

# Stability and instability in a Vlasov-Boltzman binary mixture at the phase transition

*Edinburgh - PDEs in kinetic theories: kinetic description of biological models,*

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# Outline

- Background
  - The model
  - Equilibrium, phase transition.
- Results:
  - Review of previous stability/instability results in  $\mathbb{R}$ ;
  - Stability and instability in a large finite interval with periodic boundary conditions.

Joint works with Yan Guo and Rossana Marra

# Boltzmann-Vlasov equation – B-V

$$\partial_t f^1 + v \cdot \nabla_x f^1 + F^1[f^2] \cdot \nabla_v f^1 = B(f^1, f^1) + B(f^1, f^2),$$

$$\partial_t f^2 + v \cdot \nabla_x f^2 + F^2[f^1] \cdot \nabla_v f^2 = B(f^2, f^2) + B(f^2, f^1),$$

$f^1 = f^{\text{red}}$  and  $f^2 = f^{\text{blue}}$  probability distribution functions on the phase space  $\Omega \times \mathbb{R}^3$ ; in this talk  $\Omega = \mathbb{R}$  or  $\mathcal{T}_L = (-L, L]$  with periodic b.c.

Self-consistent forces  $F^1[f^2], F^2[f^1]$ :

$$F^i[f^j](x, t) = -\nabla_x \int_{\Omega} dx' U(|x - x'|) \int_{\mathbb{R}^3} dv f^j(x', v, t), \quad i \neq j.$$

Collisions:  $v' = v - \omega[\omega \cdot (v - v_*)]$ ,  $v'_* = v_* + \omega[\omega \cdot (v - v_*)]$ .

$$B(f, g) = \int_{\mathbb{R}^3} dv_* \int_{|\omega|=1} d\omega |(v - v_*) \cdot \omega| [f(v')g(v'_*) - f(v)g(v_*)],$$

# Boltzmann-Vlasov equation – B-V

Model introduced by Bastea and Lebowitz (1997).

System of  $N_1$  red and  $N_2$  blue hard spheres.

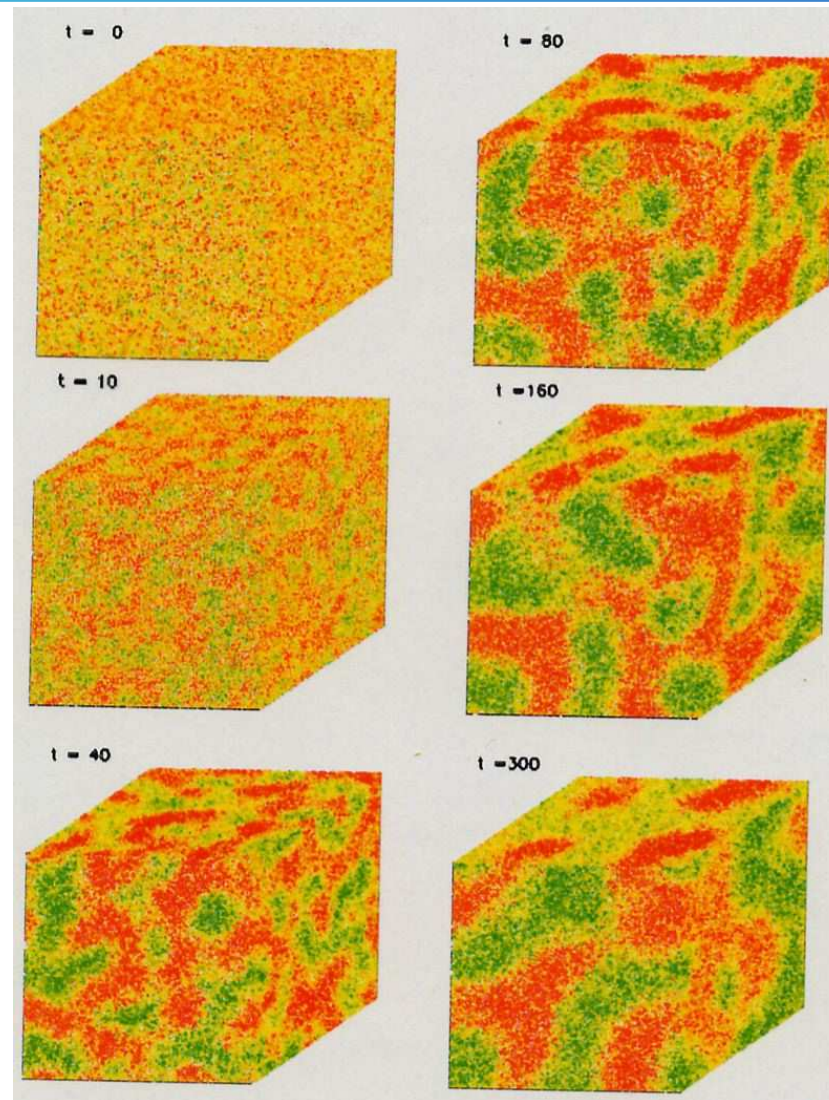
Interactions:

- elastic collisions (color blind);
- long range Kac-type repulsive force between red and blue particles with a two body potential  $U(|x|)$ .

No short times validity proof.

Vlasov-Fokker-Plank: replace collisions with a thermal bath at given temperature.

