Some kinetic models of swarming

J. A. Carrillo,

in collaboration with J. A. Cañizo and J. Rosado (UAB), to appear in M3AS.

in collaboration with F. Bolley (Paris-Dauphine) and J. A. Cañizo (UAB), preprint INI 2010 and to appear in M3AS.

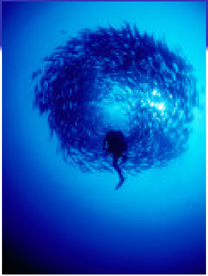
in collaboration with M. Fornasier (Linz), J. Rosado (UAB) and G. Toscani (Pavia), SIMA 2010.

ICREA - Universitat Autònoma de Barcelona

ICMS, Edinburgh 2010

Outline

Swarming



Fish schools and Birds flocks.

Outline

Types of interaction

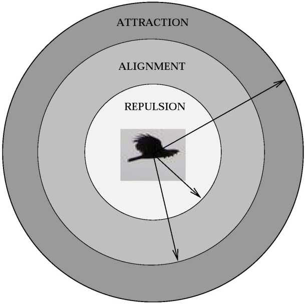
Swarming = Aggregation of agents of similar size and body type generally moving in a coordinated way.

Highly developed social organization: insects (locusts, ants, bees ...), fishes, birds, micro-organisms (myxo-bacteria, ...) and artificial robots for unmanned vehicle operation.

Interaction regions between individuals[a]

[a] Barbaro, Birnir et al. (2008).

- Repulsion Region: R_k.
- Attraction Region: A_k.
- Orientation Region: O_k.



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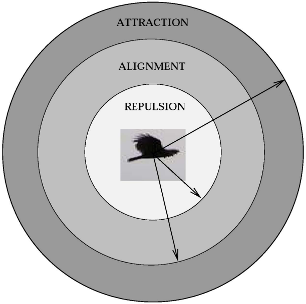

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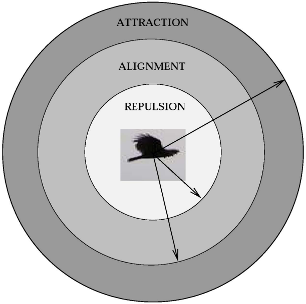


Model with an asymptotic velocity

D'Orsogna, Bertozzi et al. model (PRL 2006):

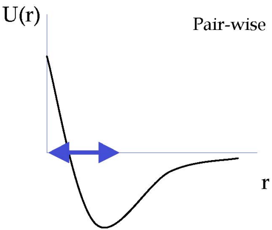
$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\ \dfrac{dv_i}{dt} = (\alpha - \beta\,|v_i|^2)v_i - \sum_{j\neq i} \nabla U(|x_i - x_j|). \end{cases}$$

Model assumptions:

- Self-propulsion and friction terms determines an asymptotic speed of $\sqrt{\alpha/\beta}$.
- Attraction/Repulsion modeled by an effective pairwise potential $U(x)$.

$$U(r) = -C_A e^{-r/\ell_A} + C_R e^{-r/\ell_R}.$$

$C = C_R/C_A > 1$, $\ell = \ell_R/\ell_A < 1$ and $C\ell^2 < 1$:



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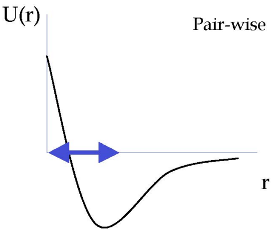

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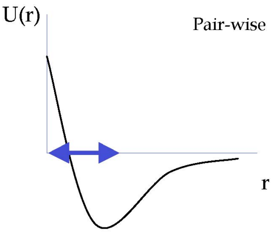
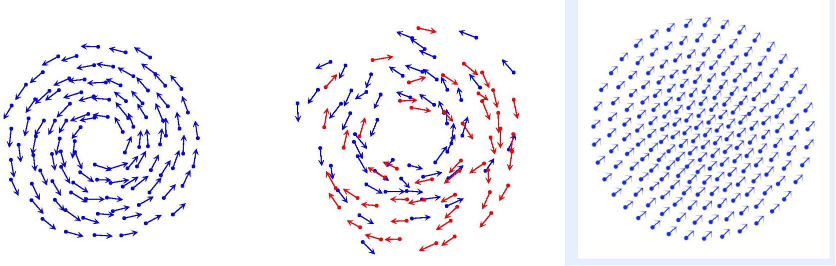


Model with an asymptotic velocity

Typical patterns: milling, double milling or flocking.



