lambda^2}$.
- Amplitude $\varepsilon_2 = \frac{a}{d_2}$
- Shallowness $\mu_2 = \frac{d_2^2}{\lambda^2}$.

Example

A model is shallow water / shallow water if $\mu_1 \ll 1$ and $\mu_2 \ll 1$

Nondimensionalized internal wave equations

Dimensionless equations ($\delta = \frac{d_1}{d_2}$)

$$\begin{cases} \partial_t \zeta - \frac{1}{\mu} G^{\mu}[\varepsilon\zeta]\psi_1 & = & 0, \\ \partial_t \big(\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1 - \gamma\nabla\psi_1 \big) + (1-\gamma)\nabla\zeta & & \\ \qquad + \frac{\varepsilon}{2}\nabla\big(|\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1|^2 - \gamma|\nabla\psi_1|^2 \big) + \varepsilon\nabla\mathcal{N}^{\mu,\delta}(\varepsilon\zeta,\psi_1) & = & 0. \end{cases}$$

Goal

Systematic derivation of asymptotic models in terms of ε_1, ε_2, μ_1, μ_2.

Earlier works:

Formal derivation in 1D: BO eqn [Benjamin67], ILW eqn [Joseph77, Kubota et al.78], SW/SW models ([Maltseva89,CamassaChoi99,NguyenDias08]), many models [CraigGuyenneKalish05], etc.
Formal derivation in 2D: many models ([CamassaChoi96]), ...

Full justification in 1D: BO eqn [OhIguchi, to appear].

The interface operator $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1$

Definition (Dimensionless interface operator)

$$\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]: \quad \Psi_1 \quad \mapsto \quad \mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\Psi_1 = \nabla\Psi_2.$$

The interface operator $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1$

Definition (Dimensionless interface operator)

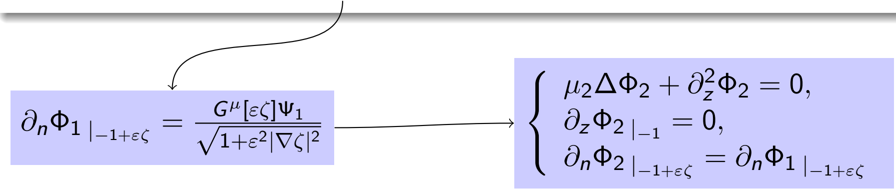
$$\mathbf{H}^{\mu,\delta}[\varepsilon\zeta] : \quad \Psi_1 \quad \mapsto \quad \mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\Psi_1 = \nabla\Psi_2.$$

$$\partial_n\Phi_1\,_{|_{-1+\varepsilon\zeta}} = \frac{G^{\mu}[\varepsilon\zeta]\Psi_1}{\sqrt{1+\varepsilon^2|\nabla\zeta|^2}}$$

The interface operator $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1$

Definition (Dimensionless interface operator)

$$\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]: \quad \Psi_1 \quad \mapsto \quad \mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\Psi_1 = \nabla\Psi_2.$$



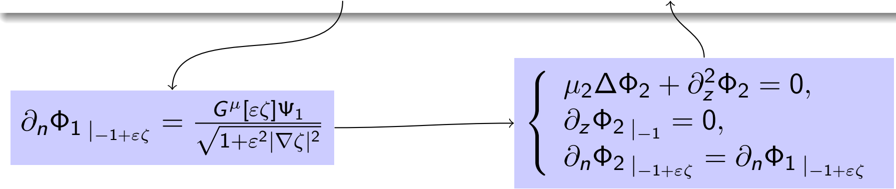
$$\partial_n \Phi_1\,_{|_{-1+\varepsilon\zeta}} = \frac{G^{\mu}[\varepsilon\zeta]\Psi_1}{\sqrt{1+\varepsilon^2|\nabla\zeta|^2}}$$

$$\begin{cases} \mu_2 \Delta\Phi_2 + \partial_z^2\Phi_2 = 0, \\ \partial_z\Phi_2\,_{|_{-1}} = 0, \\ \partial_n\Phi_2\,_{|_{-1+\varepsilon\zeta}} = \partial_n\Phi_1\,_{|_{-1+\varepsilon\zeta}} \end{cases}$$

The interface operator $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1$

Definition (Dimensionless interface operator)

$$\mathbf{H}^{\mu,\delta}[\varepsilon\zeta] : \quad \Psi_1 \quad \mapsto \quad \mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\Psi_1 = \nabla\Psi_2 .$$