# Segregation



# Free energy functional

$$\mathcal{F}_{\Omega}^{\text{kin}}(f^1, f^2) = \mathcal{E}_{\Omega}(f^1, f^2) + \beta^{-1} \mathcal{H}_{\Omega}(f^1, f^2),$$

$$\mathcal{E}_{\Omega}(f^1, f^2) = \int_{\Omega} dx \int_{\mathbb{R}^3} dv \frac{1}{2} v^2 \left[ f^1(x, v) + f^2(x, v) \right]$$

$$+ \int_{\Omega} dx \int_{\Omega} dx' U(|x - x'|) \rho_1(x) \rho_2(x'),$$

$$\rho_i(x) = \int_{\mathbb{R}^3} dv f^i(x, v),$$

$$\mathcal{H}_{\Omega}(f^1, f^2) = \sum_{i=1,2} \int_{\Omega} dx \int_{\mathbb{R}^3} dv f^i(x, v) \ln f^i(x, v).$$

# Free energy functional

The functional  $\mathcal{F}_{\Omega}^{\text{kin}}$  is non increasing for any  $\beta > 0$ .

Entropy-energy competition.

- $\beta$  small: entropy dominates. Disordered state.
- $\beta$  large: energy dominates. Segregated states.

Note that in VB case, the relevant value of  $\beta = \beta(E)$  at equilibrium is fixed by the **conserved** total energy, as discussed in [Carlen, Carvalho, R.E., Lebowitz, Marra, *Nonlinearity* **16**, 1075–1105 (2003)].

# Equilibrium

Equilibrium solutions :  $\mu_\beta = \frac{\exp[-\beta v^2]}{(2\pi\beta^{-1})^{3/2}}$

$$f^1(x, v) = \rho_1(x)\mu_\beta(v), \quad f^2(x, v) = \rho_2(x)\mu_\beta(v),$$

$$\beta^{-1} \ln \rho_1(x) + \int_{\Omega} dx' U(|x - x'|) \rho_2(x') = C_1,$$

$$\beta^{-1} \ln \rho_2(x) + \int_{\Omega} dx' U(|x - x'|) \rho_1(x') = C_2.$$

Euler-Lagrange equations for the *spatial free energy* functional

$$\begin{aligned} \mathcal{F}_\Omega[\rho_1, \rho_2] &= \beta^{-1} \int_{\Omega} dx [\rho_1(x) \ln \rho_1(x) + \rho_2(x) \ln \rho_2(x)] \\ &+ \int_{\Omega} dx \int_{\Omega} dx' U(|x - x'|) \rho_1(x) \rho_2(x') \equiv \mathcal{F}_\Omega^{\text{kin}}[\rho_1 \mu_\beta, \rho_2 \mu_\beta]. \end{aligned}$$

# Local free energy

Homogeneous minimizers: minimizers of the local free energy

$$\varphi(\rho_1, \rho_2) = \beta^{-1} [\rho_1 \ln \rho_1 + \rho_2 \ln \rho_2] + \rho_1 \rho_2$$

Indeed,

$$\begin{aligned} \mathcal{F}_\Omega[\rho_1, \rho_2] &= \int_\Omega dx \varphi(\rho_1(x), \rho_2(x)) \\ &- \frac{1}{2} \int_\Omega dx \int_\Omega dx' U(|x - x'|) (\rho_1(x) - \rho_1(x')) (\rho_2(x) - \rho_2(x')). \end{aligned}$$

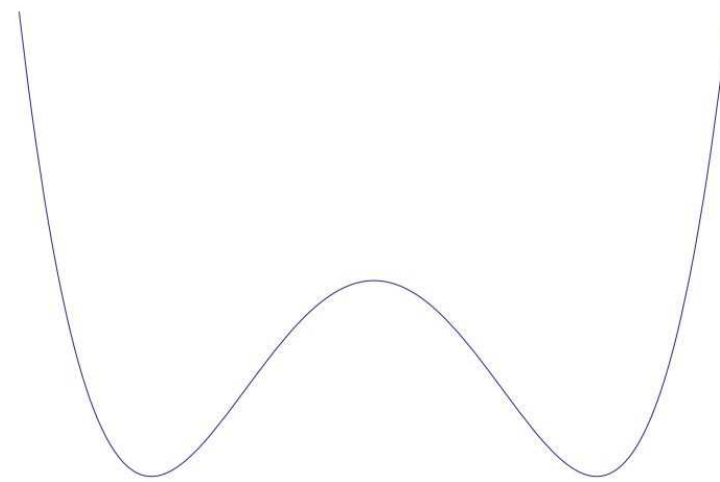
By rearrangement arguments ( $U$  decreasing) the non local term is non negative on minimizers, so minimizers are spatially homogeneous as much as they can.

# Homogeneous equilibrium states

$$\text{Set } \rho = \rho_1 + \rho_2; m = \frac{\rho_1 - \rho_2}{\rho}.$$

$$\varphi(\rho_1, \rho_2) = \beta^{-1} \rho \ln \rho + \frac{1}{4} \rho^2 + \rho^2 f_\rho(m)$$

$$f_\rho(m) = -\frac{m^2}{4} + (\rho\beta)^{-1} \left[ \frac{1-m}{2} \ln \frac{1-m}{2} + \frac{1+m}{2} \ln \frac{1+m}{2} \right]$$



Graph of  $m \rightarrow f_\rho(m)$  for  $\rho\beta > 2$ .

