Stability of periodic waves in dispersive models

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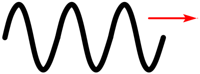
Periodic waves

- **Generalized KdV equation**

$$u_t + (u_{xx} + u^{p+1})_x = 0, \quad p \geq 1$$

  - *other dispersive models: Kawahara, BBM, NLS,...*

- **Travelling periodic waves:** $u(x,t) = q(x - ct)$



  - $q_{yy} = cq - q^{p+1} + b, \quad y = x - ct$
  - three parameter family (a, b, speed c)
  - scaling invariance $\rightsquigarrow$ $c = 1$
  - *KdV equation,* $p = 1$*: Galilean invariance* $\rightsquigarrow$ $b = 0$

Periodic waves

$$q_{a,b}(y) = P_{a,b}(k_{a,b}y)$$

with $P_{a,b}$ a 2π-periodic even solution of

$$k_{a,b}^2 v_{zz} - v + v^{p+1} = b$$

- **Small amplitude**

$$P_{a,b}(z) = Q_b + \cos(z)\, a - \frac{p+1}{4}\, a^2 + \frac{p+1}{12}\, \cos(2z)\, a^2 + O(|a|(a^2+b^2))$$

$$Q_b = 1 + \frac{1}{p}\, b - \frac{p+1}{2p^2}\, b^2 + O(|b|^3)$$

$$k_{a,b}^2 = p + (p+1)b - \frac{p(p+1)(p+4)}{12}\, a^2 - \frac{p+1}{p}\, b^2 + O(|a|^3 + |b|^3)$$

- **Question: stability ?**

Stability of periodic waves

- **periodic perturbations: orbital stability**

  [Angulo, Bona, & Scialom; Gallay & H.; Hakkaev, Iliev, & Kirchev; . . .]

- **localized/bounded perturbations: spectral stability**

  [H., Lombardi, & Scheel; Gallay & H.; Bottman & Deconinck, . . .]

Spectral stability

- **Linearized operator**

$$\mathcal{A}_{a,b}\mathbf{v} = -k_{a,b}^2\partial_{zzz}\mathbf{v} + \partial_z\mathbf{v} - (p+1)\partial_z(P_{a,b}^p\mathbf{v})$$

- **Spectrum in $L^2(\mathbb{R})$ or $C_b(\mathbb{R})$**

$$\sigma(\mathcal{A}_{a,b}) = \{\lambda \in \mathbb{C}\ ;\ \lambda - \mathcal{A}_{a,b} \text{ is not invertible }\}$$

- *The periodic wave is* **spectrally stable** *if*

$$\sigma(\mathcal{A}_{a,b}) = \{\lambda \in \mathbb{C}\ ;\ \mathrm{Re}\,\lambda \leq 0\}$$

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- **Question: locate the spectrum ?**
  - *first difficulty: continuous spectrum*

Bloch-wave decomposition

- *reduces the spectral problem for localized/bounded perturbations to the study of the spectra of an (infinite) family of operators with* **point spectra** [Reed & Simon; Scarpelini; Mielke; ...]
- **THEOREM**

$$\sigma_{L^2(\mathbb{R})}(\mathcal{A}_{a,b}) = \sigma_{C_b^0(\mathbb{R})}(\mathcal{A}_{a,b}) = \bigcup_{\gamma \in (-\frac{1}{2},\frac{1}{2}]} \sigma_{L^2(0,2\pi)}(\mathcal{A}_{a,b,\gamma})$$

where

$$\mathcal{A}_{a,b,\gamma} = -k_{a,b}^2(\partial_z + \mathrm{i}\gamma)^3 + (\partial_z + \mathrm{i}\gamma) - (p+1)(\partial_z + \mathrm{i}\gamma)(P_{a,b}^p \cdot)$$

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- **Question: locate the point spectra of $\mathcal{A}_{a,b,\gamma}$?**
  - *use the Hamiltonian structure*

Hamiltonian structure

- Bloch operators

$$\mathcal{A}_{a,b,\gamma} = \mathcal{J}_\gamma \mathcal{L}_{a,b,\gamma}$$

  where

$$\mathcal{J}_\gamma = \partial_z + \mathrm{i}\gamma, \quad \mathcal{L}_{a,b,\gamma} = -k_{a,b}^2(\partial_z + \mathrm{i}\gamma)^2 + 1 - (p+1)P_{a,b}^p$$