Double milling patterns: Carrillo, D'Orsogna, Panferov, KRM (2009).

Velocity consensus model

Cucker-Smale Model (IEEE Automatic Control 2007):

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\ \dfrac{dv_i}{dt} = \displaystyle\sum_{j=1}^{N} a_{ij}\left(v_j - v_i\right), \end{cases}$$

with the communication rate, $\gamma \geq 0$:

$$a_{ij} = a(|x_i - x_j|) = \frac{1}{(1 + |x_i - x_j|^2)^{\gamma}}.$$

Unconditional flocking: $\gamma \leq 1/2$; Ha-Tadmor, Ha-Liu, Carrillo-Fornasier-Toscani-Rosado.
Conditional flocking: $\gamma > 1/2$.

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Mesoscopic models

Model with asymptotic velocity + Attraction/Repulsion:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + \mathrm{div}_v[(\alpha - \beta|v|^2)vf] - \mathrm{div}_v\left[(\nabla_x U \star \rho)f\right] = 0.$$

Velocity consensus Model:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \nabla_v \cdot \Bigg[\underbrace{\left(\int_{\mathbb{R}^{2d}} \frac{v-w}{(1+|x-y|^2)^\gamma} f(y,w,t)\,dy\,dw \right)}_{:=\xi(f)(x,v,t)=H(x,v)\star_{(x,v)} f} f(x,v,t) \Bigg]$$

Orientation, Attraction and Repulsion:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f - \mathrm{div}_v\left[(\nabla_x U \star \rho)f\right] = \nabla_v \cdot \left[\xi(f)(x,v,t) f(x,v,t)\right].$$

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Definition of the distance[1]

Transporting measures:

Given $T : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ mesurable, we say that $\nu = T\#\mu$, if $\nu[K] := \mu[T^{-1}(K)]$ for all mesurable sets $K \subset \mathbb{R}^d$, equivalently

$$\int_{\mathbb{R}^d} \varphi \, d\nu = \int_{\mathbb{R}^d} (\varphi \circ T) \, d\mu$$

for all $\varphi \in C_o(\mathbb{R}^d)$.

Random variables:

Say that X is a random variable with law given by μ, is to say $X : (\Omega, \mathcal{A}, P) \longrightarrow (\mathbb{R}^d, \mathcal{B}_d)$ is a mesurable map such that $X\#P = \mu$, i.e.,

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Kantorovich-Rubinstein-Wasserstein Distance:

$$W_1(\mu,\nu) = \inf_\pi \left\{ \iint_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|\, d\pi(x,y) \right\} = \inf_{(X,Y)} \{\mathbb{E}\,[|X-Y|]\}$$

where the transference plan π runs over the set of joint probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ with marginals f and $g \in \mathcal{P}_1(\mathbb{R}^d)$ and (X,Y) are all possible couples of random variables with μ and ν as respective laws.

Basic Properties

1. **Translation:** $f \in \mathcal{P}_1(\mathbb{R}^d)$ and $a \in \mathbb{R}^d$, let f_a denotes the translated f with vector a, then $W_1(f_a, f) = |a|$.
2. **Convergence of measures:** $W_1(f_n, f) \to 0$ is equivalent to $f_n \rightharpoonup f$ weakly-* as measures and convergence of first moments.
3. **Completeness:** The space $\mathcal{P}_1(\mathbb{R}^d)$ endowed with the distance W_1 is a complete metric space.