# Homogeneous equilibrium states

The critical points for  $f$  have to solve

$$m = \tanh\left(\frac{1}{2}\rho\beta m\right).$$

If  $\frac{1}{2}\rho\beta > 1$ , there is  $\bar{m} > 0$  and the E-L equation is solved by

$m = 0$  and  $m = \pm\bar{m}$

Set  $\rho^\pm := \frac{1}{2}\rho(1 \pm \bar{m})$ .

# Homogeneous equilibrium states

$\rho\beta \leq 2$ :  $m = 0$ : Minimizer of  $\varphi$ :  $\rho_1 = \rho_2$  (mixed phase).

$\rho\beta > 2$ :

- Minimizer:  $\rho_1 = \rho^+$ ,  $\rho_2 = \rho^-$ , (red rich phase);
- Minimizer:  $\rho_1 = \rho^-$ ,  $\rho_2 = \rho^+$ , (blue rich phase);
- Maximizer (local):  $\rho_1 = \rho_2$ .

**Note:** the minimizing total density is always unique.

# Non homogeneous equilibrium states

Conservation of masses  $\implies$  Minimizers with the constraints

$$\frac{1}{|\Omega|} \int_{\Omega} dx \rho_i(x) = n_i, \quad i = 1, 2$$

$n_i$  given by the initial conditions.

If  $\rho^- < n_i < \rho^+$ ,  $i = 1, 2$  **non homogeneous minimizers** arise:

Regions **blue rich** and **red rich** separated by interfaces.

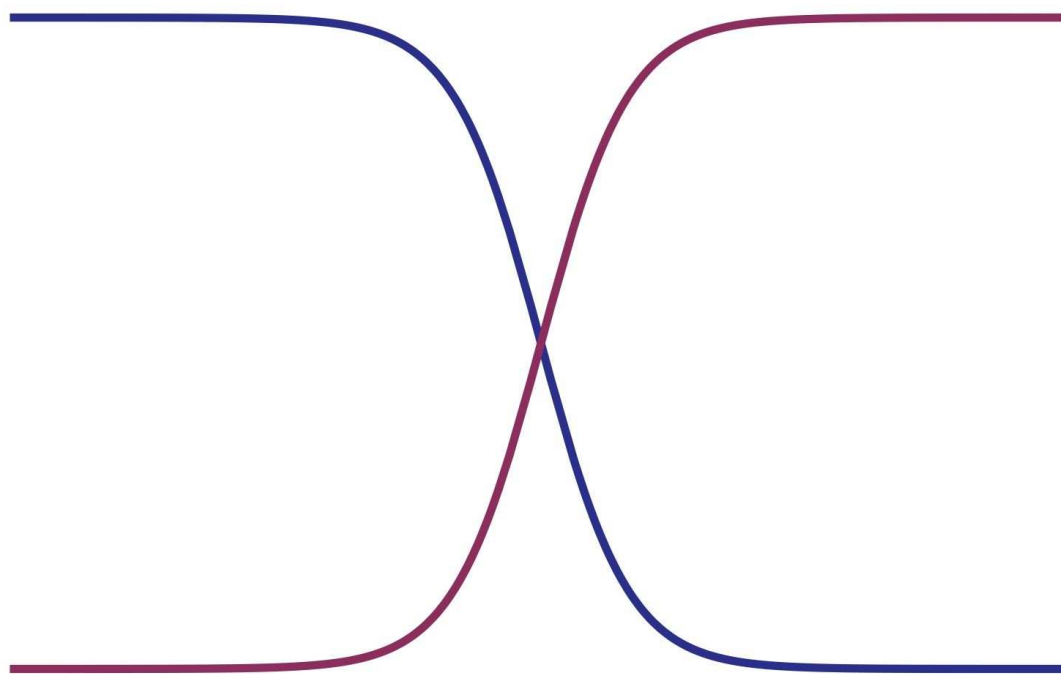
**Assume  $\Omega$  a  $d$ -dimensional torus of size  $L$ .**

From rearrangement arguments (using  $U$  monotone), for  $L$  sufficiently large, the minimizers are **symmetric monotone** with values close to  $\rho^+$  and  $\rho^-$  in regions separated by a

**small interface** [Carlen, Carvalho, R.E., Lebowitz, Marra, *Nonlinearity* **16**, 1075–1105 (2003)].

# Front solution

$$\Omega = \mathbb{R}, \quad \lim_{x \rightarrow \mp\infty} \rho_1 = \rho^{\mp}, \quad \lim_{x \rightarrow \mp\infty} \rho_2 = \rho^{\pm}$$



[Carlen, Carvalho, R.E., Lebowitz, Marra; ARMA **194**, 823–847 (2009)]

# Stability

Question: Are the equilibrium states stable w.r.t. perturbations of the initial conditions in the time evolution?

Expected answer: Minimizers are stable, maximizers are unstable.

Results: Case  $\Omega = \mathbb{R}$ ;

- V-B evolution: stability of the minimizers, instability of the maximizer.

# Results for Vlasov-Boltzmann

**Theorem**[R. E., Guo, Marra, CMP **296**,1-33 (2010)]:  $\rho = 2$ .

- $\beta < 1$ : The unique equilibrium  $(f_1, f_2) = (\mu_\beta, \mu_\beta)$  is **stable**.
- $\beta > 1$ :
  - the homogeneous equilibrium states  $(f_1, f_2) = (\rho^+ \mu_\beta, \rho^- \mu_\beta)$  and  $(f_1, f_2) = (\rho^- \mu_\beta, \rho^+ \mu_\beta)$  are **stable**;
  - the equilibrium  $(f_1, f_2) = (\bar{\rho}^1(x) \mu_\beta, \bar{\rho}^2(x) \mu_\beta)$  is **stable** w.r.t. symmetric perturbations;
  - the homogeneous equilibrium  $(f_1, f_2) = (\mu_\beta, \mu_\beta)$  is **unstable**.

Here stability and instability are in  $L^\infty(\mathbb{R} \times \mathbb{R}^3)$  and in  $H^1(\mathbb{R} \times \mathbb{R}^3)$  for  $h_i = (\sqrt{M_i})^{-1}(f_i - M_i)$ . Symmetric perturbation means  $h_1(x, v) = h_2(-x, Rv)$ , where  $Rv = (-v_x, v_y, v_z)$ .

# Remarks

- No convergence to the equilibrium is stated.
- In order to have phase transitions: force not small.  
Treating the force terms as perturbations does not work.  
Strategy based on entropy-energy arguments:  $L^2$  estimates promoted to  $L^\infty$  by analysis of the characteristics. Crucial step: spectral gap for the second derivative of the free energy.
- The instability is based on the construction of a growing mode for the linear collisionless case, perturbation arguments and persistence of the growing mode at non linear level.

# Spectral gap

Given  $\rho = (\rho_1, \rho_2)$ , define the operator  $A$  on  $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ :

$$\langle u, Au \rangle = \frac{1}{2} \frac{d^2}{ds^2} \hat{\mathcal{F}}(\rho + su) \Big|_{s=0} .$$

Whenever  $\rho$  is a minimizer for the excess free energy,  $A$  is non negative. Let  $\mathcal{P}$  be the projector on the null space of  $A$

**Lemma.[CCELM]** *There is  $\delta > 0$  such that*

$$\langle u, Au \rangle \geq \delta \|(1 - \mathcal{P})u\|^2 .$$

*If  $\rho = (\bar{\rho}_1, \bar{\rho}_2)$ , then the null space of  $A$  is  $\{c(\bar{\rho}'_1, \bar{\rho}'_2), c \in \mathbb{R}\}$ .*

The null space of the analog of  $A$  is trivial if  $\rho = (\rho^+, \rho^-)$  or  $\rho = (\rho^-, \rho^+)$  (case  $\beta > 1$ ) or  $\rho = (1, 1)$  (case  $\beta < 1$ ).

# Main proposition

**Theorem:** *Let  $w(v) = (\Sigma + |v|^2)^\gamma$ , with  $\Sigma$  sufficiently large and  $\gamma > \frac{3}{2}$ . If  $\|wg(0)\|_\infty + \sqrt{\mathcal{F}(g(0))} < \delta$  for  $\delta$  sufficiently small, then there is  $T_0 > 0$  such that*

$$\|wh(T_0)\|_\infty \leq \frac{1}{2}\|wh(0)\|_\infty + C_{T_0}\sqrt{\mathcal{F}(h(0))}.$$

The stability follows by iteration on the time interval.

# Growing mode

1) Without collisions there is a growing mode.

2) Collisions do not destroy the growing mode.

The linearization of the equation for the perturbation is:

$$\partial_t g + \mathcal{L}g = 0,$$

**Notation**:  $\mu = \mu_\beta$ ;  $\xi$  first component of the velocity  $v = (\xi, \zeta)$ ,

$$(\mathcal{L}g)_i = \xi \partial_x g_i - \beta F(\sqrt{\mu} g_{i+1}) \xi \sqrt{\mu} - \alpha L_i g,$$

$$\alpha = 1 \text{ and } L_i g = \frac{1}{\sqrt{\mu}} \left( B(\sqrt{\mu} g_i, 2\mu) + B(\mu, \sqrt{\mu}(g_1 + g_2)) \right).$$

Seek for a growing mode of the form

$$g_1(t, x, \xi, \zeta) = e^{\lambda t} e^{\mathbf{i}kx} q(\xi, \zeta), \quad g_2(t, x, \xi, \zeta) = e^{\lambda t} e^{-\mathbf{i}kx} q(-\xi, \zeta).$$

# Instability theorem

**Theorem.** Assume  $\beta > 1$ . There exist constants  $k_0 > 0$ ,  $\theta > 0$ ,  $C > 0$ ,  $c > 0$  and a family of initial  $\frac{2\pi}{k_0}$ -periodic data  $f_i^\delta(0) = \mu + \sqrt{\mu}g_i^\delta(0) \geq 0$ , with  $g^\delta(0)$  satisfying

$$\|\nabla_{x,\xi}g^\delta(0)\|_{L^2} + \|wg^\delta(0)\|_{L^\infty} \leq C\delta,$$