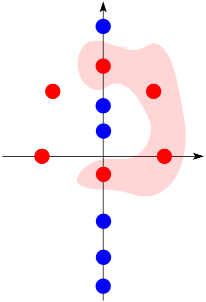
  - $\mathcal{J}_\gamma$ is **skew-adjoint** with compact resolvent, and invertible for $\gamma \neq 0$
  - $\mathcal{L}_{a,b,\gamma}$ is **self-adjoint** with compact resolvent, and invertible for a.a. γ
  - $\mathcal{L}_{a,b,\gamma}$ has a finite number of negative eigenvalues

- **Connection between the spectra of $\mathcal{A}_{a,b,\gamma}$ and $\mathcal{L}_{a,b,\gamma}$?**

Spectra of $\mathcal{A} := \mathcal{A}_{a,b,\gamma}$ and $\mathcal{L} := \mathcal{L}_{a,b,\gamma}$

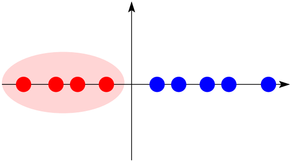
Spectrum of $\mathcal{A}$

– symmetric w.r.t. imaginary axis –



Spectrum of $\mathcal{L}$

– real –



$$\mathbf{n}(\mathcal{L}) := \#\{\lambda \in \sigma(\mathcal{L}) : \lambda < 0\}$$

$$\downarrow$$

$$k_u := \#\{\lambda \in \sigma(\mathcal{A}) : \mathrm{Re}\,\lambda > 0\}$$

$$k_i^- := \#\{\lambda \in \sigma(\mathcal{A}) : \mathrm{Re}\,\lambda = 0, \text{ with negative Krein signature}\}$$

$$\longrightarrow \quad \boxed{\mathbf{n}(\mathcal{L}) = k_u + k_i^-}$$

Krein signature: the sign of $\langle \mathcal{L}v, v\rangle$ for a simple eigenvalue with eigenvector v

Definition of k_i^-

- Take $\lambda \in \sigma(\mathcal{A})$ with $\operatorname{Re}\lambda = 0$, and consider the associated spectral subspace E_λ (finite-dimensional)
- Consider the Hermitian matrix $\mathbf{L}(\lambda)$ associated with the quadratic form $\langle \mathcal{L}|_{E_\lambda} \cdot, \cdot \rangle$ on E_λ
- Define $k_i^-(\lambda) = \mathbf{n}(\mathbf{L}(\lambda))$ (the number of negative eigenvalues of the matrix $\mathbf{L}(\lambda)$)
- Set

$$k_i^- := \sum_{\lambda \in \sigma(\mathcal{A}), \operatorname{Re}\lambda = 0} k_{\mathrm{i}}^-(\lambda)$$

gKdV equation

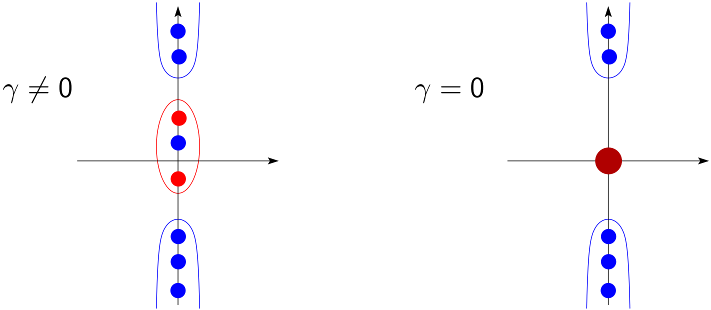
- Use $\mathbf{n}(\mathcal{L}) = k_u + k_i^-$, and
- perturbation arguments (a, b small)
  - locate the spectra for $a = b = 0$