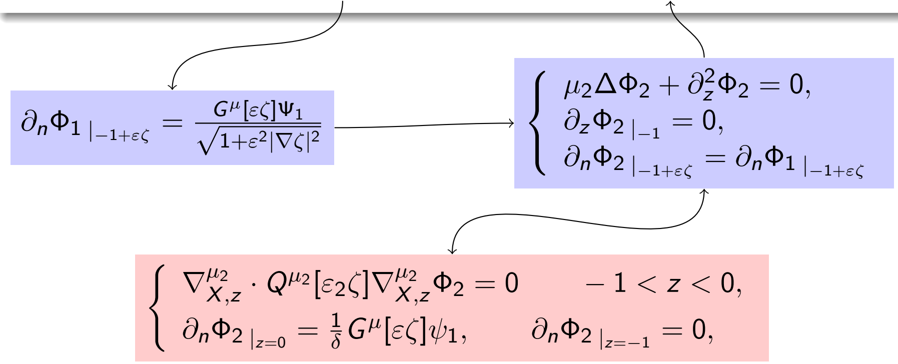
$$\partial_n\Phi_1\,|_{-1+\varepsilon\zeta} = \frac{G^\mu[\varepsilon\zeta]\Psi_1}{\sqrt{1+\varepsilon^2|\nabla\zeta|^2}}$$

$$\left\{\begin{array}{l} \mu_2\Delta\Phi_2 + \partial_z^2\Phi_2 = 0, \\ \partial_z\Phi_2\,|_{-1} = 0, \\ \partial_n\Phi_2\,|_{-1+\varepsilon\zeta} = \partial_n\Phi_1\,|_{-1+\varepsilon\zeta} \end{array}\right.$$

The interface operator $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1$

Definition (Dimensionless interface operator)

$$\mathbf{H}^{\mu,\delta}[\varepsilon\zeta] : \quad \Psi_1 \quad \mapsto \quad \mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\Psi_1 = \nabla\Psi_2.$$



$$\partial_n\Phi_1{}_{|_{-1+\varepsilon\zeta}} = \frac{G^\mu[\varepsilon\zeta]\Psi_1}{\sqrt{1+\varepsilon^2|\nabla\zeta|^2}}$$

$$\begin{cases} \mu_2\Delta\Phi_2 + \partial_z^2\Phi_2 = 0, \\ \partial_z\Phi_2{}_{|_{-1}} = 0, \\ \partial_n\Phi_2{}_{|_{-1+\varepsilon\zeta}} = \partial_n\Phi_1{}_{|_{-1+\varepsilon\zeta}} \end{cases}$$

$$\begin{cases} \nabla^{\mu_2}_{X,z} \cdot Q^{\mu_2}[\varepsilon_2\zeta]\nabla^{\mu_2}_{X,z}\Phi_2 = 0 \qquad -1 < z < 0, \\ \partial_n\Phi_2{}_{|_{z=0}} = \frac{1}{\delta}G^\mu[\varepsilon\zeta]\psi_1, \qquad \partial_n\Phi_2{}_{|_{z=-1}} = 0, \end{cases}$$

where $\nabla^{\mu_2} = (\sqrt{\mu_2}\nabla, \partial_z)^T$.

Control of the interface operator

$$\begin{cases} \nabla^{\mu_2}_{X,z}\cdot Q^{\mu_2}[\varepsilon_2\zeta]\nabla^{\mu_2}_{X,z}u = \nabla^{\mu_2}_{X,z}\cdot \mathbf{h} & -1<z<0,\\ \partial_n u_{|_{z=0}} = \sqrt{\mu_2}\nabla\cdot V + \mathbf{e_z}\cdot\mathbf{h}_{|_{z=0}}, & \partial_n u_{|_{z=-1}} = \mathbf{e_z}\cdot \mathbf{h}_{|_{z=-1}}, \end{cases}$$

$$\boxed{\left|\nabla u_{|_{z=0}}\right|_{☠} \le \frac{1}{\sqrt{\mu_2}}\mathrm{Cst}\,\left(\|\mathbf{h}\|_{☠}+|V|_{☠}\right).}$$

Corollary

Estimate on the operator norm of $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]$ *and stability to perturbations.*

Control of the interface operator

$$\begin{cases} \nabla^{\mu_2}_{X,z} \cdot Q^{\mu_2}[\varepsilon_2\zeta]\nabla^{\mu_2}_{X,z} u = \nabla^{\mu_2}_{X,z} \cdot \mathbf{h} \qquad -1 < z < 0, \\ \partial_n u_{|_{z=0}} = \sqrt{\mu_2}\nabla \cdot V + \mathbf{e_z} \cdot \mathbf{h}_{|_{z=0}}, \qquad \partial_n u_{|_{z=-1}} = \mathbf{e_z} \cdot \mathbf{h}_{|_{z=-1}}, \end{cases}$$

$$\big|\nabla u_{|_{z=0}}\big|_{\text{☠}} \leq \frac{1}{\sqrt{\mu_2}}\mathrm{Cst}\,\big(\|\mathbf{h}\|_{\text{☠}} + |V|_{\text{☠}}\big).$$

Corollary

Estimate on the operator norm of $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]$ *and stability to perturbations.*

Proof: Take $G^{\mu}[\varepsilon\zeta]\psi = \sqrt{\mu}\nabla \cdot V_{DN}$.

Asymptotic expansion of $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]$: the strategy

Recall that $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\Psi_1 = \nabla u_{|_{z=0}}$ where

$$\begin{cases} \nabla^{\mu_2}_{X,z}\cdot Q^{\mu_2}[\varepsilon_2\zeta]\nabla^{\mu_2}_{X,z}u = 0 & -1<z<0, \\ \partial_n u_{\,|_{z=0}} = \frac{1}{\delta}G^{\mu}[\varepsilon\zeta]\psi_1 = \sqrt{\mu_2}\nabla\cdot V_{DN}, & \partial_n u_{\,|_{z=-1}} = 0, \end{cases}$$

Asymptotic expansion of $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]$: the strategy

Recall that $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\Psi_1 = \nabla u_{|_{z=0}}$ where

$$\begin{cases} \nabla^{\mu_2}_{X,z} \cdot Q^{\mu_2}[\varepsilon_2\zeta]\nabla^{\mu_2}_{X,z} u = 0 & -1 < z < 0, \\ \partial_n u_{\,|_{z=0}} = \frac{1}{\delta} G^{\mu}[\varepsilon\zeta]\psi_1 = \sqrt{\mu_2}\nabla \cdot V_{DN}, & \partial_n u_{\,|_{z=-1}} = 0, \end{cases}$$

- Choose small parameter α (e.g. μ, ε, etc.)

Asymptotic expansion of $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]$: the strategy

Recall that $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\Psi_1 = \nabla u_{|_{z=0}}$ where

$$\begin{cases} \nabla^{\mu_2}_{X,z}\cdot Q^{\mu_2}[\varepsilon_2\zeta]\nabla^{\mu_2}_{X,z} u = 0 \qquad -1 < z < 0, \\ \partial_n u_{\,|_{z=0}} = \frac{1}{\delta} G^{\mu}[\varepsilon\zeta]\psi_1 = \sqrt{\mu_2}\nabla\cdot V_{DN}, \qquad \partial_n u_{\,|_{z=-1}} = 0, \end{cases}$$

- Choose small parameter α (e.g. μ, ε, etc.)
- Expand the DN operator: $V_{DN} = V_0 + \alpha V_1 + \alpha^2 V_2 + \dots$

Asymptotic expansion of $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]$: the strategy

Recall that $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\Psi_1 = \nabla u_{|_{z=0}}$ where

$$\begin{cases} \nabla^{\mu_2}_{X,z}\cdot Q^{\mu_2}[\varepsilon_2\zeta]\nabla^{\mu_2}_{X,z} u = 0 & -1 < z < 0, \\ \partial_n u_{\,|_{z=0}} = \frac{1}{\delta} G^{\mu}[\varepsilon\zeta]\psi_1 = \sqrt{\mu_2}\nabla\cdot V_{DN}, & \partial_n u_{\,|_{z=-1}} = 0, \end{cases}$$

- Choose small parameter α (e.g. μ, ε, etc.)
- Expand the DN operator: $V_{DN} = V_0 + \alpha V_1 + \alpha^2 V_2 + \dots$
- Construct u_{app} such that

$$\begin{cases} \nabla^{\mu_2}_{X,z}\cdot Q^{\mu_2}[\varepsilon_2\zeta]\nabla^{\mu_2}_{X,z} u_{app} = \alpha^n \nabla^{\mu_2}_{X,z}\cdot \mathbf{r} & -1 < z < 0, \\ \partial_n u_{app\,|_{z=0}} = \sqrt{\mu_2}\nabla\cdot (V_0 + \dots + \alpha^{n-1} V_{n-1}) + \alpha^n \mathbf{e_z}\cdot \mathbf{r}_{|_{z=0}}, \\ \partial_n u_{app\,|_{z=-1}} = \alpha^n \mathbf{e_z}\cdot\mathbf{r}_{|_{z=-1}}, \end{cases}$$

Asymptotic expansion of $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]$: the strategy

Recall that $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\Psi_1 = \nabla u_{|_{z=0}}$ where

$$\begin{cases} \nabla^{\mu_2}_{X,z} \cdot Q^{\mu_2}[\varepsilon_2\zeta]\nabla^{\mu_2}_{X,z} u = 0 & -1 < z < 0, \\ \partial_n u_{\,|_{z=0}} = \frac{1}{\delta} G^{\mu}[\varepsilon\zeta]\psi_1 = \sqrt{\mu_2}\nabla \cdot V_{DN}, & \partial_n u_{\,|_{z=-1}} = 0, \end{cases}$$

- Choose small parameter α (e.g. μ, ε, etc.)
- Expand the DN operator: $V_{DN} = V_0 + \alpha V_1 + \alpha^2 V_2 + \dots$
- Construct u_{app} such that
$$\begin{cases} \nabla^{\mu_2}_{X,z} \cdot Q^{\mu_2}[\varepsilon_2\zeta]\nabla^{\mu_2}_{X,z} u_{app} = \alpha^n \nabla^{\mu_2}_{X,z} \cdot \mathbf{r} & -1 < z < 0, \\ \partial_n u_{app\,|_{z=0}} = \sqrt{\mu_2}\nabla \cdot (V_0 + \dots + \alpha^{n-1} V_{n-1}) + \alpha^n \mathbf{e_z} \cdot \mathbf{r}_{|_{z=0}}, & \\ \partial_n u_{app\,|_{z=-1}} = \alpha^n \mathbf{e_z} \cdot \mathbf{r}_{|_{z=-1}}, & \end{cases}$$
- Use proposition with $V = V_{DN} - (V_0 + \dots + \alpha^{n-1} V_{n-1})$ and $\mathbf{h} = \alpha^n \mathbf{r}$.

Expansion of $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]$ in the SA/SA case: $\varepsilon_1 \ll 1$, $\varepsilon_2 \ll 1$

Expansion of $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]$ in the SA/SA case: $\varepsilon_1 \ll 1$, $\varepsilon_2 \ll 1$

- Defining $\mathcal{T}_{0,\mu} = \frac{\tanh(\sqrt{\mu}|D|)}{|D|}$ and $\mathcal{T}_{1,\mu}[\zeta] = -\nabla\mathcal{T}_{0,\mu}(\zeta\mathcal{T}_{0,\mu}\nabla^T)$, one has

$$V_{DN} = \mathcal{T}_{0,\mu}\nabla\psi_1 + \varepsilon_2\sqrt{\mu_2}(-\zeta + \mathcal{T}_{1,\mu}[\zeta])\nabla\psi_1 + O(\frac{1}{\delta^2}\varepsilon_2^2\sqrt{\mu_2}),$$

Expansion of $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]$ in the SA/SA case: $\varepsilon_1 \ll 1$, $\varepsilon_2 \ll 1$

- Defining $\mathcal{T}_{0,\mu} = \frac{\tanh(\sqrt{\mu}|D|)}{|D|}$ and $\mathcal{T}_{1,\mu}[\zeta] = -\nabla\mathcal{T}_{0,\mu}(\zeta\mathcal{T}_{0,\mu}\nabla^T)$, one has

$$V_{DN} = \mathcal{T}_{0,\mu}\nabla\psi_1 + \varepsilon_2\sqrt{\mu_2}(-\zeta + \mathcal{T}_{1,\mu}[\zeta])\nabla\psi_1 + O(\tfrac{1}{\delta^2}\varepsilon_2^2\sqrt{\mu_2}),$$

- We look for $u_{app} = u_0 + \varepsilon_2 u_1$:

$$\begin{cases} \Delta^{\mu_2}_{X,z} u_0 = 0, \\ \partial_z u_0\,_{|z=0} = \sqrt{\mu_2}\nabla\cdot(\mathcal{T}_{0,\mu}\nabla\psi_1), \qquad \partial_z u_0\,_{|z=-1} = 0, \end{cases}$$

$\Longrightarrow$ $u_0(X,z) =$ ☠☠☠

and u_1 solves a nonhomogeneous Laplace equation.

Proposition (Small Amplitude/Small Amplitude expansion)

$$\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1 = -\frac{\tanh(\sqrt{\mu}|D|)}{\tanh(\sqrt{\mu_2}|D|)}\nabla\psi_1 + \varepsilon_2 B(\zeta,\nabla\psi_1) + O(\frac{\varepsilon_2^2+\varepsilon^2}{\sqrt{\mu_2}}).$$

Expansion of $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]$ in the sw/sw case: $\mu_1 \ll 1$, $\mu_2 \ll 1$

We want to construct an approximate solution to

$$\begin{cases} \nabla^{\mu_2}_{X,z} \cdot Q^{\mu_2}[\varepsilon_2\zeta]\nabla^{\mu_2}_{X,z} u = 0 \qquad -1 < z < 0, \\ \partial_n u\,_{|_{z=0}} = \frac{1}{\delta} G^{\mu}[\varepsilon\zeta]\psi_1, \qquad \partial_n u\,_{|_{z=-1}} = 0. \end{cases}$$

Expansion of $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]$ in the sw/sw case: $\mu_1 \ll 1$, $\mu_2 \ll 1$

We want to construct an approximate solution to

$$\begin{cases} \nabla^{\mu_2}_{X,z} \cdot Q^{\mu_2}[\varepsilon_2\zeta]\nabla^{\mu_2}_{X,z} u = 0 & -1 < z < 0, \\ \partial_n u_{\,|z=0} = \frac{1}{\delta} G^{\mu}[\varepsilon\zeta]\psi_1, & \partial_n u_{\,|z=-1} = 0. \end{cases}$$

- $\frac{1}{\delta} G^{\mu}[\varepsilon\zeta]\psi_1 = \delta\mu_2 \nabla \cdot (h_1 \nabla \psi_1) + O(\frac{\mu^2}{\delta})$

Expansion of $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]$ in the sw/sw case: $\mu_1 \ll 1$, $\mu_2 \ll 1$

We want to construct an approximate solution to

$$\begin{cases} \nabla^{\mu_2}_{X,z} \cdot Q^{\mu_2}[\varepsilon_2\zeta]\nabla^{\mu_2}_{X,z} u = 0 & -1 < z < 0, \\ \partial_n u_{\,|_{z=0}} = \frac{1}{\delta} G^{\mu}[\varepsilon\zeta]\psi_1, & \partial_n u_{\,|_{z=-1}} = 0. \end{cases}$$

- $\frac{1}{\delta} G^{\mu}[\varepsilon\zeta]\psi_1 = \delta\mu_2 \nabla\cdot(h_1\nabla\psi_1) + O(\frac{\mu^2}{\delta})$
- We seek $u = u_0 + \mu_2 u_1$ and remark that

$$\nabla^{\mu_2}_{X,z} \cdot Q^{\mu_2}[\varepsilon_2\zeta]\nabla^{\mu_2}_{X,z} = \frac{1}{h_2}\partial_z^2 + \mu_2 \nabla_{X,z}\cdot Q_1\nabla_{X,z}, \qquad (h_2 = 1 + \varepsilon_2\zeta)$$

Expansion of $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]$ in the sw/sw case: $\mu_1 \ll 1$, $\mu_2 \ll 1$

We want to construct an approximate solution to

$$\begin{cases} \nabla^{\mu_2}_{X,z} \cdot Q^{\mu_2}[\varepsilon_2\zeta]\nabla^{\mu_2}_{X,z} u = 0 & -1 < z < 0, \\ \partial_n u_{\,|z=0} = \frac{1}{\delta} G^{\mu}[\varepsilon\zeta]\psi_1, & \partial_n u_{\,|z=-1} = 0. \end{cases}$$

- $\frac{1}{\delta} G^{\mu}[\varepsilon\zeta]\psi_1 = \delta\mu_2 \nabla \cdot (h_1 \nabla\psi_1) + O(\frac{\mu^2}{\delta})$
- We seek $u = u_0 + \mu_2 u_1$ and remark that

$$\nabla^{\mu_2}_{X,z} \cdot Q^{\mu_2}[\varepsilon_2\zeta]\nabla^{\mu_2}_{X,z} = \frac{1}{h_2}\partial_z^2 + \mu_2 \nabla_{X,z} \cdot Q_1 \nabla_{X,z}, \qquad (h_2 = 1 + \varepsilon_2\zeta)$$

- At first order

$$\begin{cases} \partial_z^2 u_0 = 0, \\ \partial_z u_{0\,|z=0} = 0, \qquad \partial_z u_{0\,|z=-1} = 0, \end{cases}$$

which is obviously solved by any $u_0(X,z) = u_0(X)$.

Expansion of $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]$ in the sw/sw case: $\mu_1 \ll 1$, $\mu_2 \ll 1$

- At next order

$$\begin{cases} \partial_z^2 u_1 = -h_2^2 \Delta u_0, \\ \partial_z u_1{}_{|_{z=0}} = h_2\big(\varepsilon_2 \nabla\zeta \cdot \nabla u_0 + \delta \nabla \cdot (h_1 \nabla \psi_1)\big), \qquad \partial_z u_1{}_{|_{z=-1}} = 0, \end{cases}$$

and thus

$$u_1 = -\frac{z^2}{2} h_2^2 \Delta u_0 + z(\partial_z u_1{}_{|_{z=0}})$$

if the following compatibility condition is satisfied

$$\boxed{\nabla \cdot (h_2 \nabla u_0) = -\delta \nabla \cdot (h_1 \nabla \psi_1),}$$

with $h_1 = 1 - \varepsilon_1 \zeta$ and $h_2 = 1 + \varepsilon_2 \zeta$.

Expansion of $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]$ in the sw/sw case: $\mu_1 \ll 1$, $\mu_2 \ll 1$

The compatibility condition

$$\nabla\cdot(h_2\nabla u_0) = -\delta\nabla\cdot(h_1\nabla\psi_1).$$

Lemma

Assume that $|\varepsilon_2\zeta|_\infty < 1$.

- *In dimension* $d=1$, *one has* $\partial_x u_0 = -\delta\frac{h_1}{h_2}\partial_x\psi_1$.
- *In dimension* $d=2$, *one has*

$$\nabla u_0 = -\delta(I+\Pi(\varepsilon_2\zeta\Pi\cdot))^{-1}\Pi(h_1\nabla\psi_1),$$

  where $\Pi = \frac{\nabla\nabla^T}{\Delta}$.

Corollary

$$\mathbf{H}^{\mu,\delta}[\varepsilon\zeta] = -\delta(I+\Pi(\varepsilon_2\zeta\Pi\cdot))^{-1}\Pi(h_1\nabla\psi_1) + O(\delta(\mu+\mu_2)).$$

Consistency results

Dimensionless equations ($\delta = \frac{d_1}{d_2}$)

$$\begin{cases} \partial_t \zeta - \frac{1}{\mu} G^{\mu}[\varepsilon\zeta]\psi_1 & = & 0, \\ \partial_t \big(\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1 - \gamma\nabla\psi_1 \big) + (1-\gamma)\nabla\zeta & & \\ \qquad + \frac{\varepsilon}{2}\nabla\big(|\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1|^2 - \gamma|\nabla\psi_1|^2 \big) + \varepsilon\nabla\mathcal{N}^{\mu,\delta}(\varepsilon\zeta,\psi_1) & = & 0. \end{cases}$$

Consistency results

Dimensionless equations ($\delta = \frac{d_1}{d_2}$)

$$\begin{cases} \partial_t \zeta - \frac{1}{\mu} G^{\mu}[\varepsilon\zeta]\psi_1 & = & 0, \\ \partial_t \big(\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1 - \gamma \nabla \psi_1 \big) + (1-\gamma)\nabla\zeta & & \\ \qquad + \frac{\varepsilon}{2} \nabla \big(|\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1|^2 - \gamma |\nabla\psi_1|^2 \big) + \varepsilon \nabla \mathcal{N}^{\mu,\delta}(\varepsilon\zeta, \psi_1) & = & 0. \end{cases}$$

- Rewrite the equations in terms of ζ and $\mathbf{v}$ where $\mathbf{v} = \mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1 - \gamma\nabla\psi_1$.

Consistency results

Dimensionless equations ($\delta = \frac{d_1}{d_2}$)

$$\begin{cases} \partial_t \zeta - \frac{1}{\mu} G^\mu[\varepsilon\zeta]\psi_1 & = & 0, \\ \partial_t \big(\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1 - \gamma\nabla\psi_1 \big) + (1-\gamma)\nabla\zeta & & \\ \qquad + \frac{\varepsilon}{2}\nabla\big(|\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1|^2 - \gamma|\nabla\psi_1|^2 \big) + \varepsilon\nabla\mathcal{N}^{\mu,\delta}(\varepsilon\zeta,\psi_1) & = & 0. \end{cases}$$

- Rewrite the equations in terms of ζ and $\mathbf{v}$ where $\mathbf{v} = \mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1 - \gamma\nabla\psi_1$.
- Expand $\nabla\psi_1$ and $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1$ in terms of ζ and $\mathbf{v}$.

Consistency results

Dimensionless equations ($\delta = \frac{d_1}{d_2}$)

$$\begin{cases} \partial_t \zeta - \frac{1}{\mu} G^{\mu}[\varepsilon\zeta]\psi_1 & = & 0, \\ \partial_t \big(\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1 - \gamma \nabla\psi_1 \big) + (1-\gamma)\nabla\zeta & & \\ \qquad + \frac{\varepsilon}{2}\nabla\big(|\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1|^2 - \gamma|\nabla\psi_1|^2 \big) + \varepsilon\nabla\mathcal{N}^{\mu,\delta}(\varepsilon\zeta,\psi_1) & = & 0. \end{cases}$$

- Rewrite the equations in terms of ζ and $\mathbf{v}$ where $\mathbf{v} = \mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1 - \gamma\nabla\psi_1$.
- Expand $\nabla\psi_1$ and $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1$ in terms of ζ and $\mathbf{v}$.
- Control the residual

Consistency results

Dimensionless equations ($\delta = \frac{d_1}{d_2}$)

$$\begin{cases} \partial_t \zeta - \frac{1}{\mu} G^\mu[\varepsilon\zeta]\psi_1 & = \quad 0, \\ \partial_t \big(\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1 - \gamma \nabla \psi_1 \big) + (1-\gamma)\nabla\zeta & \\ \qquad + \frac{\varepsilon}{2} \nabla \big(|\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1|^2 - \gamma |\nabla\psi_1|^2 \big) + \varepsilon \nabla \mathcal{N}^{\mu,\delta}(\varepsilon\zeta, \psi_1) & = \quad 0. \end{cases}$$

- Rewrite the equations in terms of ζ and $\mathbf{v}$ where $\mathbf{v} = \mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1 - \gamma\nabla\psi_1$.
- Expand $\nabla\psi_1$ and $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1$ in terms of ζ and $\mathbf{v}$.
- Control the residual
- Do this for all the possible regimes!

The SW/SW model

Theorem

The internal waves equations are consistent with the SW/SW equations

$$\begin{cases} \partial_t \zeta + \nabla \cdot \big(h_1 \mathfrak{R}[\zeta]\mathbf{v} \big) = 0, \\ \partial_t \mathbf{v} + (1-\gamma)\nabla\zeta + \dfrac{\varepsilon_1}{2}\nabla\Big(\big|\mathbf{v} - \gamma\mathfrak{R}[\zeta]\mathbf{v}\big|^2 - \gamma\big|\mathfrak{R}[\zeta]\mathbf{v}\big|^2 \Big) = 0, \end{cases}$$

where $h_1 = 1 - \varepsilon_1\zeta$, $h_2 = 1 + \varepsilon_2\zeta$ *and*

$$\mathfrak{R}[\zeta]\mathbf{u} = \text{☠}$$

The SW/SW model

Theorem

The internal waves equations are consistent with the SW/SW equations

$$\begin{cases} \partial_t \zeta + \nabla \cdot \big(h_1 \mathfrak{R}[\zeta]\mathbf{v} \big) = 0, \\ \partial_t \mathbf{v} + (1-\gamma)\nabla\zeta + \frac{\varepsilon_1}{2}\nabla\Big(\big|\mathbf{v} - \gamma\mathfrak{R}[\zeta]\mathbf{v}\big|^2 - \gamma\big|\mathfrak{R}[\zeta]\mathbf{v}\big|^2 \Big) = 0, \end{cases}$$

where $h_1 = 1 - \varepsilon_1\zeta$, $h_2 = 1 + \varepsilon_2\zeta$ *and*

$$\mathfrak{R}[\zeta]\mathbf{u} = ☠$$

The water-waves case: $\gamma = 0$, $\delta = 1$.

☠ $\rightsquigarrow$ ☺!!!

$$\begin{cases} \partial_t \zeta + \nabla \cdot \big(h_2 \mathbf{v} \big) = 0, \\ \partial_t \mathbf{v} + \nabla\zeta + \frac{\varepsilon_1}{2}\nabla|\mathbf{v}|^2 = 0, \end{cases}$$

The SW/SW model

Theorem

The internal waves equations are consistent with the SW/SW equations

$$\begin{cases} \partial_t \zeta + \nabla \cdot \big(h_1 \mathfrak{R}[\zeta]\mathbf{v}\big) = 0, \\ \partial_t \mathbf{v} + (1-\gamma)\nabla\zeta + \dfrac{\varepsilon_1}{2}\nabla\Big(\big|\mathbf{v} - \gamma\mathfrak{R}[\zeta]\mathbf{v}\big|^2 - \gamma\big|\mathfrak{R}[\zeta]\mathbf{v}\big|^2\Big) = 0, \end{cases}$$

where $h_1 = 1 - \varepsilon_1\zeta$, $h_2 = 1 + \varepsilon_2\zeta$ *and*

$$\mathfrak{R}[\zeta]\mathbf{u} = \text{☠}$$

The case $d = 1$:

☠ $\rightsquigarrow$ ☺!!!

$$\begin{cases} \partial_t \zeta + \partial_x\big(\frac{h_1 h_2}{\delta h_1 + \gamma h_2}\mathbf{v}\big) = 0, \\ \partial_t \mathbf{v} + (1-\gamma)\partial_x\zeta + \frac{\varepsilon}{2}\partial_x\big(\frac{(\delta h_1)^2 - \gamma h_2^2}{(\delta h_1 + \gamma h_2)^2}\mathbf{v}^2\big) = 0, \end{cases}$$

The SW/SW model

Theorem

The internal waves equations are consistent with the SW/SW equations

$$\begin{cases} \partial_t \zeta + \nabla \cdot \big(h_1 \mathfrak{R}[\zeta]\mathbf{v} \big) = 0, \\ \partial_t \mathbf{v} + (1-\gamma)\nabla\zeta + \dfrac{\varepsilon_1}{2}\nabla\Big(\big|\mathbf{v} - \gamma \mathfrak{R}[\zeta]\mathbf{v}\big|^2 - \gamma \big|\mathfrak{R}[\zeta]\mathbf{v}\big|^2 \Big) = 0, \end{cases}$$

where $h_1 = 1 - \varepsilon_1 \zeta$, $h_2 = 1 + \varepsilon_2 \zeta$ *and*

$$\mathfrak{R}[\zeta]\mathbf{u} = \text{☠}$$

Other remarks

- Full justification of the SW/SW regime for existing internal-waves
- Kelvin-Helmoltz instability!!!

The Boussinesq/Boussinesq model

The internal wave equations are consistent with the B/B systems:

$$\begin{cases} \Big(1-\mu b\Delta\Big)\partial_t\zeta+\frac{1}{\gamma+\delta}\nabla\cdot\mathbf{v}_\beta+\varepsilon\frac{\delta^2-\gamma}{(\gamma+\delta)^2}\nabla\cdot(\zeta\mathbf{v}_\beta)+\mu a\nabla\cdot\Delta\mathbf{v}_\beta=0 \\ \Big(1-\mu d\Delta\Big)\partial_t\mathbf{v}_\beta+(1-\gamma)\nabla\zeta+\frac{\varepsilon}{2}\frac{\delta^2-\gamma}{(\delta+\gamma)^2}\nabla|\mathbf{v}_\beta|^2+\mu c\Delta\nabla\zeta=0, \end{cases}$$

where $\mathbf{v}_\beta=(1-\mu\beta\Delta)^{-1}\mathbf{v}$ and

$$a=\frac{(1-\alpha_1)(1+\gamma\delta)-3\delta\beta(\gamma+\delta)}{3\delta(\gamma+\delta)^2},\quad b=\alpha_1\tfrac{1+\gamma\delta}{3\delta(\gamma+\delta)},$$
$$c=\beta\alpha_2,\qquad d=\beta(1-\alpha_2),$$

($\alpha_1\geq 0$, $\beta\geq 0$ and $\alpha_2\leq 1$).

Remark: The dispersion relation is

$$\omega^2=|\mathbf{k}|^2\frac{(\frac{1}{\gamma+\delta}-\mu a|\mathbf{k}|^2)(1-\gamma-\mu c|\mathbf{k}|^2)}{(1+\mu b|\mathbf{k}|^2)(1+\mu d|\mathbf{k}|^2)}.$$