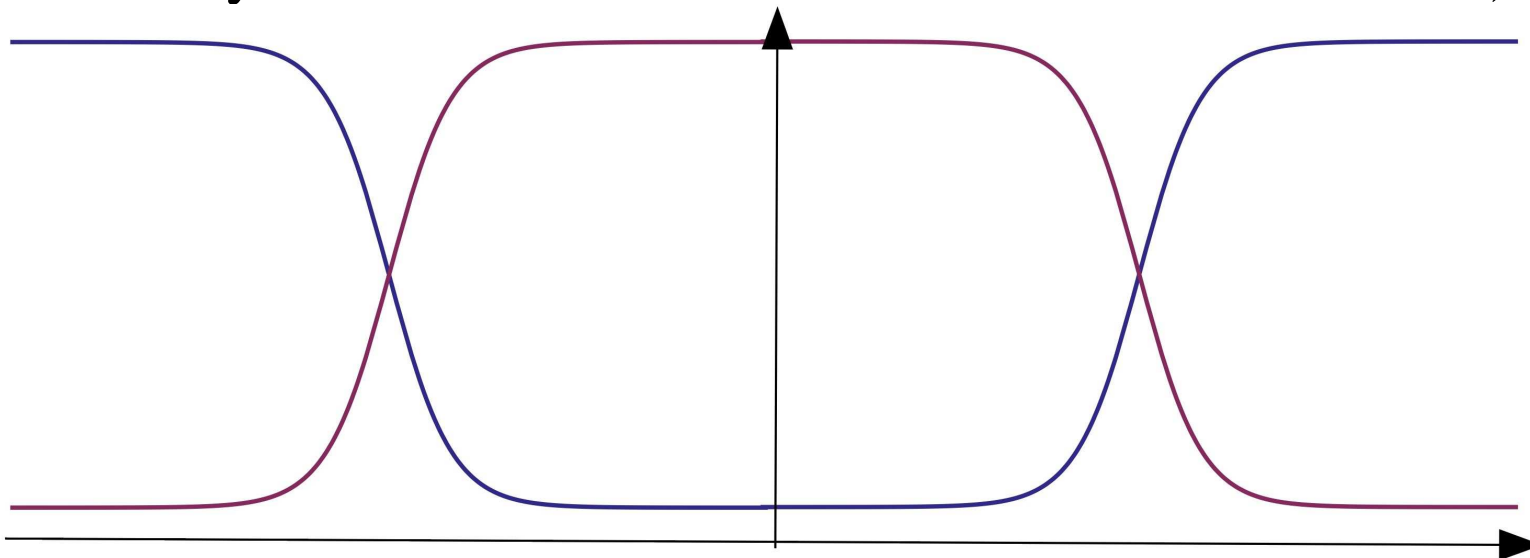
for  $\delta$  sufficiently small, but the solution  $g^\delta(t)$  satisfies

$$\sup_{0 \leq t \leq T^\delta} \|wg^\delta(t)\|_{L^\infty} \geq c \sup_{0 \leq t \leq T^\delta} \|g^\delta(t)\|_{L^2} \geq c\theta > 0.$$

Here the escape time is  $T^\delta = \frac{1}{\operatorname{Re}\lambda} \ln \frac{\theta}{\delta}$ ,

# V-B on a periodic interval

- Finite volume, **1d torus** (with Y. Guo and R. Marra)  
Stability of the non constant minimizer “double front”



- Operator  $A$  on  $L^2(\mathcal{T}_L)$ ,  $\mathcal{T}_L$  the 1-d torus of size  $2L$ . The derivative of the front is in the null space of  $A$ .  
**What is the null space? There is still a spectral gap?**

# Operator $A$

Let  $w = (w_1, w_2)$  be a centered front on the circle.

$$\langle u, Au \rangle = \frac{1}{2} \frac{d^2}{ds^2} \mathcal{F}(w + su) \Big|_{s=0}.$$

$$\langle u, Au \rangle \geq \delta \|(1 - \mathcal{P})u\|^2.$$

$\mathcal{P}$  projector on the null space of  $A$ .

$$(Au)_1 = \frac{u_1}{w_1} + \beta U * u_2, \quad (Au)_2 = \frac{u_2}{w_2} + \beta U * u_1.$$

# Spectral gap

$A$  has a **negative eigenvalue**. By the E-L equations,  $Aw' = 0$ , which means