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Well-posedness in probability measures[2]

Existence, uniqueness and stability

Take a potential $U \in \mathcal{C}^2(\mathbb{R}^d)$, H locally Lipschitz, with

$$|\nabla U(x)| \leq C(1 + |x|) \quad \text{for all } x \in \mathbb{R}^d,$$

$$|H(x, v)| \leq C(1 + |x| + |v|) \quad \text{for all } x, v \in \mathbb{R}^d,$$

and f_0 a measure on $\mathbb{R}^d \times \mathbb{R}^d$ with compact support. There exists a solution $f \in \mathcal{C}([0, +\infty); \mathcal{P}_1(\mathbb{R}^d))$ in the sense of solving the equation through the characteristics: $f_t := P^t \# f_0$ with P^t the flow map associated to the equation.

Moreover, given any two solutions f and g with initial data f_0 and g_0, there is an increasing function depending on the size of the support of the solutions and the parameters, such that

$$W_1(f_t, g_t) \leq \alpha(t)\, W_1(f_0, g_0)$$

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Convergence of the particle method

- Empirical measures: if $x_i, v_i : [0,T) \to \mathbb{R}^d$, for $i = 1, \dots, N$, is a solution to the ODE system,

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$$f_N(t) := \sum_{i=1}^{N} m_i \delta_{(x_i(t), v_i(t))}$$

  is the solution corresponding to initial atomic measures.

- Convergence of approximations of measures by particles due to the stability at any given time T as an alternative derivation of the kinetic models.

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Let us consider the N_p-particle system:

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Due to translation invariancy, w.l.o.g. the mean velocity is zero and thus the center of mass is preserved along the evolution, i.e.,

$$\sum_{i=1}^{N_p} m_i v_i(t) = 0 \qquad \text{and} \qquad \sum_{i=1}^{N_p} m_i x_i(t) = x_c$$

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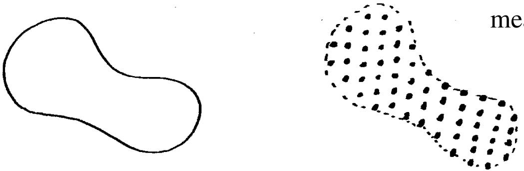
Asymptotic Flocking



Find a bound independent of the number of particles for the time it takes for all the particles to travel at the mean velocity.

Asymptotic Flocking

Unconditional Non-universal Flocking Result for Particles

The unique measure-valued solution for the CS kinetic model with $\gamma \leq 1/2$, with a finite number of particles given by

$$\tilde{\mu}(t) = \sum_{i=1}^{N_p} m_i \, \delta(x - x_i(t)) \, \delta(v - v_i(t)),$$

satisfies that

$$\lim_{t\to\infty} W_1\left(\tilde{\mu}(t), \tilde{\mu}^\infty\right) = 0$$

with

$$\tilde{\mu}^\infty = \sum_{i=1}^{N_p} m_i \, \delta(x - x_i^\infty - mt) \, \delta(v - m)$$

with m the initial mean velocity of the particles.

Asymptotic Flocking

Unconditional Non-universal Flocking Result for general measures

Given $\mu_0 \in \mathcal{M}(\mathbb{R}^{2d})$ compactly supported, then the unique measure-valued solution to the CS kinetic model with $\gamma \leq 1/2$, satisfies the following bounds on their supports:

$$\text{supp } \mu(t) \subset B(x_c(0) + mt, R^x(t)) \times B(m, R^v(t))$$

for all $t \geq 0$, with $R^x(t) \leq \bar{R}$ and $R^v(t) \leq R_0\, e^{-\lambda t}$ with $\bar{R}^x$ depending only on the initial support radius.

Moreover,

$$\lim_{t\to\infty} W_1(\mu_x^{mt}(t), L_\infty(\mu_0)) = 0,$$

where the measure $L_\infty(\mu_0)$ is defined as

$$\int_{\mathbb{R}^d} \zeta(x)\, dL_\infty(\mu_0)(x) = \int_{\mathbb{R}^{2d}} \zeta\left(x + \int_0^\infty [V(s;x,v) - m]\, ds\right) d\mu_0(x,v),$$

for all $\zeta \in \mathcal{C}_b^0(\mathbb{R}^d)$.

