Large amplitude internal waves

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Summary

Collaborators

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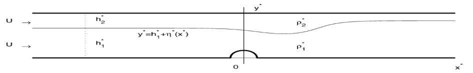
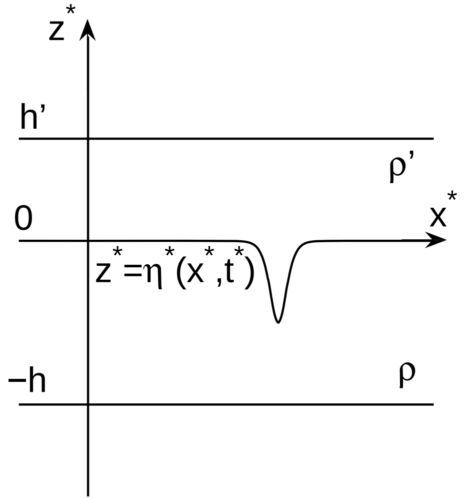
- J.-M. Vanden-Broeck (University College London, UK)
- R. Grimshaw (Loughborough University, UK)
- G. Iooss (University of Nice, France)
- H. Michallet (LEGI, Grenoble)
- T. McClimans (Trondheim, Norway)
- Former PhD students : D. Dutykh (University of Savoie), H. Nguyen (University of Rennes), E. Parau (University of East Anglia), O. Laget (IFP), P. Christodoulides (Cyprus Technical University)

Experiments performed in Trondheim, Norway

Fresh water on top of salt water - Vanden-Broeck, McClimans & FD



Sketch of 2D interfacial waves



Water wave problem in 2D
In dimensional variables

- Continuity equations

$$\Delta\phi = 0, \quad (x,z) \in \Omega_t \qquad \Delta\phi' = 0, \quad (x,z) \in \Omega'_t$$

- Kinematic conditions at the bottom and at the top

$$\frac{\partial\phi}{\partial z} = 0, \quad z = -h \qquad \frac{\partial\phi'}{\partial z} = 0, \quad z = h'$$

- Kinematic conditions at the interface

$$\frac{\partial\phi}{\partial z} = \frac{\partial\eta}{\partial t} + \nabla\phi\cdot\nabla\eta, \quad z = \eta(x,t), \qquad \frac{\partial\phi'}{\partial z} = \frac{\partial\eta}{\partial t} + \nabla\phi'\cdot\nabla\eta$$

- Dynamic condition at the interface $z = \eta(x,t)$

$$\rho\left(\frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + gz\right) = \rho'\left(\frac{\partial\phi'}{\partial t} + \frac{1}{2}|\nabla\phi'|^2 + gz\right)$$

Particle trajectories

- Let (X_0, Z_0) and (X_0', Z_0') denote the mean particle coordinates in the bottom and top layers respectively
- k is the wave number
- C is an arbitrary amplitude
- r is the density ratio

Streamlines and trajectories of 2D interfacial waves

From J.S. Turner's book on "Buoyancy Effects in Fluids" (Cambridge Monographs on Mechanics)

LINEAR INTERNAL WAVES 17

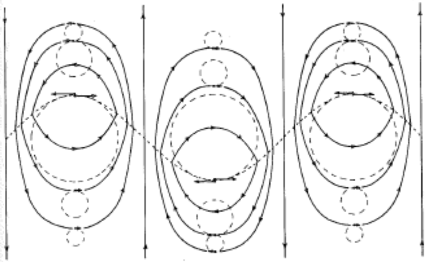


Fig. 2.2. Streamlines and orbits in a progressive internal wave travelling from left to right along the interface between two fluids. From Defant (1961).

(in the linear approximation) are circular, with amplitude decreasing exponentially with distance from the interface. (See fig. 2.2.)

Trajectories of the particles of the lower layer

$$\frac{(X-X_0)^2}{(\cosh k(h+Z_0))^2}+\frac{(Z-Z_0)^2}{(\sinh k(h+Z_0))^2}=\left(\frac{kC}{\omega\cosh kh}\right)^2$$