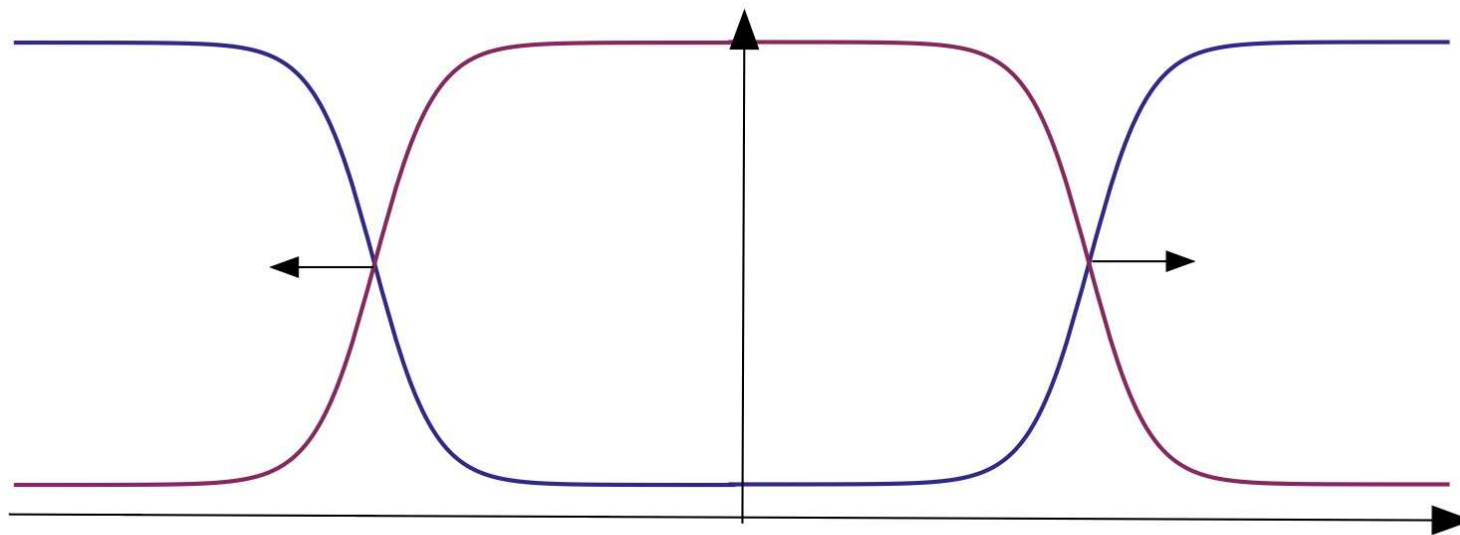
$$\frac{w'_1}{w_1} = -\beta U * w'_2, ; \quad \frac{w'_2}{w_2} = -\beta U * w'_1.$$

Define  $\tilde{w} = (|w'_1|, -|w'_2|)$ . By using above equations,

$$\begin{aligned} (\tilde{w}, A\tilde{w}) &= \int_{\mathcal{T}_L} |w'_1| \left( \frac{|w'_1|}{w_1} - \beta U * |w'_2| \right) + |w'_2| \left( \frac{|w'_2|}{w_2} - \beta U * |w'_1| \right) \\ &= -2\beta \left[ \int_0^L w'_1 U * (|w'_2| + w'_2) + \int_{-L}^0 w'_2 U * (|w'_1| + w'_1) \right] < 0. \end{aligned}$$

# Negative eigenvalue

Perturbing  $w$  with the negative eigenvector expands the bump.

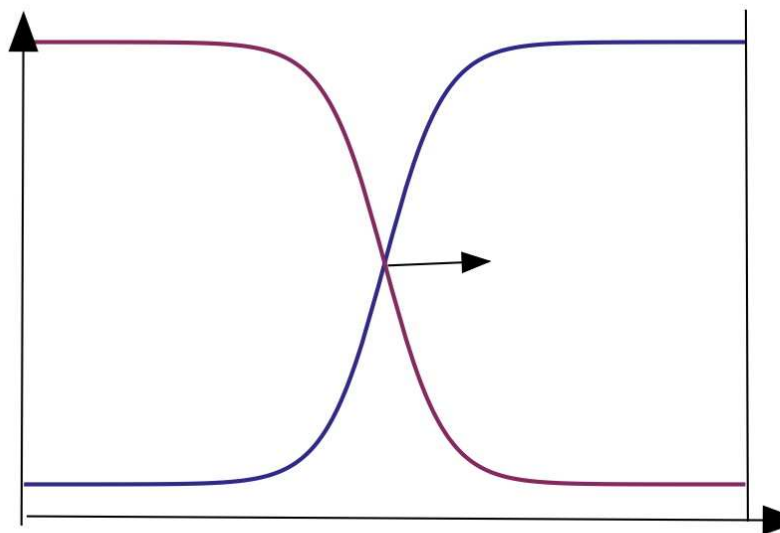


Idea: the **mass constraint** kills the negative eigenvalue.

# Neumann b. c.

Spectral gap true for **anti-symmetric** (by reflection) functions.

Problem on the circle for **symmetric** functions reduced to the case of **Neumann boundary conditions on  $[0, L]$** :



# Neumann b. c.

$$(\hat{A}g)_1 = \beta^{-1} \frac{g_1}{w_1} + \hat{U} * g_2, \quad (\hat{A}g)_2 = \beta^{-1} \frac{g_2}{w_2} + \beta \hat{U} * g_1$$

$$\hat{U}(x, x') = U(x, x') + U(x, R_0 x') + U(x, R_L x')$$

$R_0$  reflection around zero and  $R_L$  reflection around  $L$ .

$\lambda_L < 0$       minimum eigenvalue,

$\hat{e}$       corresponding eigenvector.

# Spectral gap

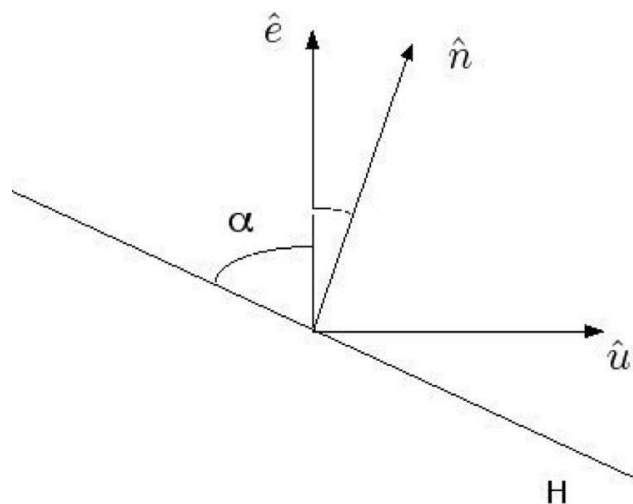
We need spectral gap for functions in the hyperplane

$$H = \{h : \int_0^L h_i = 0, \}, i = 1, 2.$$

We expect to prove spectral gap for functions in the orthogonal to  $\hat{e}$ :

$$(u, \hat{A}u) \geq \delta(u, u), \quad \text{if } (u, \hat{e}) = 0.$$

If the angle  $\alpha$  between  $\hat{e}$  and  $H$  is not too small we get the spectral gap:



# Spectral gap

Decompose  $h \in H$  as  $a\hat{e} + b\hat{u}$  with  $\hat{u}$  orthogonal to  $\hat{e}$

$$(h, \hat{A}h) = a^2 \lambda_L + b^2 (\hat{u}, \hat{A}\hat{u})$$

$a^2 = \cos^2 \alpha$ ,  $b^2 = \sin^2 \alpha$ . with  $\sin \alpha = \frac{1}{\sqrt{L}} \int_0^L \hat{e}$ .

If  $\hat{e}$  decays fast enough  $b^2 \approx \frac{1}{L}$ .  $\lambda_L$  is negative

**Competition**  $(h, \hat{A}h) \geq -|\lambda_L| + \frac{c}{L}\delta$

If  $\lambda_L$  decays faster than  $\frac{1}{L}$  we can prove spectral gap for  **$L$  large**

$(h, \hat{A}h) > d(h, h)$  for any  $h \in H$ .

G. Manzi (2007)

# Spectrum

Analysis of the spectrum using **Markov chains**. Generalize method by De Masi, Olivieri, Presutti (1998), Ising case.

- exponential bounds on the minimum eigenvalue  $\lambda_L$

$$-c_1 e^{-\gamma L} \leq \lambda_L \leq c_2 e^{-\gamma L}$$

- exponential bound on the minimum eigenfunction  $\hat{e}$

$$-c e^{-\gamma|L-x|} \leq \hat{e}(x) < 0$$

- **spectral gap** in a suitable weighted  $L_\infty$  for functions  $u$  in the orthogonal to  $\hat{e}$ . Implies spectral gap in  $L_2$ -norm.

# Spectral gap

Operator  $S$   $(Su)_i = w_i \hat{U} * u_j, \quad i \neq j, i = 1, 2$

$$(u, \hat{A}u) = \sum_i \int_{\mathcal{T}_L} dx w_i u_i (\hat{A}u)_i = (u, u) + (u, Su)$$

$$(u, S^2u) = \sum_i \int_{\mathcal{T}_L} u_i w_i \hat{U} * (w_j \hat{U} * u_i) = (u, Tu)$$

Negative eigenvalue for  $\hat{A}$  means eigenvalue for  $S$  greater than 1. We study the operators

$$T_1 h = w_1 \hat{U} * (w_2 \hat{U} * h), \quad T_2 h = w_2 \hat{U} * (w_1 \hat{U} * h).$$

# Markov chain

$$M(x, x') = w_1(x) \int dz \hat{U}(x - z) w_2(z) \hat{U}(z - x').$$

For  $\lambda_L > 0$  and  $\hat{e}(x)$  positive, define the Markov kernel

$$K(x, x') = \frac{M(x, x') \hat{e}(x')}{\lambda_0 \hat{e}(x)}.$$

By using the theory of the Markov kernels one gets the required bounds.

Hence the front does exist and the corresponding operator  $A$  has the required spectral gap in the relevant functional space. We denote it again by  $\bar{\rho}(x) = (\bar{\rho}_1(x), \bar{\rho}_2(x))$ .

# Free energy functional

Given the equilibrium state  $(M_1, M_2) = (\rho_1 \mu_\beta, \rho_2 \mu_\beta)$ , let  $g = (g_1, g_2)$  with  $g_i = \frac{f_i - M_i}{\sqrt{M_i}}$  be the deviation from the equilibrium. Define:

$$\begin{aligned}\mathcal{H}(g) &= \sum_{i=1}^2 \int_{\mathcal{T}_L} dx \int dv \left[ f_i \log f_i - M_i \log M_i \right], \\ \mathcal{E}(g) &= \sum_{i=1}^2 \int_{\mathcal{T}_L} dx \int dv \frac{v^2}{2} g_i \sqrt{M_i} \\ &+ \int_{\mathcal{T}_L \times \mathcal{T}_L} dx dy U(|x - y|) \left( \rho_{f_1}(x) \rho_{f_2}(y) - \rho_1(x) \rho_2(y) \right), \\ \rho_{f_i} &= \int dv f_i(x, v).\end{aligned}$$

# Free energy functional

The free energy functional is

$$\mathcal{F}(g) = \mathcal{H}(g) + \beta\mathcal{E}(g),$$

The free energy functional does not increase:

$$\mathcal{F}(g(t)) \leq \mathcal{F}(g(0)) \quad \text{for any } t > 0.$$

Quadratic approximation. There is  $\alpha \in (0, 1)$  s.t. with  $\tilde{f}_i = \alpha_i f_i + (1 - \alpha_i)M_i$ ,  $\alpha_i \in (0, 1)$ :

$$\mathcal{F}(g) = \sum_{i=1}^2 \int_{\mathcal{T}_L} dx \int_{\mathbb{R}^3} dv \frac{(f_i - M_i)^2}{2\tilde{f}_i}$$

$$+ \beta \int_{\mathcal{T}_L} dx \int_{\mathcal{T}_L} dy U(|x - y|) (\rho_{f_1}(t, x) - \rho_1(x)) (\rho_{f_2}(t, y) - \rho_2(y)),$$

# Free energy functional

**Lemma.** *If  $u_i = \rho_{f_i} - \rho_i$  s.t.  $\mathcal{P}(u_1, u_2) = 0$ , then there are  $c > 0$  and  $\kappa$  sufficiently small so that*

$$c \sum_{i=1,2} \int_{\mathcal{T}_L} dx \int_{\mathbb{R}^3} dv \left\{ \frac{(f_i(t) - M_i)^2}{M_i} \mathbf{1}_{\{|f_i(t) - M_i| \leq \kappa M_i\}} + |f_i(t) - M_i| \mathbf{1}_{\{|f_i(t) - M_i| \geq \kappa M_i\}} \right\} \leq \mathcal{F}(g(0)).$$

**Remark:** It is crucial that the initial perturbation is **orthogonal** to the null space of  $A$ . This is ensured by the **symmetry condition**.

# Stability for V-B on the circle

**Theorem:** Assume  $\rho = 2$  and  $L$  sufficiently large. Then.

- $\beta < 1$ : The unique equilibrium  $(f_1, f_2) = (\mu_\beta, \mu_\beta)$  is **stable**.
- $\beta > 1$ :
  - the homogeneous equilibrium states  $(f_1, f_2) = (\rho^+ \mu_\beta, \rho^- \mu_\beta)$  and  $(f_1, f_2) = (\rho^- \mu_\beta, \rho^+ \mu_\beta)$  are **stable**;
  - the equilibrium  $(f_1, f_2) = (\bar{\rho}^1(x) \mu_\beta, \bar{\rho}^2(x) \mu_\beta)$  is **stable** w.r.t. symmetric perturbations;
  - the homogeneous equilibrium  $(f_1, f_2) = (\mu_\beta, \mu_\beta)$  is **unstable**.

Stability in  $L^\infty(\mathcal{T}_L \times \mathbb{R}^3)$  and  $H^1(\mathcal{T}_L \times \mathbb{R}^3)$

# Convergence to equil. on the circle

On  $\mathbb{R}$  we did not state any approach to equilibrium and rate for the V-B case. On the circle we have such a statement in  $L_2(\mathbb{T}_L \times \mathbb{R}^3)$  with a suitable weight.

**Theorem:** *Suppose that the masses and energy of the perturbation  $h$  vanish at time  $t = 0$  and the perturbation is small in the sense of the previous theorem. Then there is  $\lambda > 0$  such that, for any  $t > 0$*

$$\exp[\lambda t] \|g(t)\|_{L_2(\mathcal{T}_L \times \mathbb{R}^3)} \leq C \|g(0)\|_{L_2(\mathcal{T}_L \times \mathbb{R}^3)}.$$

The proof is based on the study of the linearized equation

$$\begin{aligned} \partial_t g_i + w_i \xi \partial_x (A \rho_g)_i + \xi \partial_x \tilde{g}_i - (U * w'_j) \partial_\xi \tilde{g}_i &= (L \tilde{g})_i, \\ g_i &= \rho_g \sqrt{M} + \tilde{g}_i. \end{aligned}$$

# Convergence to equil. on the circle

Main steps:

- Establish an  $L_2$ -bound for  $e^{\lambda t} g(t)$  under the linearized evolution
- Bound the nonlinear term using the  $L_\infty$ -bound on the perturbation established in the stability theorem:

Essential for the study of the linearized equation are the spectral gaps for  $L$  and  $A$  and the following

# Convergence to equil. on the circle

**Lemma** *If  $g = (g_1, g_2)$  is solution to the linearized equation, there is a constant  $K$  s.t.*

$$\int_0^1 ds \|Pg(s)\|^2 \leq K \int_0^1 ds \|(1 - P)g(s)\|^2.$$

Equation for  $h(t) = e^{\lambda t}g(t)$ :

$$\partial_t h_i + w_i \xi \partial_x (A\rho h)_i + \xi \partial_x \tilde{h}_i - (U * w'_j) \partial_\xi \tilde{h}_i = (L\tilde{h})_i + \lambda h_i,$$

The term  $\lambda Ph_i$  is controlled in term of  $\lambda(1 - P)h_i$  using the Lemma and a time iteration, the part  $\lambda(1 - P)h_i$  by the spectral gap if  $\lambda$  is sufficiently small.

# Convergence to equil. on the circle

The energy bound is established by using the norm

$$|||f|||^2 = \langle \rho_f, A\rho_f \rangle + \|\tilde{f}\|^2,$$

This norm turns out to be equivalent to the usual  $L_2$ -norm because of the spectral gap for  $A$  and the symmetry properties that ensures the orthogonality of  $\rho_f$  to  $w'$ , whose span is the null of  $A$ .

**Thank you!**