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Asymptotic Flocking

Let us fix any $R_0^x > 0$ and $R_0^v > 0$, such that all the initial velocities lie inside the ball $B(0, R_0^v)$ and all positions inside $B(x_c, R_0^x)$.

Let us define the function $R^v(t)$ to be

$$R^v(t) := \max\left\{|v_i(t)|\,,\, i = 1, \ldots, N_p\right\}.$$

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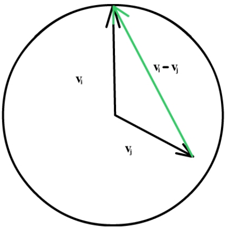
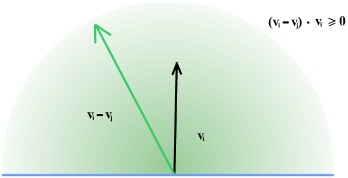
Asymptotic Flocking

Choosing the label i to be the one achieving the maximum, we get

$$\frac{d}{dt}R^v(t)^2 = \frac{d}{dt}|v_i|^2 = -2\sum_{j\neq i} m_j\left[(v_i - v_j)\cdot v_i\right]\, a(|x_i - x_j|)\,.$$

Because of the choice of the label i, we have that $(v_i - v_j)\cdot v_i \geq 0$ for all $j \neq i$ that together with $a \geq 0$ imply $R^v(t) \leq R^v_0$ for all $t \geq 0$.





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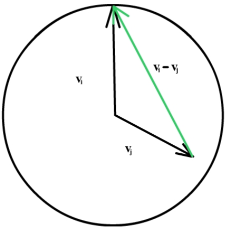

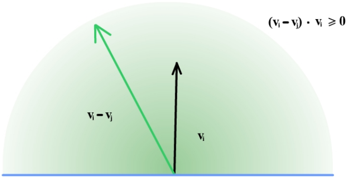

Asymptotic Flocking

Coming back to the equation for the positions,

$$|x_i(t) - x_i^0| \le R_0^v t \quad \text{for all } t \ge 0 \text{ and all } i = 1, \dots, N_p.$$

$$a(|x_i - x_j|) \ge \frac{1}{[1 + 4R_0^2(1+t)^2]^\gamma} \quad \text{for all } t \ge 0 \text{ and all } i, j = 1, \dots, N_p,$$

with $R_0 = \min(R_0^x, R_0^v)$.

Coming back to the equation for the maximal velocity

$$\begin{aligned}
\frac{d}{dt} R^v(t)^2 &= -2 \sum_{j \neq i} m_j \left[(v_i - v_j) \cdot v_i\right] a(|x_i - x_j|) \\
&\le -\frac{2}{[1 + 4R_0^2(1+t)^2]^\gamma} \sum_{j \neq i} m_j \left[(v_i - v_j) \cdot v_i\right] \\
&= -\frac{2}{[1 + 4R_0^2(1+t)^2]^\gamma} R^v(t)^2 := -f(t)\, R^v(t)^2,
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Asymptotic Flocking

Gronwall's lemma:

$$R^v(t) \le R_0^v \exp\left\{-\frac{1}{2}\int_0^t f(s)\,ds\right\}.$$

For $\gamma \le 1/2$, the function $f(t)$ is not integrable at ∞ and therefore

$$\lim_{t\to\infty}\int_0^t f(s)\,ds = +\infty$$

and $R^v(t) \to 0$ as $t \to \infty$ giving the convergence to a single point, its mean velocity, of the support for the velocity.
Again for the position variables, we get

$$\begin{cases} \displaystyle\int_0^t |v_i(s)|\,ds \le C_1 \int_0^t (1+s)^{-1-\epsilon}\,ds & \gamma < 1/2 \\ \displaystyle\int_0^t |v_i(s)|\,ds \le C \int_0^t \frac{1}{1+s}\,ds = C\ln(1+t) & \gamma = 1/2, \end{cases}.$$