Trajectories of the particles of the upper layer

$$\frac{(X'-X'_0)^2}{(\cosh k(h'-Z'_0))^2}+\frac{(Z'-Z'_0)^2}{(\sinh k(h'-Z'_0))^2}=\left(\frac{Ck\tanh kh}{\omega\sinh kh'}\right)^2$$

where $\omega^2=\dfrac{(1-r)gk\tanh kh\tanh kh'}{\tanh kh'+r\tanh kh}$

Trajectories of the particles in the lower and upper layer are ellipses

Computation of particle trajectories

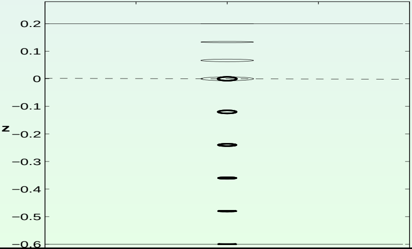
- For a particle on the interface

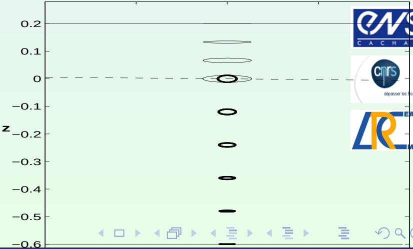
$$\tilde{X} = -\frac{kC}{\omega}\cos(kX-\omega t) \qquad \tilde{X}' = \frac{kC}{\omega}\frac{\tanh kh}{\tanh kh'}\cos(kX-\omega t),$$

$$\tilde{Z} = -\frac{kC}{\omega}\tanh kh \sin(kX-\omega t) \qquad \tilde{Z}' = -\frac{kC}{\omega}\tanh kh \sin(kX-\omega t)$$

$\tilde{Z} = \tilde{Z}'$; $\tilde{X} \neq \tilde{X}'$ $\Longrightarrow$ Continuous normal velocity; tangential velocity changes sign across the interface

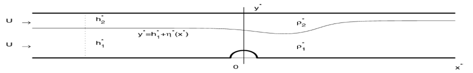
- $h = .6\text{m}$, $h' = .2\text{m}$, $r = .9$, $C = .008$, (left) $k = .6\text{m}^{-1}$; (right) $k = 1.2\text{m}^{-1}$





Experiments performed in Trondheim, Norway

Fresh water on top of salt water - Vanden-Broeck, McClimans & FD



Three-layer configuration

- Dispersion relation

$$a_4 c^4 - a_2 \left(\frac{g}{k}\right) c^2 + a_0 \left(\frac{g}{k}\right)^2 = 0 \quad \text{with}$$

$$\begin{aligned}
a_4 &= 1 + R T_1 T_2 + \frac{S}{T_3}\left(T_1 + \frac{T_2}{R}\right) \\
a_2 &= T_1 + T_2 + \frac{T_2 S}{R}\left(\frac{T_1}{T_3} - 1\right) - T_1 S \left(\frac{T_2}{T_3} + 1\right) \\
a_0 &= T_1 T_2 \left(1 - R + S - \frac{S}{R}\right)
\end{aligned}$$

$$R = \rho_2/\rho_1, \quad S = \rho_3/\rho_1, \quad T_i = \tanh(k h_i)$$

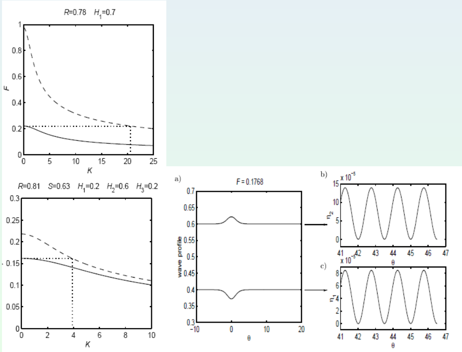
- Typical values

$$R = 0.81, \quad S = 0.63, \quad H_1 = 0.2, \quad H_2 = 0.6, \quad H_3 = 0.2$$

Three-layer configuration

From “Non-linear resonance between short and long waves”, Michallet & FD (1999)

Numerical solution of generalized solitary waves (full Euler)



Two-layer with free surface - deep bottom layer

From “Interfacial periodic waves of permanent form”, Parau & FD, JFM (2001)

Numerical solution of mode–1 and –2 solitary waves (full Euler)

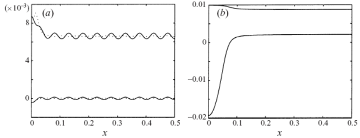


FIGURE 10. Profiles of long free-surface and interfacial internal waves for two density ratios. (a) $R = 0.1$, $H = 0.0067$; $F_\lambda = 0.0078$ (—) and 0.0083 (· · ·); (b) $R = 0.9$, $H = 0.0075$, $F_\lambda = 0.002$ ($K^e/K \approx 79.57$).