    $\longrightarrow$ operators with constant coefficients

    $\longrightarrow$ use Fourier analysis

Spectra at $a = b = 0$

- **Spectrum of $\mathcal{A}_{0,0,\gamma}$**



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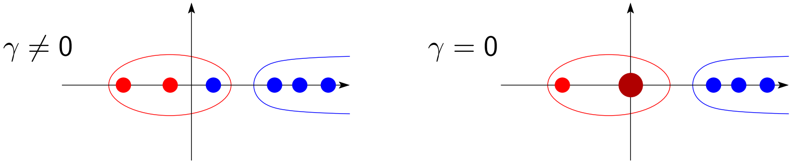
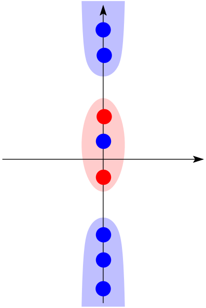
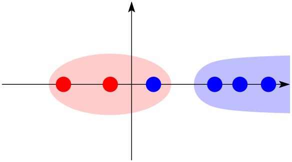


Perturbation arguments: (a, b) small

Spectrum of $\mathcal{A}_{a,b,\gamma}$



Spectrum of $\mathcal{L}_{a,b,\gamma}$

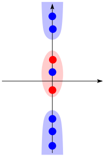


- *Spectral decomposition*

$$\sigma(\mathcal{A}_{a,b,\gamma}) = \sigma_1(\mathcal{A}_{a,b,\gamma}) \cup \sigma_2(\mathcal{A}_{a,b,\gamma})$$

- *The eigenvalues in $\sigma_1(\mathcal{A}_{a,b,\gamma})$ have positive Krein signature*
- $\mathbf{n}(\mathcal{L}) = k_u + k_i^- \quad \Longrightarrow \quad \boxed{\boldsymbol{\sigma_1(\mathcal{A}_{a,b,\gamma}) \subset i\mathbb{R}}}$

Location of $\sigma_2(\mathcal{A}_{a,b,\gamma})$



- $\boxed{|\gamma| \geq \gamma_*}$

- $\boxed{\gamma \sim \mathbf{0}}$

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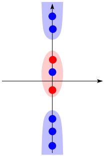
[1] **Difficulty:** *three small parameters; use the results for* $a = b = 0$ *and* $\gamma = 0$.

Location of $\sigma_2(\mathcal{A}_{a,b,\gamma})$



- $\boxed{|\gamma| \geq \gamma_*}$
  - *The three eigenvalues in* $\sigma_2(\mathcal{A}_{a,b,\gamma})$ *are simple*
  - *The spectrum is symmetric w.r.t. the imaginary axis*

$$\Longrightarrow \quad \boxed{\boldsymbol{\sigma_2(\mathcal{A}_{a,b,\gamma}) \subset i\mathbb{R}}}$$

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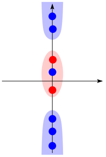

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- $\boxed{\gamma \sim \mathbf{0}}$
  - compute a **basis** $\left\{\xi^0_{a,b,\gamma},\ \xi^1_{a,b,\gamma},\ \xi^2_{a,b,\gamma}\right\}$ for the three dimensional spectral subspace;
  - compute the 3×3-matrix $\mathcal{M}_{a,b,\gamma}$ representing the **action of** $\mathcal{A}_{a,b,\gamma}$ on this subspace;
  - locate the three **eigenvalues** of this matrix [1]

$$\Longrightarrow \quad \boxed{\textbf{purely imaginary if } \mathbf{p < 2}}$$

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[1] **Difficulty:** *three small parameters; use the results for* $a = b = 0$ *and* $\gamma = 0$.

THEOREM

Consider the generalized KdV equation

$$u_t + (u_{xx} + u^{p+1})_x = 0, \quad p \geq 1$$

and the periodic travelling wave $q_{a,b}$ *for* a *and* b *sufficiently small.*

- $\boxed{p < 2}$
  - $k_u(\gamma) = 0,\ k_i^-(\gamma) = 2,$ *for any* $\gamma \in \left[-\frac{1}{2}, \frac{1}{2}\right) \setminus \{0\}$
  - $\sigma(\mathcal{A}_{a,b,\gamma}) \subset \mathrm{i}\mathbb{R}$
  - **the periodic wave is spectrally stable**
- $\boxed{p > 2}$
  - $k_u(\gamma) = 1,\ k_i^-(\gamma) = 1,$ *for sufficiently small* $\gamma = o(|a|)$
  - $\sigma(\mathcal{A}_{a,b,\gamma}) \cap \{\lambda \in \mathbb{C},\ \mathrm{Re}\,\lambda > 0\} \neq \emptyset$
  - **the periodic wave is spectrally unstable**