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There exists $R_1^x > 0$ such that

$$|x_i(t) - x_i^0| \le R_1^x$$

Now, $a(|x_i(t) - x_j(t)|) \ge a(2\bar{R}^x)$,

$$\frac{d}{dt}R^v(t)^2 = -2\sum_{j\neq i} m_j \left[(v_i - v_j)\cdot v_i\right] a(|x_i - x_j|)$$
$$\le -2a(2\bar{R}^x)\sum_{j\neq i} m_j \left[(v_i - v_j)\cdot v_i\right] = -2a(2\bar{R}^x)R^v(t)^2$$

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Outline

Stochastic Particle System

General Interacting Particle System with Noise:

N interacting $\mathbb{R}^{2d}$-valued processes $(X_t^i, V_t^i)_{t\geq 0}$ with $1 \leq i \leq N$ solution of

$$\begin{cases} dX_t^i = V_t^i dt, \\ dV_t^i = \sqrt{2} dB_t^i - F(X_t^i, V_t^i)dt - \dfrac{1}{N}\displaystyle\sum_{j=1}^N H(X_t^i - X_t^j, V_t^i - V_t^j)dt, \end{cases}$$

with independent and commonly distributed initial data (X_0^i, V_0^i) with $1 \leq i \leq N$.

Empirical Measure:

$$\hat{f}_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{(X_t^i, V_t^i)}$$

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Coupling Method 1

Stochastic Particle System Associated to PDE:

N interacting processes $(\overline{X}_t^i, \overline{V}_t^i)_{t\geq 0}$ solutions of the kinetic McKean-Vlasov type equation on $\mathbb{R}^{2d}$:

$$\begin{cases} d\overline{X}_t^i = \overline{V}_t^i \, dt \\ d\overline{V}_t^i = \sqrt{2}\, dB_t^i - F(\overline{X}_t^i, \overline{V}_t^i)dt - H * f_t(\overline{X}_t^i, \overline{V}_t^i)dt, \\ (\overline{X}_0^i, \overline{V}_0^i) = (X_0^i, V_0^i), \quad f_t = \mathrm{law}(\overline{X}_t^i, \overline{V}_t^i). \end{cases}$$

The stochastic processes are independent and identically distributed according to

$$\partial_t f_t + v \cdot \nabla_x f_t = \Delta_v f_t + \nabla_v \cdot ((F + H * f_t) f_t), \quad t > 0, x, v \in \mathbb{R}^d.$$

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Coupling Method 2

Conjecture: The N interacting processes $(X_t^i, V_t^i)_{t\geq 0}$ behave as $N \to \infty$ like the processes $(\overline{X}_t^i, \overline{V}_t^i)_{t\geq 0}$ associated to the PDE.

More precisely, the objective is to estimate the convergence as $N \to \infty$ of

$$\mathbb{E}\left[|X_t^i - \overline{X}_t^i|^2 + |V_t^i - \overline{V}_t^i|^2\right] \leq \varepsilon(N)$$

Consequences

1. $f_t^{(1)}$ of any of the particles X_t^i at time t converges to f_t as N goes to infinity:

$$W_2^2(f_t^{(1)}, f_t) \leq \mathbb{E}\left[|X_t^i - \overline{X}_t^i|^2 + |V_t^i - \overline{V}_t^i|^2\right] \leq \varepsilon(N)\,.$$

