Entropy-entropy dissipation inequalities for fragmentation-drift equations

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Fragmentation

Consider a set of particles of different sizes that can break into several pieces, in some random fashion.

$f(t, x) \equiv f(x) \equiv$ density of clusters of size $x > 0$ at time $t \geq 0$.

$b(x, y) \equiv$ probability of obtaining a cluster of size y when a cluster of size x breaks.

Fragmentation equation

$$\partial_t f(x) = Ff(x) := \int_x^\infty b(y, x) f(y)\, dy - f(x) \int_0^x \frac{y}{x} b(x, y)\, dy$$

Typical coefficients

The coefficient of the negative term is

$$B(x) := \int_0^x \frac{y}{x} b(x,y)\, dy,$$

the *total fragmentation rate*. It is the one for which fragmentation conserves the *total mass*:

$$\int_0^\infty x\, Ff(x)\, dx = 0.$$

Typical fragmentation coefficient: think of...

$$b(x,y) = x^{\gamma-1} \quad \text{for some } \gamma \geq 0, \text{or}$$

$$b(x,y) = \delta_{y=x/2} \quad (\textit{mitosis case})$$

Self-similar fragmentation

Under fragmentation, the size of clusters becomes smaller and smaller. We rescale to study the long-time shape of the distribution, as it concentrates around size 0:

$$n(t,x) := e^{-2t} f(e^{\gamma t} - 1, e^{-t}x) \qquad (t, x > 0).$$

Then,

Self-similar fragmentation equation

$$\partial_t n + x\,\partial_x n + 2n = \gamma\, Fn$$

Fragmentation-growth

Now, consider a set of particles of different sizes (cells?) that can randomly break into several pieces, and that also grow at a fixed rate.

$f(t,x), b(x,y)$ as before

Fragmentation-growth equation

$$\partial_t f = -\partial_x f + Ff$$

Rescaled growth-fragmentation

Distribution grows large in time: we rescale as $n(t,x) := e^{-\lambda t} f(t,x)$ to get

Rescaled fragmentation-growth equation

$$\partial_t n = -\lambda n - \partial_x n + Fn$$

The precise λ must be chosen for this equation to have equilibria: take λ to be the largest eigenvalue of the operator in the r.h.s. (Malthusian growth rate of the population)

The dual equation: eigenproblem

Find λ, ϕ such that:

$$-\partial_x \phi = \int_0^x b(x,y)\phi(y)\,dy - (B(x)+\lambda)\,\phi(x),$$

$$\phi \geq 0, \quad \int_0^\infty N(x)\phi(x)\,dx = 1.$$

Estimates can be obtained through a maximum/minimum principle.

The eigenproblem

In the case of fragmentation, the dual equation is very easy: $\phi(x) = x$.
In the case of growth-fragmentation, the dual equation is in general difficult to solve: no evident conserved quantity.

- The case of a constant total fragmentation rate is the simplest: $\lambda = 2$, $\phi(x) \equiv 1$.
- Otherwise, one has to adjust the λ to the right value and solve for ϕ, a problem studied by Michel (2006), Doumic & Gabriel (2010). With careful estimates, we are able to show that $\phi(x)$ is bounded below, and looks like x for x large.

Asymptotic behavior

So once rescaled, we look at either

$$\partial_t n = -2n - x\,\partial_x n + \gamma\, Fn$$

or

$$\partial_t n = -\lambda n - \partial_x n + Fn$$

As $t \to +\infty$, n converges to a limiting shape (self-similar profile).

How fast?

or:

Do the linear operators on the right hand sides have spectral gaps?

Existing results: fragmentation

- Existence and uniqueness of self-similar profiles, and some of their properties (behavior near $x = 0$ and $x = +\infty$) was proved by Escobedo, Mischler and Rodríguez Ricard (2004/2005).
- Convergence to equilibrium was also proved here using entropy methods (more later).
- No results on rate.

Existing results: growth-fragmentation

- Perthame and Ryzhik (2005) proved the above for the mitosis case, with constant total fragmentation rate, or perturbations of these. They showed an exponential rate of convergence for these coefficients.
- Michel (2006) and Doumic & Gabriel (2010) studied, for general coefficients, the dual eigenproblem for the growth-fragmentation equation, which essentially implies convergence to equilibrium via entropy methods.
- Laurençot and Perthame (2009) proved exponential decay for a constant total fragmentation rate, allowing for more general kinds of breakage than mitosis.

Entropy method

In a quite general setting, linear equations which come from a Markov process have many entropy functionals: in our case, take the quadratic one:

$$\frac{d}{dt}\int_0^\infty \left(\frac{n(x)}{N(x)}-1\right)^2 N(x)\phi(x)\,dx = \\ -\int_0^\infty \int_y^\infty \phi(y)b(x,y)N(x)\left(\frac{n(x)}{N(x)}-\frac{n(y)}{N(y)}\right)^2\,dx\,dy$$

$\phi \equiv$ a function obtained from the solution of an eigenvalue problem (more to follow),
$N \equiv$ equilibrium, normalized so that $\int N\phi = 1$.

General relative entropy

A general way to find entropy functionals in many linear equations was given by Michel, Mischler & Perthame (2004/2005). If you have a linear equation with a differential and a “scattering” part (like ours)

$$\partial_t n = Ln$$

assume you can find a stationary profile N for our equation, and ϕ of the dual operator L^*:

$$LN = 0, \quad L^* \phi = 0.$$

Then, for any convex H, this is an entropy of the equation:

$$\int H\left(\frac{n(x)}{N(x)}\right) N(x)\phi(x)\,dx.$$

Entropy method (II)

Call H, D the entropy and dissipation of entropy, resp. Then,

$$\frac{d}{dt}H = -D,$$

and if we prove the following functional inequality

$$H \leq CD$$

for some constant C, we have a speed of convergence to equilibrium:

$$H(t) \leq e^{-Ct}H(0).$$

(Observe that $\sqrt{H} = \|n - N\|_2$, where the L^2 norm is w.r.t. the measure $\phi(x)/N(x)\,dx$)

Functional inequality

So we are led to look at the following inequality:

Entropy-entropy dissipation inequality

$$\int_0^\infty (u(x)-1)^2 N(x)\phi(x)\,dx \le$$
$$C\int_0^\infty \int_y^\infty \phi(y)b(x,y)N(x)\left(u(x)-u(y)\right)^2\,dx\,dy,$$

valid for functions u with $\int uN\phi = 1$.
Can we prove it, for some $C > 0$?
Not always: there are examples of b for which $D = 0$ does not even imply $H = 0$ (e.g., mitosis: $b(x,y) = 2\delta_{y=x/2}$).
What happens with $b(x,y) = x^{\gamma-1}$?

A general type of inequality

Step away a bit from the problem: we want inequalities between the (quadratic) entropy and its dissipation **for a Markov process.**

Let us see some examples.

Some examples

1. The kind of inequality we want:

$$\int_x (u(x) - 1)^2 \, N(x) \leq C \int_x \int_y N(x) K(x,y) \, (u(x) - u(y))^2$$

2. Fokker-Planck:

$$\int_{\mathbb{R}^d} (u(x) - 1)^2 N(x) \, dx \leq C \int_{\mathbb{R}^d} N(x) |\nabla u(x)|^2$$

3. Discrete Markov process:

$$\sum_i (u_i - 1)^2 N_i \leq C \sum_{i,j} N_i \, K_{i,j} \, (u_i - u_j)^2$$

A general strategy

First, “double the variables”:

$$\sum_i N_i\,(u_i - 1)^2 = \frac{1}{2}\sum_{i,j} N_i\,N_j\,(u_i - u_j)^2.$$

(The normalization is $\sum u_i N_i = 1$). Now it looks more similar to the dissipation:

$$\sum_{i,j} N_i\,K_{i,j}\,(u_i - u_j)^2$$

- If we can compare the integrands, it works!
- Otherwise, the inequality is trickier, or may be false.

A general strategy (II)

An idea from Diaconis & Stroock (Ann. Appl. Prob. '91), in the discrete case: for any (i,j), choose a path of jumps from i to j, with jumps $(i,i_1),(i_1,i_2),\dots,(i_m,j)$. Call it γ_{ij}.

$$\begin{aligned}\sum_{i,j} N_i\,N_j\,(u_i-u_j)^2 &= \sum_{i,j} N_i\,N_j\left(\sum_{(k,l)\in\gamma_{ij}}(u_k-u_l)\right)^2\\ &\leq \sum_{i,j} N_i\,N_j\,\|\gamma_{ij}\|\sum_{(k,l)\in\gamma_{ij}} N_k K_{k,l}(u_k-u_l)^2\\ &= \sum_{k,l} N_k K_{k,l}(u_k-u_l)^2 \sum_{\gamma_{ij}\ni(k,l)} N_i\,N_j\,\|\gamma_{ij}\|,\end{aligned}$$

where the "length" of a path γ_{ij} is $\|\gamma_{ij}\| := \sum_{(k,l)\in\gamma_{ij}}(N_k K_{k,l})^{-1}$.

A general strategy: remarks

- Making a good choice of paths is particular to each problem!
- The strategy for the discrete case suggests how to treat continuous ones.
- Finding a constant in the inequality for a finite system which does not depend on the size of the system amounts to showing inequalities in continuous cases.
- An additional possibility: instead of choosing *one* path for each pair, choose *a set of paths* for each pair, and take the mean in the sum above. This may improve the result! (See later).
- A similar strategy is used in the work of Baranger & Mouhot on explicit spectral gaps for the linearized Boltzmann equation.

An illustration: weighted Poincaré inequalities

Same idea, in a continuous setting: for any two points on $\mathbb{R}^d$, choose a continuous path that joins them; e.g., a segment:

$$\begin{aligned}
&\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} N(x) N(y) (u(x) - u(y))^2 \, dx \, dy \\
&\qquad = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} N(x) N(y) \left(\int_y^x \nabla u(z) \, dz \right)^2 \, dx \\
&\qquad \leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} N(x) N(y) |x - y| \int_y^x |\nabla u(z)|^2 \, dz \, dx \\
&\qquad = \int_{\mathbb{R}^d} |\nabla u(z)|^2 \int_{[x,y] \ni z} N(x) N(y) |x - y| \, d(x,y) \, dz.
\end{aligned}$$

Back to fragmentation

For fragmentation-drift, we wanted to show that:

$$\int_0^\infty \int_y^\infty \phi(y)\phi(x)N(y)N(x)\left(u(x)-u(y)\right)^2\,dx\,dy \\ \leq C \int_0^\infty \int_y^\infty \phi(y)b(x,y)N(x)\left(u(x)-u(y)\right)^2\,dx\,dy.$$

Here, $\phi(y)$ is approximately equal to y (exactly equal for self-similar fragmentation), N is a nice function with exponential decay at $+\infty$, and the tentative inequality should be

$$\int_0^\infty \int_y^\infty x\,y\,N(y)N(x)\left(u(x)-u(y)\right)^2\,dx\,dy \\ \leq C \int_0^\infty \int_y^\infty y\,x^{\gamma-1}N(x)\left(u(x)-u(y)\right)^2\,dx\,dy.$$

Can we compare the integrands in our model cases? So, can we compare...

$$x\,y\,N(y)N(x) \le Cyx^{\gamma-1}N(x) \qquad (x, y > 0)$$

this is,

$$N(y) \le Cx^{\gamma-2} \qquad (x, y > 0)?$$

This clearly fails for x large!

Idea of the proof

Let us try the strategy explained before: we pass through "intermediate reactions" in order to compare terms which do not correspond to the same x, y:

$$u(x) - u(y) = u(x) - u(z) + u(z) - u(y).$$

We need the additional trick: we average among a range of possible splittings:

$$\begin{aligned} u(x) - u(y) &= \int_2^3 (u(x) - u(y/s) + u(y/s) - u(y))\, ds \\ &= \int_{y/3}^{y/2} \frac{y}{z^2} (u(x) - u(z) + u(z) - u(y))\, dz. \end{aligned}$$

Then, by Cauchy-Schwarz's inequality,

$$\begin{aligned}(u(x)-u(y))^2 &\leq \frac{y}{6}\int_{y/3}^{y/2}\frac{y^2}{z^4}(u(x)-u(z)+u(z)-u(y))^2\,dz\\ &\leq \frac{27}{2\,y}\int_{y/3}^{y/2}(u(x)-u(z)+u(z)-u(y))^2\,dz\\ &\leq \frac{27}{y}\int_{y/3}^{y/2}(u(x)-u(z))^2\,dz+\frac{27}{y}\int_{y/3}^{y/2}(u(z)-u(y))^2\,dz.\end{aligned}$$

One then has to plug these two terms in the l.h.s. of the inequality, and exchange the order of integration. But the gain is clear here: one can gain up to one order in the comparison we tried before.

First term

$$
\begin{aligned}
&\int_0^\infty \int_{\max\{4x,M\}}^\infty x\,N(x)N(y) \int_{y/3}^{y/2} (u(x)-u(z))^2\,dz\,dy\,dx \\
&= \int_0^\infty \int_{\max\{\frac{4x}{3},\frac{M}{3}\}}^\infty x\,N(x)(u(x)-u(z))^2 \int_{\max\{4x,2z,M\}}^{3z} N(y)\,dy\,dz\,dx \\
&\leq C \int_0^\infty \int_{\max\{\frac{4x}{3},\frac{M}{3}\}}^\infty x\,N(x)(u(x)-u(z))^2 \frac{1}{1+z} \int_{2z}^\infty y\,N(y)\,dy\,dz\,dx \\
&\qquad \leq C \int_0^\infty \int_{\frac{4x}{3}}^\infty x\,N(x)\,\frac{N(z)}{1+z}(u(x)-u(z))^2\,dz\,dx \leq C\,D,
\end{aligned}
$$

Second term

$$
\begin{aligned}
&\int_0^\infty \int_{\max\{4x,M\}}^\infty x\, N(x)N(y) \int_{y/3}^{y/2} (u(z)-u(y))^2 \,dz\,dy\,dx \\
&\quad \leq C \int_0^\infty \int_{4x}^\infty x\, N(x)N(y) \int_{y/3}^{y/2} (u(z)-u(y))^2 \,dz\,dy\,dx \\
&\quad = C \int_0^\infty \int_{y/3}^{y/2} N(y)(u(z)-u(y))^2 \int_0^{y/4} xN(x)\,dx\,dz\,dy \\
&\quad \leq C \int_0^\infty \int_{y/3}^{y/2} N(y)(u(z)-u(y))^2 \,dz\,dy \\
&\quad \leq C \int_0^\infty \int_{y/3}^{y/2} \frac{z}{1+y} N(y)(u(z)-u(y))^2 \,dz\,dy \leq C\, D.
\end{aligned}
$$

Theorem

The entropy-entropy dissipation inequality mentioned before is true for $b(x,y) = y^{\gamma-1}$ *for the range of* γ *below.*

Hence: assume

- $0 \leq \gamma < 2$ *for growth-fragmentation.*
- $0 < \gamma < 2$ *for self-similar fragmentation.*

Then, for any solution n to one of these models,

$$\|n - N\|_2 \leq e^{-Ct} \|n_0 - N\|_2,$$

for some constant $C > 0$*.*

Estimates we need

For the previous proof to work for a given stationary profile *N* and dual profile ϕ we need:

1. Exponential decay estimates on *N* (by above and below).
2. Accurate estimates on ϕ (above and below by some power).

Let us do them in the growth-fragmentation case (more difficult for this).

Estimates on N: moments

First, study the moments of N: if M_k is the moment of order k,

$$(1 - \frac{1}{k}) M_{\gamma + k} \leq k M_{k-1}.$$

Summing to get exponential moments,

$$\int_0^\infty \exp(a x^{\gamma + 1}) N(x) \, dx < +\infty.$$

Estimates on N: bounds above and below

Trick: define

$$\Lambda(x) := \lambda x + \int_0^x B(x)\,dx.$$

When $b(x,y) = 2x^{\gamma-1}$, then $N(x) = e^{-\Lambda(x)}$ is is *an exact solution*.
Otherwise, we calculate:

$$(Ge^{a\Lambda})' = (a-1)(B+\lambda)Ge^{a\Lambda} + e^{a\Lambda}\mathcal{L}_+(G).$$

- Right hand side integrable $\Longrightarrow$ upper bound.
- Right hand side nonnegative $\Longrightarrow$ lower bound.

Estimates on N: result

Finally, this gives

$$C_1\, e^{-a_1 x^{\gamma+1}} \le G(x) \le C_2\, e^{-a_2\, x^{\gamma+1}} \qquad (x > \delta)$$

for some constants $a_1, a_2, C_1, C_2 > 0$.

Maximum principle for ϕ

ϕ is limit of solutions of

$$0 = \mathcal{S}\phi_L(x) := -\phi_L'(x) + (\lambda_L + B(x))\,\phi_L(x) - \int_0^x b(x,y)\,\phi_L(y)\,dy.$$

Lemma (Maximum principle for $\mathcal{S}$)

Take $A \geq 1/\lambda_L$. If $\mathcal{S}w \geq 0$ on (A, L), $w \geq 0$ on $[0, A]$ and $w(L) \geq 0$ then $w \geq 0$ on $[A, L]$.

Maximum principle: proof

Assume $x_0 \in (A, L)$ with $w(x_0) < 0$, and such that $w(x_0)/x_0$ is minimum. Then $w'(x_0) = w(x_0)/x_0$, so

$$\begin{aligned}\mathcal{S}(x_0) &= -\frac{w(x_0)}{x_0} + (\lambda_L + B(x_0))\, w(x_0) - \int_0^{x_0} b(x_0, y)\, w(y)\, dy \\ &\le -\frac{w(x_0)}{x_0} + (\lambda_L + B(x_0))\, w(x_0) - \frac{w(x_0)}{x_0} \int_0^{x_0} b(x_0, y)\, y\, dy \\ &= w(x_0)\Big(\lambda_L - \frac{1}{x_0}\Big) < 0,\end{aligned}$$

which contradicts that $\mathcal{S}w \ge 0$ on (A, L).

Estimates for ