Non-symmetric periodic three-dimensional gravity water waves

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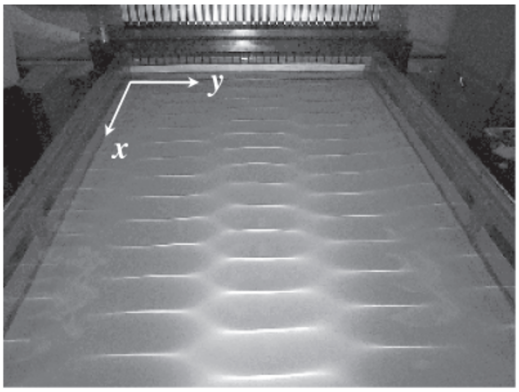
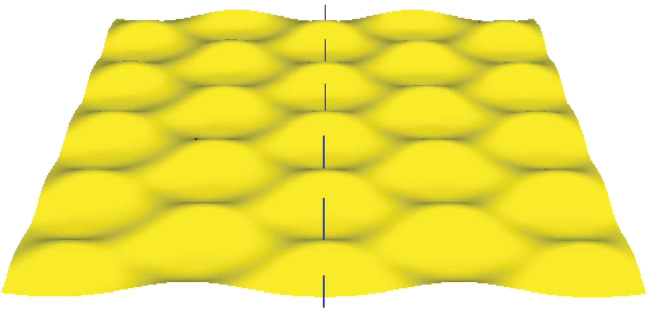
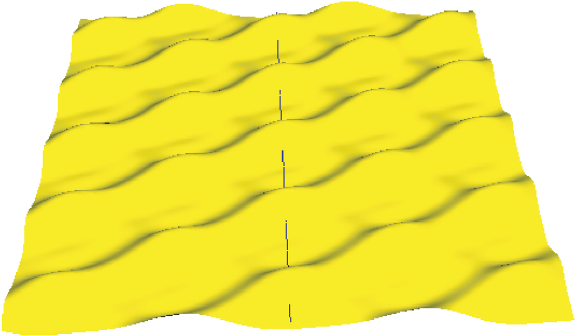
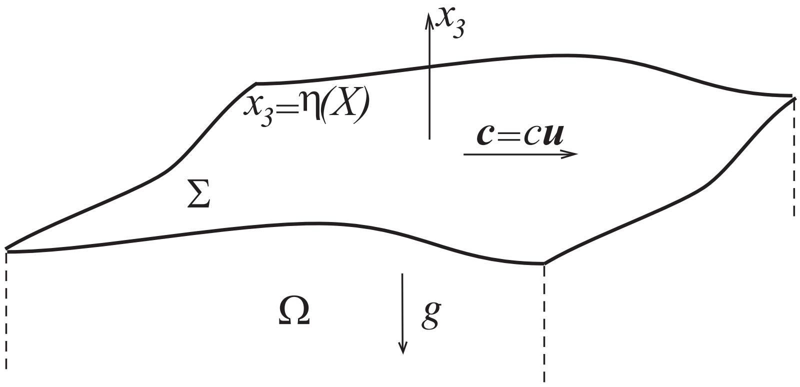


D.Henderson et al JFM 532, 2005







moving frame

$$\begin{array}{rcl} \Delta\varphi & = & 0 \qquad\qquad x_3 < \eta(X) \\ \nabla\varphi & \to & 0 \ \text{ as } x_3 \to -\infty \end{array}$$

Boundary conditions on $x_3 = \eta(X)$

$$\nabla_X \eta \cdot (\mathbf{u} + \nabla_X \varphi) - \frac{\partial \varphi}{\partial x_3} = 0$$

$$\mathbf{u} \cdot \nabla_X \varphi + \frac{(\nabla\varphi)^2}{2} + \mu\eta = 0$$

$$\mu = \frac{gL}{c^2}$$

Basic solution: (flat free surface)

$$\varphi = 0, \quad \eta = 0.$$

Bi-periodic functions

lattice Γ of periods: $\boldsymbol{\lambda}_j \in \mathbb{R}^2$, $j = 1, 2$

$$\Gamma = \{\boldsymbol{\lambda} = m_1\boldsymbol{\lambda}_1 + m_2\boldsymbol{\lambda}_2 : m_j \in \mathbb{Z}\}$$

lattice Γ′ of wave vectors: $K_j \in \mathbb{R}^2$, $j = 1, 2$

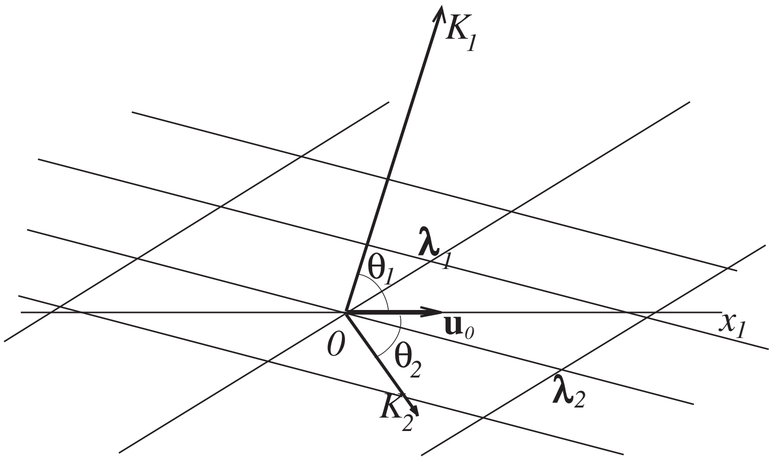
$$\Gamma' = \{K = n_1K_1 + n_2K_2 : n_j \in \mathbb{Z}, \quad \boldsymbol{\lambda}_j \cdot K_l = 2\pi\delta_{jl}\}$$

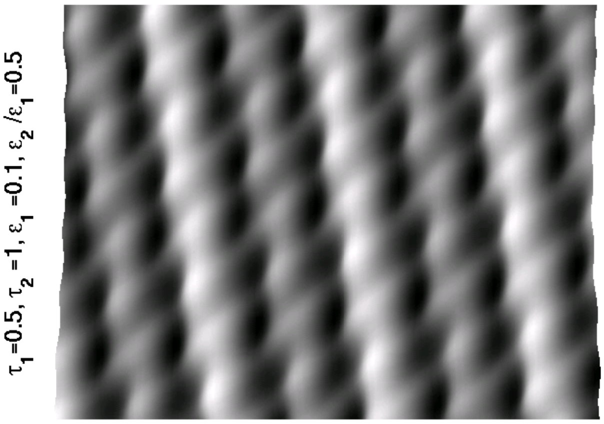
η bi-periodic in $X \in \mathbb{R}^2$

$$\eta(X) = \sum_{K \in \Gamma'} u_K e^{iK\cdot X}$$

Γ' generated by $K_1, K_2 \in \mathbb{R}^2$

Γ generated by $\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2 \in \mathbb{R}^2$





Basic formulation

Dirichlet-Neumann linear operator $\mathcal{G}_\eta$

$$\mathcal{G}_\eta \psi = \Big(1 + (\nabla\eta)^2\Big)^{1/2} \frac{d\varphi}{dn}|_{x_3=\eta(X)}$$

n normal exterior to Ω, and φ solution of the Dirichlet problem

$$\begin{array}{rcl} \Delta\varphi & = & 0, \quad x_3 < \eta(X) \\ \varphi & = & \psi, \quad x_3 = \eta(X) \\ \nabla\varphi & \rightarrow & 0 \text{ as } x_3 \rightarrow -\infty. \end{array}$$

Basic formulation: $\mu > 0, \mathbf{u} \in \mathbb{S}_1$

$$\mathcal{F}(U, \mu, \mathbf{u}) = 0, \quad \mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$$

$$U = (\psi, \eta), \quad \psi = \varphi(X, \eta(X))$$

$$\begin{array}{rcl} \mathcal{F}_1(U, \mu, \mathbf{u}) & = & : \mathcal{G}_\eta(\psi) - \mathbf{u}\cdot\nabla\eta, \\ \mathcal{F}_2(U, \mu, \mathbf{u}) & = & : \mathbf{u}\cdot\nabla\psi + \mu\eta + \dfrac{(\nabla\psi)^2}{2} \\ & & -\dfrac{1}{2(1 + (\nabla\eta)^2)}\{\nabla\eta\cdot(\nabla\psi + \mathbf{u})\}^2 \end{array}$$

$$U \in \mathbb{H}^m(\mathbb{R}^2/\Gamma) =: H_0^m(\mathbb{R}^2/\Gamma) \times H^m(\mathbb{R}^2/\Gamma)$$

Mathematical results on 3-dim travelling waves

First formal studies

R.Fuchs. U.S.Natl. Bur. Stand. Circ. 521 (1952), 187-200.

L.Sretenskii. Dokl. Akad. Nauk SSSR (N.S.) 89 (1953), 25-28.

Existence results (always with surface tension)

J.Reeder,M. Shinbrot. Nonlinear Anal., T.M.A., 5 (1981), 3, 303-323.

W.Craig, D.Nicholls. SIAM J. Math Anal. 32 (2000) 323-359.

M.Groves, A.Mielke. Proc. Roy. Soc. Edin. A 131 (2001), 83-136.

M.Groves, M.Haragus. J.Nonlinear Sci. (2003) 13, 397-447.

Existence result without surface tension (diamond waves)

G.Iooss, P.Plotnikov. Memoirs of A.M.S.(to appear)

Linearized system at the origin

$$\begin{aligned} \mathcal{G}^{(0)}\psi - \mathbf{u}\cdot\nabla\eta &= 0, \quad \mathcal{G}^{(0)} = (-\Delta)^{1/2} \\ \mathbf{u}\cdot\nabla\psi + \mu\eta &= 0 \end{aligned}$$

Dispersion relation:

$$\mu|K| - (K\cdot\mathbf{u})^2 = 0$$

Assume the following solution of the dispersion equation

$$\begin{aligned} \mathbf{u} &= \mathbf{u}_0 = (1,0), \\ K_1 &= (1,\tau_1), \quad \tau_1 = \tan\theta_1 \\ K_2 &= \lambda(1,-\tau_2), \quad \tau_2 = \tan\theta_2 \end{aligned}$$

Then τ_1, τ_2 and λ are linked:

$$\begin{aligned} \mu_c &= \cos\theta_1 = \frac{1}{|K_1|} = \frac{\lambda^2}{|K_2|} \\ \lambda &= \frac{\cos\theta_1}{\cos\theta_2} \end{aligned}$$

For $\lambda = 1$ the lattice Γ is a *diamond pattern*

Kernel of the linearized operator

linearized operator (acts on $U = (\psi, \eta)$)

$$\mathcal{L}_0 = \begin{pmatrix} \mathcal{G}^{(0)} & -\mathbf{u}_0 \cdot \nabla \\ \mathbf{u}_0 \cdot \nabla & \mu_c \end{pmatrix}$$

Non resonant assumption:

$\pm K_1$ and $\pm K_2$ are the only solutions in Γ of the dispersion equation for $\mathbf{u} = \mathbf{u}_0$, $\mu = \mu_c$:

$$\mu_c |K| - (K \cdot \mathbf{u}_0)^2 = 0$$

4-dim kernel:

$$\begin{aligned} \zeta_{K_1} &= (i, \frac{1}{\mu_c}) e^{iK_1 \cdot X}, \quad \zeta_{-K_1} = \bar{\zeta}_{K_1} \\ \zeta_{K_2} &= (i, \frac{\lambda}{\mu_c}) e^{iK_2 \cdot X}, \quad \zeta_{-K_2} = \bar{\zeta}_{K_2} \end{aligned}$$

Formal Asymptotic expansion

$$\begin{aligned} 0 &= \mathcal{L}_0 U + \widetilde{\mu}\mathcal{L}_1 U + \mathcal{L}_2(\boldsymbol{\omega}, U) + \\ & \quad + \mathcal{N}_2(U,U) + \mathcal{N}_3(U,U,U) + .. \end{aligned}$$

$$U = (\psi, \eta), \quad \widetilde{\mu} = \mu - \mu_c, \quad \boldsymbol{\omega} = \mathbf{u} - \mathbf{u}_0$$

Equivariance under symmetries:

$$\begin{aligned} \mathcal{T}_{\mathbf{v}} &: \quad \text{shift } X \mapsto X + \mathbf{v},\ \mathbf{v} \in \mathbb{R}^2 \\ \mathcal{S}_0 &: \quad \mathcal{S}_0(\psi, \eta)(X) = (-\psi, \eta)(-X) \end{aligned}$$

Formal Lyapunov-Schmidt method

$$\begin{aligned} U &= W + V, \quad W \in \ker \mathcal{L}_0 \\ W &= A\zeta_{K_1} + B\zeta_{K_2} + \bar{A}\bar{\zeta}_{K_1} + \bar{B}\bar{\zeta}_{K_2} \end{aligned}$$

1st step: solve formally w.r. to V

$$V = -\widetilde{\mathcal{L}}_0^{-1}\mathcal{N}_2(W,W) + O(||W||^3 + (|\widetilde{\mu}| + |\boldsymbol{\omega}|)||W||)$$

small divisor problem in the denominator of $\widetilde{\mathcal{L}}_0^{-1}$:

$$\mu_c|K| - (K \cdot \mathbf{u}_0)^2, \quad K \in \Gamma \backslash \{\pm K_1, \pm K_2\}$$

2nd step: obtain 2 complex "bifurcation equations"

$$\begin{aligned} A\phi_1(\widetilde{\mu}, \boldsymbol{\omega}, |A|^2, |B|^2) &= 0 \\ B\phi_2(\widetilde{\mu}, \boldsymbol{\omega}, |A|^2, |B|^2) &= 0 \end{aligned}$$

Asymptotic expansion of non-symmetric 3-dim waves

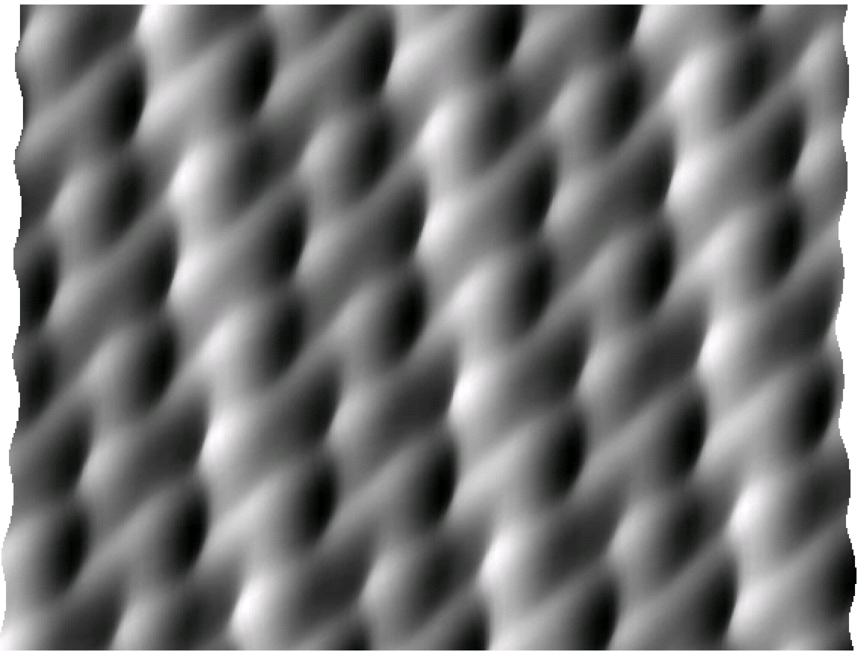
$$U = (\psi, \eta) = \sum_{p+q+r+s \geq 1} A^p \bar{A}^q B^r \bar{B}^s U_{pqrs}$$

$$\begin{aligned} A &= \varepsilon_1 e^{iK_1 \cdot \mathbf{v}}, \quad B = \varepsilon_2 e^{iK_2 \cdot \mathbf{v}} \\ \mathbf{v} \text{ arbitrary in } \mathbb{R}^2 &\sim \text{horizontal shift } X \mapsto X + \mathbf{v} \\ \mathbf{v} &= 0 \Rightarrow \psi \text{ odd, } \eta \text{ even in } X \end{aligned}$$

$$U_{1000} = \zeta_{K_1}, \; U_{0010} = \zeta_{K_2}, \; U_{pqrs} = \bar{U}_{qpsr}$$

$$\begin{aligned} \mu - \mu_c &= \alpha_1 \varepsilon_1^2 + \alpha_2 \varepsilon_2^2 + O(\varepsilon_1^2 + \varepsilon_2^2)^2 \\ \mathbf{u} - \mathbf{u}_0 &= \boldsymbol{\omega} = (\omega_1, \omega_2), \; \omega_1 = -\frac{\omega_2^2}{2} + .. \\ \omega_2 &= \beta_1 \varepsilon_1^2 + \beta_2 \varepsilon_2^2 + O(\varepsilon_1^2 + \varepsilon_2^2)^2 \end{aligned}$$

α_j, β_j known functions of τ_1 and τ_2



$\tau_1=0.5,\ \tau_2=1,\ \varepsilon_1=0.1,\ \varepsilon_2/\varepsilon_1=1$

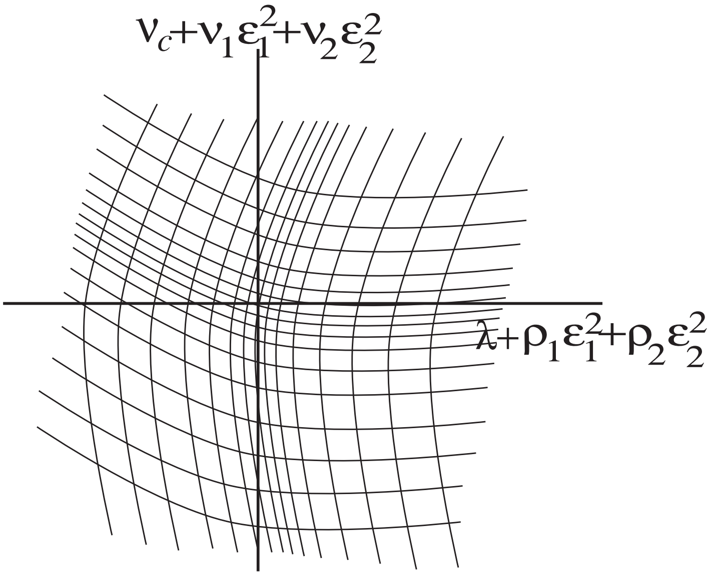
Existence region

Conditions on $(\tau_1, \tau_2, \varepsilon_1^2, \varepsilon_2^2)$: for $(m,n) \in \mathbb{Z}^2 \backslash \{0\}$

$$\begin{aligned} |m + \rho n| &\geq \frac{c}{(|m| + |n|)^{1+\delta}} \\ |\nu(m+\rho n)^2 - |n|| &\geq \frac{c}{(|m|+|n|)^{\delta+1/2}} \end{aligned}$$

$$\rho = \lambda + \rho_1 \varepsilon_1^2 + \rho_2 \varepsilon_2^2 + \mathcal{R}(\varepsilon)$$

$$\nu = \nu_c + \nu_1 \varepsilon_1^2 + \nu_2 \varepsilon_2^2 + \mathcal{V}(\varepsilon)$$



Existence problem

$$\mathcal{F}(U,\mu)=0$$

Approximate solution (with ψ odd, η even in X):

$$\begin{aligned} U_{\varepsilon_1,\varepsilon_2} &: \text{ truncated expan. at order } N. \\ \varepsilon &= |\varepsilon_1|+|\varepsilon_2| \\ \mathcal{F}(U_{\varepsilon_1,\varepsilon_2},\mu_{\varepsilon_1,\varepsilon_2}) &= \varepsilon^{N+1}Q_\varepsilon \end{aligned}$$

Perturbation:

$$U=U_{\varepsilon_1,\varepsilon_2}+\varepsilon^N W,\quad N\geq 3,$$

$$\Phi(W,\varepsilon_1,\varepsilon_2)\overset{def}{=}\varepsilon^{-N}\mathcal{F}(U_{\varepsilon_1,\varepsilon_2}+\varepsilon^N W,\mu_{\varepsilon_1,\varepsilon_2})$$

$$\Phi(W,\varepsilon_1,\varepsilon_2)=0,\quad \Phi(0,\varepsilon_1,\varepsilon_2)=\varepsilon Q_{\varepsilon_1,\varepsilon_2}$$

Adapt Nash-Moser theorem (based on Newton iteration method)

$$\begin{aligned} &\partial_W\Phi(W_p,\varepsilon_1,\varepsilon_2)\{W_{p+1}(\varepsilon_1,\varepsilon_2)-W_p(\varepsilon_1,\varepsilon_2)\} \\ = &-\Phi(W_p(\varepsilon_1,\varepsilon_2),\varepsilon_1,\varepsilon_2) \end{aligned}$$

$$W_0(\varepsilon_1,\varepsilon_2)=0;$$

$\partial_W\Phi(W,\varepsilon_1,\varepsilon_2)$ linear integro-differential operator of second order, periodic coefficients.

Inversion of the Differential of Φ

$U=(\psi,\eta)=U_{\varepsilon_1,\varepsilon_2}+\varepsilon^N W$ being given, $U_{\varepsilon_1,\varepsilon_2}$:truncated expan. at order N, (with ψ odd, η even in X), solve w.r. to $\delta\phi\in H^{s-l}_{odd}(\mathbb{R}^2/\Gamma)$ the linear equation

$$-\mathcal{J}^*\left(\frac{1}{\mathfrak{a}}\mathcal{J}(\delta\phi)\right)+\mathcal{G}_\eta(\delta\phi)=h\in H^s_{odd}(\mathbb{R}^2/\Gamma)$$

$$\begin{aligned}\mathcal{J} &= V\cdot\nabla(\cdot),\ \ V=\nabla\psi+\mathbf{u}_0-\mathfrak{b}\nabla\eta\\ \mathfrak{a} &= V\cdot\nabla\mathfrak{b}+\mu\\ \mathfrak{b} &= \frac{1}{1+\nabla\eta^2}\{\nabla\eta\cdot(\mathbf{u}_0+\nabla\psi)\}\end{aligned}$$

$$\mathcal{G}_\eta=\mathcal{G}_1+\mathcal{G}_0+\mathcal{G}_{-1}$$

$\mathcal{G}_j$ pseudodifferential operators of order j. For $(\psi,\eta)=0$, and $\mu=\mu_c$

$$\{\mu_c^{-1}(\partial_{x_1})^2+(-\Delta)^{1/2}\}(\delta\phi)=h$$

1st step: find a diffeomorphism of the torus $X \in \mathbb{R}^2/\Gamma \mapsto Y \in (\mathbb{R}/2\pi\mathbb{Z})^2$

$$Y = (K_1{\cdot}X, K_2{\cdot}X + u(X)) + (1,\rho)v(K_1{\cdot}X, K_2{\cdot}X + u(X))$$

such that

$$[\mathcal{G}_\eta\phi - \mathcal{J}^*(\frac{1}{\mathfrak{a}}\mathcal{J}\phi)](X) =$$

$$= \kappa(Y)(\mathbb{I} + \mathfrak{Q}_{-1})[\mathfrak{L}_{0,\varepsilon} + \mathfrak{A}_\varepsilon\mathcal{D}_1 + \mathfrak{B}_\varepsilon + \mathfrak{L}_{-1}]\check{\phi}(Y)$$

where $\phi(X(Y)) = \check{\phi}(Y)$

$$\begin{array}{rcl} \mathfrak{L}_{0,\varepsilon} & = & \nu\mathcal{D}_1^2 + (-\widetilde{\Delta})^{1/2},\ \mathcal{D}_1 = \partial_{y_1} + \rho\partial_{y_2} \\ L(k) & = & -\nu(k_1+\rho k_2)^2 + \\ & & \{a_0(k_1+\rho k_2)^2 + 2b_0(k_1+\rho k_2)k_2 + k_2^2\}^{1/2} \end{array}$$

$\mathfrak{A}_\varepsilon$ and $\mathfrak{B}_\varepsilon$ pseudodiff operators of order 0

$\mathfrak{L}_{-1}$ of order -1

- ρ : rotation number of the field V on $\mathbb{R}^2/\Gamma$

$$\begin{aligned} V \cdot \nabla_X u &= \rho K_1 \cdot V - K_2 \cdot V, \\ &\quad \text{where } u \ \Gamma- \text{ periodic} \end{aligned}$$

$$\rho = \lambda + \rho_1 \varepsilon_1^2 + \rho_2 \varepsilon_2^2 + h.o.t.$$

Identify powers of $\varepsilon_1, \varepsilon_2$ for obtaining the formal expansion of $(\rho, u(X))$

- Diophantine condition on ρ

$$|m + \rho n| \geq \frac{c}{(|m| + |n|)^{1+\delta}}, (m, n) \in \mathbb{Z}^2 \backslash \{0\}$$

$\Rightarrow$restriction on $\lambda + \rho_1 \varepsilon_1^2 + \rho_2 \varepsilon_2^2$ to a set of positive measure, asymptotically of full measure as $|\varepsilon| \to 0$

- Solve in $\nu \in \mathbb{R}$ and $v(Y)$ $2\pi-$ periodic and odd in (y_1, y_2), an equation of the form (linear small divisor problem)

$$\begin{aligned} 1 + \partial_{y_1} v + \rho \partial_{y_2} v &= \nu^{1/2} f \\ f &: \text{ known, even, periodic in } Y \end{aligned}$$

$$\nu = \nu_c + \nu_1 \varepsilon_1^2 + \nu_2 \varepsilon_2^2 + h.o.t.$$

2nd step: invert $\mathfrak{L}_{0,\varepsilon} - \varkappa \mathbb{I}$ where $\varkappa = O(|\varepsilon|^2)$

Diophantine condition for $(m,n) \in \mathbb{Z}^2 \backslash \{0\}$

$$|\nu(m+\rho n)^2 - \{\mathbb{T}(m,n)\}^{1/2} + \varkappa| \geq \frac{c|\varepsilon|^{2+\beta}}{(|m|+|n|)^{\delta+1/2}},$$

where

$$\mathbb{T}(m,n) = a_0(m+\rho n)^2 + 2b_0(m+\rho n)n + n^2$$

⇒restriction on $\nu_c + \nu_1\varepsilon_1^2 + \nu_2\varepsilon_2^2$ to a set of positive measure, asymptotically of full measure as $|\varepsilon| \to 0$

then

for $f \in H^{s+1}_{odd} \cap \{\ker \mathfrak{L}_{0,0}\}^{\perp}$

$$||(\mathfrak{L}_{0,\varepsilon} - \varkappa \mathbb{I})^{-1} f||_{H^s_{odd}} \leq \frac{c(s)}{|\varepsilon|^{2+\beta}} ||f||_{H^{s+1}_{odd}}$$

(loss of 1 degree of regularity)

Descent method

Find bounded operators $\mathfrak{C}$ and $\mathfrak{E}$ such that

$$(\mathfrak{L}_{0,\varepsilon}+\mathfrak{A}_\varepsilon\mathcal{D}_1+\mathfrak{B}_\varepsilon)(1+\mathfrak{C})u = (1+\mathfrak{E})(\mathfrak{L}_{0,\varepsilon}-\varkappa)u+\mathfrak{F}u$$

$\mathfrak{F}$ bounded operator of order -1, $\varkappa = O(\varepsilon_1^2+\varepsilon_2^2)$ scalar

General algebraic procedure, based on properties of pseudodifferential operators of 0 order - sort of "averaging".

Inverting

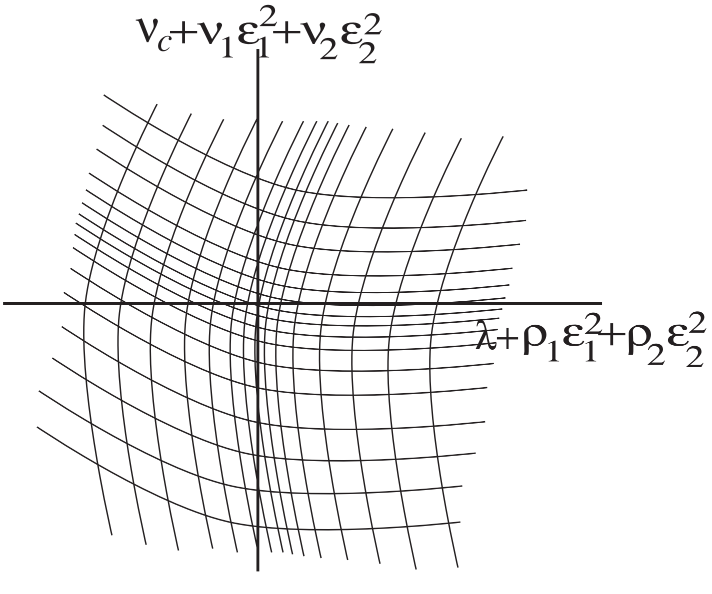
$$\mathfrak{L}_{0,\varepsilon}+\mathfrak{A}_\varepsilon\mathcal{D}_1+\mathfrak{B}_\varepsilon+\mathfrak{L}_{-1}$$

then reduces to invert

$$\mathfrak{L}_{0,\varepsilon}-\varkappa+\widetilde{\mathfrak{L}}_{-1}$$

$\widetilde{\mathfrak{L}}_{-1}$: smoothing reminder. Then we obtain

$$||(\mathfrak{L}_{0,\varepsilon}-\varkappa+\widetilde{\mathfrak{L}}_{-1})^{-1}||_{\mathcal{L}(H^{s+1}_{odd},H^{s}_{odd})} \le \frac{c}{\varepsilon_1^2\varepsilon_2^2|\varepsilon|^{2+\beta}}.$$


$\nu_c+\nu_1\varepsilon_1^2+\nu_2\varepsilon_2^2$
$\lambda+\rho_1\varepsilon_1^2+\rho_2\varepsilon_2^2$