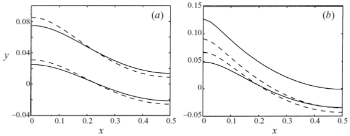
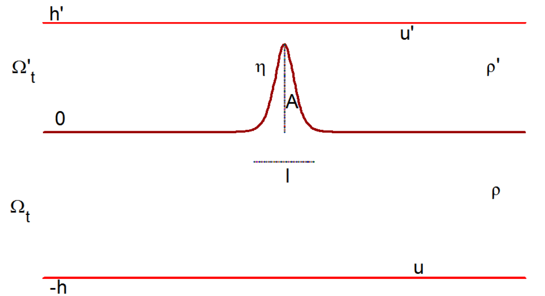


FIGURE 11. Profiles of interfacial and free-surface ‘external’ periodic waves. (a) $H = 0.04$, $F_\lambda = 0.165$; $R = 0.9$ (—) and 0.1 (– –). (b) $R = 0.9$, $H = 0.04$, $F_\lambda = 0.186$ (—) and $R = 0.1$, $H = 0.01$, $F_\lambda = 0.18$ (– –).

Regime under investigation

2D problem; 2 inviscid fluid layers; rigid top and bottom; irrotational flows



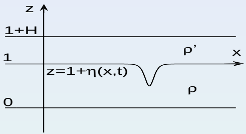
- ℓ: wavelength
- A: amplitude
- ρ, ρ': densities, $r = \rho'/\rho$
- h,h': depths, $H = h'/h$
- $u(x,t), u'(x,t)$: bottom and top horizontal velocities

- $\eta(x,t)$: deviation of interface from its undisturbed position
- $\Omega_t = \{(x,z) : -h \le z \le \eta\}$: lower fluid

- Nonlinearity $\alpha = \frac{A}{h} \ll 1$
- Dispersion $\beta = \left(\frac{h}{\ell}\right)^2 \ll 1$
- Stokes # $S = \frac{\alpha}{\beta} \approx 1$

Water wave problem in dimensionless variables

Variable scaling: $x = \ell \tilde{x}$,
$z = h(\tilde{z} - 1)$, $\eta = A\tilde{\eta}$, $t = \ell \tilde{t}/c_0$,
$\phi = gA\ell\tilde{\phi}/c_0$, $c_0 = \sqrt{gh}$



- Continuity equations

$$\beta\phi_{xx} + \phi_{zz} = 0, \quad (x,z) \in \Omega_t, \quad \beta\phi'_{xx} + \phi'_{zz} = 0, \quad (x,z) \in \Omega'_t$$

- Kinematic cond. at bottom $\phi_z(0) = 0$ and top $\phi'_z(1+H) = 0$
- Kinematic conditions at the interface

$$\frac{1}{\beta}\phi_z = \eta_t + \alpha\phi_x\eta_x, \quad z = 1 + \alpha\eta(x,t), \quad \frac{1}{\beta}\phi'_z = \eta_t + \alpha\phi'_x\eta_x$$

- Dynamic condition at the interface $z = 1 + \alpha\eta(x,t)$

$$\left(\eta + \phi_t + \frac{\alpha}{2}\phi_x^2 + \frac{1}{2}\frac{\alpha}{\beta}\phi_z^2\right) - r\left(\eta + \phi'_t + \frac{\alpha}{2}\phi_x'^2 + \frac{1}{2}\frac{\alpha}{\beta}\phi_z'^2\right) = 0$$

Asymptotic expansion

- We search velocity potentials of the form

$$\phi(x,z,t) = \sum_{m=0}^{\infty} f_m(x,t) z^m, \quad \phi'(x,z,t) = \sum_{m=0}^{\infty} f'_m(x,t)(1+H-z)^m$$

- By using continuity equations and kinematic conditions

$$\phi(x,z,t) = \sum_{k=0}^{\infty} \frac{(-1)^k \beta^k}{(2k)!} \frac{\partial^{2k} F(x,t)}{\partial x^{2k}} z^{2k}$$

$$\phi'(x,z,t) = \sum_{k=0}^{\infty} \frac{(-1)^k \beta^k}{(2k)!} \frac{\partial^{2k} F'(x,t)}{\partial x^{2k}} (1+H-z)^{2k}$$

$F(x,t)$ velocity potential at the bottom $z=0$, $F'(x,t)$ at the top $z = 1+H$

System of 3 equations

System of 3 equations with horizontal velocities at the bottom and at the top

- Let
  $u = \dfrac{\partial F(x,t)}{\partial x}$ horizontal velocity at the bottom $z = 0$
  $u' = \dfrac{\partial F'(x,t)}{\partial x}$ horizontal velocity at the top $z = 1 + H$
- Substitute ϕ, ϕ' in kin. and dyn. conditions at interface
  Retain only terms of order $O(\beta)$

$$(1-r)\eta_x + u_t - ru'_t - \frac{1}{2}\beta u_{xxt} + \frac{1}{2}\beta r H^2 u'_{xxt} + \alpha(uu_x - ru'u'_x) = 0$$

$$\eta_t + u_x + \alpha(u\eta)_x - \frac{1}{6}\beta u_{xxx} = 0$$

$$\eta_t - Hu'_x + \alpha(u'\eta)_x + \frac{1}{6}\beta H^3 u'_{xxx} = 0$$

Derivation of a system of 2 equations

- Can be more general by expressing the horizontal velocities at levels θ and θ' with $0 \leq \theta \leq 1, 0 \leq \theta' \leq H$: $u \to w(x,t)$, $u' \to w'(x,t)$
- Let

$$W = w - rw'$$

- One has

$$w = \frac{H}{r+H} W + O(\beta), \quad w' = \frac{-1}{r+H} W + O(\beta)$$

- Substitute into system of 3 equations, obtain a system of 2 equations with 2 variables η et W

System of 2 equations
In dimensionless variables

$$\begin{cases} \eta_t = -\dfrac{HW_x}{r+H} - \alpha\dfrac{H^2-r}{(r+H)^2}(W\eta)_x - \beta\Big(\dfrac{1}{2}\dfrac{H^2S}{(r+H)^2} + \dfrac{1}{3}\dfrac{H^2(1+rH)}{(r+H)^2}\Big)W_{xxx} \\ W_t = -(1-r)\eta_x - \alpha\dfrac{H^2-r}{(r+H)^2}WW_x - \dfrac{\beta}{2}\dfrac{HS}{r+H}W_{xxt} \end{cases}$$

- with

$$S = (\theta^2 - 1) + \frac{r}{H}\left(\theta'^2 - H^2\right)$$

- The value of S depends on the water levels used for reference. S influences the dispersion relation.
- The condition for existence of waves $\forall k$ is

$$-(1+rH) \leq S \leq -\frac{2}{3}(1+rH)$$

Some remarks on the literature

- The system of 2 equations is essentially equivalent to the system in Bona, Chen, Saut (2002)
- In fact, there is more in Bona et al. (2002): general abcd system
- Bona, Lannes, Saut (2008) derived the abcd system for interfacial waves
- The value of S which best approximates the Choi & Camassa (1999) equations is $S = -\frac{2}{3}(1 + rH)$ (see Nguyen & FD (2008) for an explanation)

Numerical study

- It is more convenient to work with dimensionless variables scaled by $x^* = \frac{x}{h}$, $\eta^* = \frac{\eta}{h}$, $t^* = \frac{c_0}{h}t$, $W^* = \frac{W}{c_0}$, $c_0^2 = \frac{ghh'(\rho - \rho')}{\rho' h + \rho h'} = \frac{ghH(1-r)}{r+H}$
- System of equations in dimensionless variables

$$\begin{cases} \eta_t = -\frac{HW_x}{r+H} - \frac{H^2 - r}{(r+H)^2}(W\eta)_x - \left(\frac{1}{2}\frac{H^2 S}{(r+H)^2} + \frac{1}{3}\frac{H^2(1+rH)}{(r+H)^2}\right) W_{xxx} \\ W_t = -\frac{r+H}{H}\eta_x - \frac{H^2 - r}{(r+H)^2} WW_x - \frac{1}{2}\frac{HS}{r+H} W_{xxt} \end{cases}$$

- Approximate solutions

$$\eta = \eta_0 \left(\operatorname{sech}(Qx - Ut + x_0)\right)^2$$

Initial condition

- Q and U depend on r, H, η_0, θ, θ':

  $$Q^2 = \frac{I\eta_0}{6\left(K - L(1 + \frac{2I\eta_0}{3})\right)}, \qquad U = (1 + \frac{2I\eta_0}{3})Q$$

  where

  $$I = \frac{3}{4}\frac{(H^2 - r)}{H(r+H)}, \quad K = \frac{1}{2}\left(\frac{1}{2}\frac{HS}{r+H} + \frac{1}{3}\frac{H(1+rH)}{r+H}\right), \quad L = \frac{1}{4}\frac{HS}{r+H}$$

- Constraints for η_0

$H^2 - r > 0$	$0 < \eta_0 < H$
$H^2 - r < 0$	$-1 < \eta_0 < 0$

Result I

- A solitary wave propagating to the right with $\theta = 0, \theta' = H$

Solitary wave of elevation

$H = 0.7$, $r = 0.4$ $\eta_0 = 0.1$

Solitary wave of depression

$H = 0.7$, $r = 0.6$ $\eta_0 = -0.1$

Result II

- Counterpropagating solitary wave collision: Symmetric case

Solitary waves of elevation

$H = 0.7,\ r = 0.4,\ \eta_{01} = \eta_{02} = 0.1$

Solitary waves of depression

$H = 0.7,\ r = 0.6,\ \eta_{01} = \eta_{02} = -0.1$

- Observe: During the collisions, the solution rises to a slightly larger amplitude than the sum of the amplitudes of the two incident solitary waves

Result III

- Counterpropagating solitary wave collision: Asymmetric case

Solitary waves of elevation

$H = 0.7, r = 0.4,$
$\eta_{01} = 0.3, \eta_{02} = 0.1$

Solitary waves of depression

$H = 0.7, r = 0.6, \eta_{01} = -0.1, \eta_{02} = -0.$

During collisions, solution rises to a slightly larger amplitude than the sum of the amplitudes of the two incident waves

Result IV

- Copropagating solitary wave interaction (different sizes)

Solitary waves of elevation

- Observe: The maximum amplitude of the solution at any time during the interaction is strictly less than the maximum amplitude of the largest individual solitary wave

Critical case

- When $H^2 \approx r$, the nonlinear effect becomes very small. One has to use a higher-order asymptotic expansion to obtain next order nonlinear terms
- System of 2 equations with cubic terms

$$\begin{cases} \eta_t = -\frac{H}{r+H} W_x - \frac{H^2 - r}{(r+H)^2} (W\eta)_x - \left(\frac{1}{2} \frac{H^2 S}{(r+H)^2} + \frac{1}{3} \frac{H^2(1+rH)}{(r+H)^2} \right) W_{xxx} \\ \qquad + \frac{r(1+H)^2}{(r+H)^3} (W\eta^2)_x \\ W_t = -\frac{r+H}{H} \eta_x - \frac{H^2 - r}{(r+H)^2} W W_x - \frac{1}{2} \frac{HS}{r+H} W_{xxt} + \frac{r(1+H)^2}{(r+H)^3} (W^2 \eta)_x \end{cases}$$

- For 1-way propagation, system reduces to eKdV (Gardner's) equation

On the derivation of the eBoussinesq equation

A lot of terms drop out because they are of higher order. Keep terms of order α^2 and α^4.

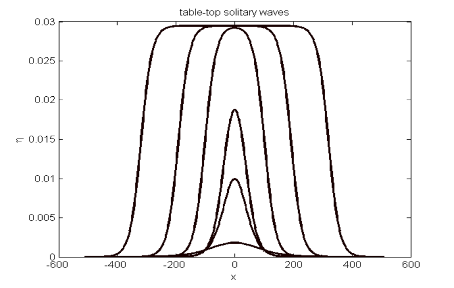
$$\begin{aligned}
&g(1-r)\eta^*_{x^*} + W^*_{t^*} + \frac{H^2-r}{(r+H)^2}W^*W^*_{x^*} + \frac{1}{2}h^2\frac{H(\theta^2-1)+r(\theta'^2-H^2)}{r+H}W^*_{x^*x^*t^*} \\
&\quad - \frac{1}{h}\frac{r(1+H)^2}{(r+H)^3}(W^{*2}\eta^*)_{x^*} + \frac{1}{2}h^2 rH(1+H)\frac{(\theta'^2-\frac{1}{3}H^2)-(\theta^2-\frac{1}{3})}{(r+H)^3}(W^*W^*_{x^*x^*})_{x^*} \\
&\quad - h\frac{H(1-r)}{r+H}(\eta^*W^*_{x^*t^*})_{x^*} + \frac{1}{2}h^2\frac{H^2(\theta^2-1)-r(\theta'^2-H^2)}{(r+H)^2}W^*W^*_{x^*x^*x^*} \\
&\quad + \frac{1}{2}h^2\frac{H^2(\theta^2+1)-r(\theta'^2+H^2)}{(r+H)^2}W^*_{x^*}W^*_{x^*x^*} - \frac{1}{2}hrH(1+H)\frac{(\theta^2-1)-(\theta'^2-H^2)}{(r+H)^2}(W^*\eta^*)_{x^*x^*t^*} \\
&\quad + h^4\left(\frac{rH((\theta^2-1)-(\theta'^2-H^2))((\theta'^2-\frac{1}{3}H^2)-(\theta^2-\frac{1}{3}))}{4(r+H)^2}\right. \\
&\quad \left. + \frac{H(\theta^2-1)(5\theta^2-1)+r(\theta'^2-H^2)(5\theta'^2-H^2)}{2(r+H)}\right)W^*_{x^*x^*x^*x^*t^*} = 0.
\end{aligned}$$

The specific scaling for small values of $|H^2-r|$,

$$\frac{x^*}{h} = \frac{x}{\beta}, \qquad \frac{t^*}{h/c_0} = \frac{t}{\alpha}, \qquad \frac{\eta^*}{h} = \alpha\eta, \qquad \frac{W^*}{gh/c_0} = \alpha W, \qquad H^2 - r = \alpha\mathcal{C},$$

Initial condition

- Approximate solutions



$$H = 0.95, r = 0.8,$$
$$c_1 = 0.544e - 004, 2.544e - 004, 3.944e - 004$$
$$c_1 = 4.544e - 004, 4.5443830e - 004, 4.544383023634e - 004$$

Result I

- A solitary wave propagating to the right

Solitary wave of elevation

$H = 0.95, r = 0.8,$
$c_1 = 3.544383023634e - 004$

Table-top profile

$H = 0.95, r = 0.8,$

Resultat III

- Collision of 2 solitary waves of equal size

Solitary waves of elevation

$H = 0.95$, $r = 0.8$,
$c_1 = 3.54436e - 004$

Solitary waves of depression

$H = 0.9, r = 0.8$,
$c_1 = 7.2e - 005$

Resultat IV

- Collision of 2 table-top solitary waves

Solitary waves of elevation

$H = 0.95, r = 0.8,$
$c_1 = 4.544383023634e - 004$

Solitary waves of depression

$H = 0.9, r = 0.8,$
$c_1 = 7.24204732e - 005$

Weakly nonlinear models for three layers

From “Non-linear resonance between short and long waves”, Michallet & FD 1999

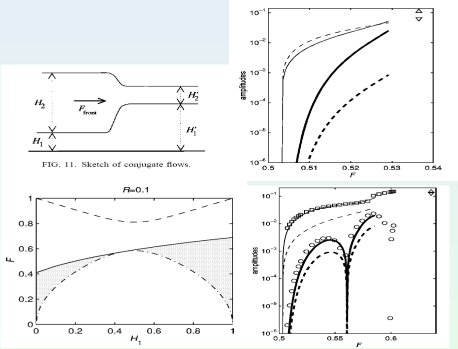


FIG. 11. Sketch of conjugate flows.

Coupled Korteweg–de Vries systems

From “Generalized solitary waves and fronts in coupled KdV systems”, Fochesato, FD & Grimshaw 2005

Stationary solutions of coupled KdV equations

$$\begin{cases} \Delta_1 u + \frac{1}{2}\alpha_1 u^2 + \frac{1}{3}\beta_1 u^3 + \lambda_1 u_{xx} + \kappa_1 v & = & 0, \\ \Delta_2 v + \frac{1}{2}\alpha_2 v^2 + \frac{1}{3}\beta_2 v^3 + \lambda_2 v_{xx} + \kappa_2 u & = & 0, \end{cases}$$

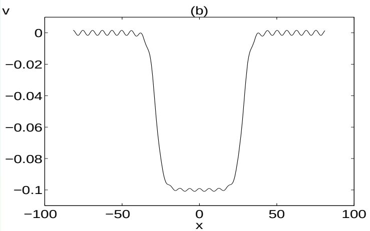
Simplified model

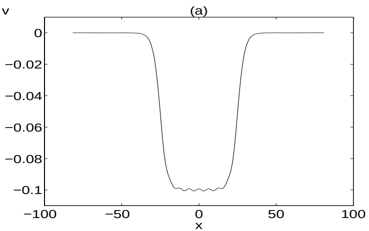
$$v_{xx} + q^2 v = -\kappa u_s, \quad \text{with}$$

$$u_s(x) = \left(\frac{\alpha_1}{-\beta_1}\right) \frac{1-\epsilon^2}{1+\epsilon \cosh\left[(-\Delta_1/\lambda_1)^{1/2} x\right]}$$

Simplified model: solution without ripples in the tails

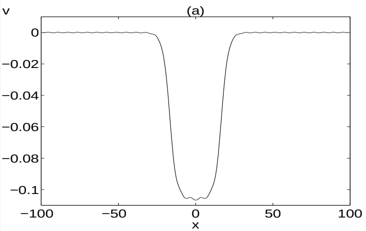
Standard KdV forcing: there are always oscillations; Table-top forcing: ripples can disappear in the central core and in the tail, but not simultaneously

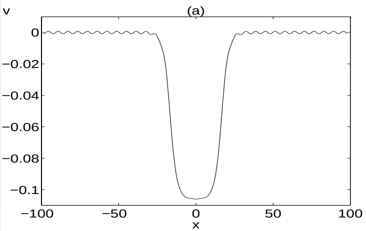




Coupled KdV model: solution without ripples in the central core

Table-top solutions: ripples can disappear in the central core and in the tail



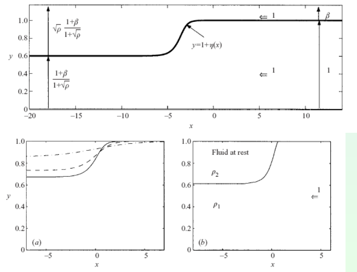


Internal fronts in a rigid lid configuration

From “On internal fronts” by FD & Vanden-Broeck 2003

$$F_{\text{front}}^2 = \frac{(1+\beta)(1-\sqrt{\rho})}{1+\sqrt{\rho}}$$

Front of depression for $\beta < \sqrt{\rho}$; Front of elevation for $\beta > \sqrt{\rho}$



Depression fronts in a rigid lid configuration

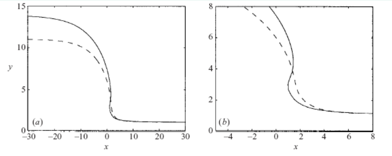
- In the limit as the top layer becomes thinner and thinner, the interface will touch the top wall
- There is a one-parameter family of solutions with the heavy fluid occupying the whole channel
- Only one is the limiting configuration of the front, the one corresponding to $F = F_{\text{front}}$
- See numerical computations of Rusas & Grue (2002)

Elevation fronts computed numerically

From "On internal fronts" by FD & Vanden-Broeck 2003

$$F_{\text{front}}^2 = \frac{(1+\beta)(1-\sqrt{\rho})}{1+\sqrt{\rho}}$$

Front of elevation for $\beta < \sqrt{\rho}$; Front of elevation for $\beta > \sqrt{\rho}$



Elevation fronts in a rigid lid configuration

- In the limit as the top layer becomes thicker and thicker, the behavior is different from the previous one
- A maximum value of β is ultimately reached
- Overhanging develops
- See again numerical computations of Rusas & Grue (2002)

Fronts in a rigid lid configuration

- Stability of fronts is not so easily solved
- Fronts are locally subject to the Kelvin–Helmholtz instability downstream
- It is not known at present whether or not the local instability can destroy the front
- See Choi's talk on Monday

Fronts in the limit $\rho \to 1$

From "On internal fronts" by FD & Vanden-Broeck 2003

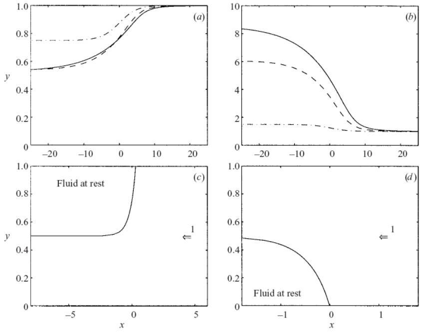


FIGURE 4. Fronts in the Boussinesq limit ($\rho \to 1$). (a) Depression fronts: $\beta = 1/2$ (dash-dotted line), $\beta = 1/11$ (dashed line), $\beta = 1/16$ (solid line). (b) Elevation fronts: $\beta = 2$ (dash-dotted line), $\beta = 11$ (dashed line), $\beta = 16$ (solid line). (c) Limiting depression front. (d) Limiting elevation front. In (a–c), the top wall lies at $y = 1 + \beta$; in (d), the y-coordinate has been rescaled so that the top wall lies at $y = 1$.

Compressible flows: Free-surface models

Mass conservation in each fluid: $\partial_t(\rho) + \nabla \cdot (\rho \vec{u}) = 0$

Momentum equation in each fluid:

$$\partial_t(\rho \vec{u}) + \nabla \cdot \left(\rho \vec{u} \otimes \vec{u} + p\mathbb{I}\right) = \rho \vec{g}$$

On the unknown free surface $z = \eta(x, y, t)$, one has the kinematic condition on both sides of the surface $z = \eta$

$$\eta_t + u_x \eta_x + u_y \eta_y = u_z$$

and the pressure is continuous across the interface.
Concerning thermodynamics, assume that each fluid has its own equation of state

$$EOS_{\pm}(p, \rho^{\pm}) = 0$$

Inviscid two-fluid models with single velocity for homogeneous flows

From "A two-fluid model for violent aerated flows" by FD, Dutykh & Ghidaglia (2008)

Mass conservation for each phase:

$$\partial_t(\alpha^\pm \rho^\pm) + \nabla \cdot (\alpha^\pm \rho^\pm \vec{u}) = 0$$

Momentum equation: $\partial_t(\rho\vec{u}) + \nabla \cdot \left(\rho\vec{u} \otimes \vec{u} + p\mathbb{I}\right) = \rho\vec{g}$

Energy conservation: $\partial_t\big(\rho E\big) + \nabla \cdot \big(\rho H\vec{u}\big) = \rho\vec{g} \cdot \vec{u}$

The superscripts $\pm$ are used to denote the two fluids respectively. α^+ is the volume fraction of fluid $+$. Obviously $\alpha^+ + \alpha^- = 1$.

$$EOS_\pm(p, \rho^\pm, e^\pm) = 0\,, \quad T^+(\rho^+, e^+) = T^-(\rho^-, e^-)\,.$$

Inviscid two-fluid models with single velocity
Definitions

$$
\begin{aligned}
(1+\alpha)\rho^{+} + (1-\alpha)\rho^{-} &= 2\rho\,, \\
(1+\alpha)\rho^{+}e^{+} + (1-\alpha)\rho^{-}e^{-} &= 2\rho\, e
\end{aligned}
$$

Simple two-fluid model

$$\begin{aligned}
\partial_t(\alpha^\pm\rho^\pm) + \nabla\cdot(\alpha^\pm\rho^\pm\vec{u}) &= 0\\
\partial_t(\rho\vec{u}) + \nabla\cdot\left(\rho\vec{u}\otimes\vec{u} + p\mathbb{I}\right) &= \rho\vec{g}\\
\partial_t\left(\rho E\right) + \nabla\cdot\left(\rho H\vec{u}\right) &= \rho\vec{g}\cdot\vec{u}
\end{aligned}$$

In this model we show that:

$$p = \mathcal{P}(\alpha, \rho, e)\,, \quad \alpha \equiv \alpha^+ - \alpha^-$$

For given values of the pressure $p > 0$ and the temperature $T > 0$, we denote by $\mathcal{R}^\pm(p, T)$ and $\mathcal{E}^\pm(p, T)$ the solutions $(\rho^\pm, e^\pm)$ to:

$$\mathcal{P}^\pm(\rho^\pm, e^\pm) = p\,, \quad \mathcal{T}^\pm(\rho^\pm, e^\pm) = T$$

Equation of state

Then:

$$\rho = \frac{1+\alpha}{2}\mathcal{R}^+(p,T) + \frac{1-\alpha}{2}\mathcal{R}^-(p,T),$$

$$\rho\, e = \frac{1+\alpha}{2}\mathcal{R}^+(p,T)\,\mathcal{E}^+(p,T) + \frac{1-\alpha}{2}\mathcal{R}^-(p,T)\,\mathcal{E}^-(p,T).$$

Finally the inversion of this system of equations leads to $p = \mathcal{P}(\alpha, \rho, e)$ and $T = \mathcal{T}(\alpha, \rho, e)$.

Application

The fluid $-$ is an ideal gas

$$p^- = (\gamma^- - 1)\rho^- e^-, \qquad e^- = C_V^- T^-$$

The fluid $+$ obeys to the stiffened gas law

$$p^+ + \pi^+ = (\gamma^+ - 1)\rho^+ e^+, \qquad e^+ = C_V^+ T^+ + \frac{\pi^+}{\gamma^+ \rho^+}$$

For pure water $\gamma^+ = 7$ and $\pi^+ = 2.1 \times 10^9$ Pa. Then

$$\mathcal{P}(\alpha, \rho, e) + \pi(\alpha) = (\gamma(\alpha) - 1)\rho\, e\,,$$

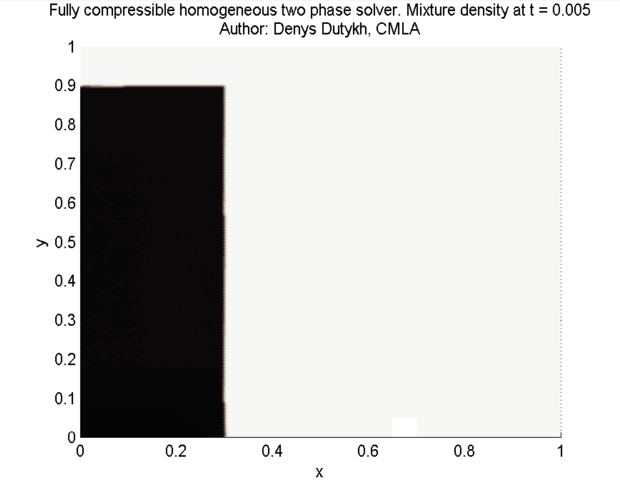
$$e = C_V(\alpha)\, \mathcal{T}(\alpha, \rho, e) + e_0(\alpha)$$

Compare to:

$$p^\pm + \pi^\pm = (\gamma^\pm - 1)\rho^\pm e^\pm, \qquad e^\pm = C_V^\pm T^\pm + \frac{\pi^\pm}{\gamma^\pm \rho^\pm}$$

Water column or dam break test case

Gravity acceleration $g = 100$ m/s^2, in heavy fluid $\alpha^+ = 0.95$, in light fluid $\alpha^+ = 0.05$



Water column test case

Maximum pressure on the right wall