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Coupling Method 3

Consequences

2. Propagation of chaos: The law $f_t^{(k)}$ of any k particles (X_t^i, V_t^i) converges to the tensor product $f_t^{\otimes k}$ as N goes to infinity:

$$W_2^2(f_t^{(k)}, f_t^{\otimes k}) \le k\varepsilon(N).$$

3. Convergence of the empirical measure $\hat{f}_t^N$ to f_t: if φ is a Lipschitz map on $\mathbb{R}^{2d}$, then

$$\mathbb{E}\left[\left|\frac{1}{N}\sum_{i=1}^{N}\varphi(X_t^i, V_t^i) - \int_{\mathbb{R}^{2d}}\varphi\, df_t\right|^2\right]$$

$$\le 2\,\mathbb{E}\left[|\varphi(X_t^i, V_t^i) - \varphi(\overline{X}_t^i, \overline{V}_t^i)|^2 + \left|\frac{1}{N}\sum_{i=1}^{N}\varphi(\overline{X}_t^i, \overline{V}_t^i) - \int_{\mathbb{R}^{2d}}\varphi\, df_t\right|^2\right]$$

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Main Result

Previous Results: If the functions involved F and H are globally Lipschitz then there are classical results by Snitzman and Meleard, implying that

$$\varepsilon(N) = O\left(\frac{1}{N}\right)$$

The typical F and H in our Cucker-Smale and D'Orsogna etal model are not globally Lipschitz.

Hypotheses:

Assume that F and H with $H(-x,-v) = -H(x,v)$, satisfy

$$-(v-w)\cdot(F(x,v)-F(x,w)) \leq A\,|v-w|^2$$
$$|F(x,v)-F(y,v)| \leq L\min\{|x-y|,1\}(1+|v|^p)$$

for all x, y, v, w in $\mathbb{R}^d$, and analogously for H instead of F.

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Main Result 2

Properties of the Stochastic Processes and PDE:

Assume that the particle system and the processes have global solutions on $[0, T]$ with initial data (X_0^i, V_0^i) such that the uniform moment condition holds:

$$\sup_{0\le t\le T}\left\{\int_{\mathbb{R}^{4d}}|H(x-y,v-w)|^2 df_t(x,v)df_t(y,w)+\int_{\mathbb{R}^{2d}}(|x|^2+e^{a|v|^p})df_t(x,v)\right\} < +\infty$$

with $f_t = law(\overline{X}_t^i, \overline{V}_t^i)$.

Main Result 3

Claim:

There exists a constant $C > 0$ such that

$$\mathbb{E}\left[|X_t^i - \overline{X}_t^i|^2 + |V_t^i - \overline{V}_t^i|^2\right] \leq \frac{C}{N^{e^{-Ct}}}$$

for all $0 \leq t \leq T$ and $N \geq 1$.

Moreover, if additionally the uniform moment condition holds with $p' > p$ then for all $0 < \epsilon < 1$ there exists a constant C such that

$$\mathbb{E}\left[|X_t^i - \overline{X}_t^i|^2 + |V_t^i - \overline{V}_t^i|^2\right] \leq \frac{C}{N^{1-\epsilon}}$$

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Outline

Step 0.- Fluctuations:

Fluctuations: $x_t^i := X_t^i - \overline{X}_t^i$, $v_t^i := V_t^i - \overline{V}_t^i$, $i = 1, \ldots, N$.

Coupling: the Brownian motions $(B_t^i)_{t\geq 0}$ are equal for the stochastic interacting particle system and for the processes

$$dx^i = v^i \, dt \,,$$

$$dv^i = -\left(F(X^i, V^i) - F(\overline{X}^i, \overline{V}^i)\right) dt - \frac{1}{N}\sum_{j=1}^{N}\left(H(X^i - X^j, V^i - V^j) - (H * f_t)(\overline{X}^i, \overline{V}^i)\right) dt.$$

Consider the quantity

$$\alpha(t) = \mathbb{E}\left[|x^i|^2 + |v^i|^2\right]$$

independent of the label i by symmetry.

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$$\frac{1}{2}\frac{d}{dt}\mathbb{E}\left[|x^i|^2\right] = \mathbb{E}\left[x^i \cdot v^i\right] \le \frac{1}{2}\alpha(t)$$

and

$$\frac{1}{2}\frac{d}{dt}\mathbb{E}\left[|v^i|^2\right] = -\mathbb{E}\left[v^i \cdot \left(F(X^i,V^i) - F(\overline{X}^i,\overline{V}^i)\right)\right] - \frac{1}{N}\mathbb{E}\left[\sum_{j=1}^N v^i \cdot \left(H(X^i - X^j, V^i - V^j) - H * f_t(\overline{X}^i,\overline{V}^i)\right)\right] =: I_1 + I_2.$$

Step 1.- Localization Estimate for I_1:

Using the hypotheses on F:

$$I_1 \leq A\,\mathbb{E}\left[|v^i|^2\right] + L\,\mathbb{E}\left[|v^i| \min\{|x^i|, 1\}\left(1 + |\overline{V}^i|^p\right)\right] := I_{11} + L\,I_{12}\,.$$

Localizing in V + Markov's inequality:

$$I_{12} \leq \,(1 + R^p)\alpha(t) + \frac{1}{2}\left(\mathbb{E}\left[|\overline{V}^i|^{4p}\right]\right)^{1/2}\left(e^{-aR^p}\,\mathbb{E}\left[e^{a|\overline{V}_t^i|^p}\right]\right)^{1/2}$$

Final Estimate: given $T > 0$, there exists $C > 0$ such that

$$I_1 \leq C(1 + r)\,\alpha(t) + C\,e^{-r}$$

holds for all $r > 0$ and all $0 \leq t \leq T$.

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Step 2.- Localization Estimate for I_2:

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\begin{aligned}
I_2 = & -\frac{1}{N}\mathbb{E}\left[\sum_{j=1}^{N} v^i \cdot \left(H(X^i - X^j, V^i - V^j) - H(\overline{X}^i - \overline{X}^j, \overline{V}^i - \overline{V}^j)\right)\right] \\
& -\frac{1}{N}\mathbb{E}\left[v^i \cdot \left(H(0,0) - (H * f_t)(\overline{X}^i, \overline{V}^i)\right)\right] \\
& -\frac{1}{N}\mathbb{E}\left[\sum_{j\neq i}^{N} v^i \cdot \left(H(\overline{X}^i - \overline{X}^j, \overline{V}^i - \overline{V}^j) - (H * f_t)(\overline{X}^i, \overline{V}^i)\right)\right] \\
=: & \; I_{21} + I_{22} + I_{23}.
\end{aligned}
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Localization in I_{21}: given $T > 0$, there exists $C > 0$ such that

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Moment bounds:

$$I_{22} \leq \frac{1}{N}\left(\mathbb{E}\left[|v^i|^2\right]\right)^{1/2}\left(\mathbb{E}\left[\left|(H*f_t)(\overline{X}^i,\overline{V}^i)\right|^2\right]\right)^{1/2} \leq \frac{C}{N}\sqrt{\alpha(t)}\,.$$

Standard Argument of Law of Large Numbers: (Snitzman)

$$I_{23} \leq \frac{1}{N}\left(\mathbb{E}\left[|v^1|^2\right]\right)^{1/2}\left(\mathbb{E}\left[\left|\sum_{j=2}^{N} Y^j\right|^2\right]\right)^{1/2} \leq \frac{C}{\sqrt{N}}\sqrt{\alpha(t)}$$

where $Y^j := H(\overline{X}^1-\overline{X}^j,\overline{V}^1-\overline{V}^j)-(H*f_t)(\overline{X}^1,\overline{V}^1)$ for $j\geq 2$.

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Step 3.- Conclusion:

First Result: given $T > 0$, there exists $C > 0$ such that

$$\alpha'(t) \le C\,(1+r)\,\alpha(t) + Ce^{-r} + \frac{C}{\sqrt{N}}\sqrt{\alpha(t)} \le C\,(1+r)\,\alpha(t) + Ce^{-r} + \frac{C}{N}$$

for all $t \in [0, T]$, all $N \ge 1$ and all $r > 0$. This implies due to changes of variables:

$$u' \le -u \ln u + \frac{1}{N}$$

implying the first statement.

Second Result: A better localization implies that given $T > 0$, there exists $C > 0$ such that

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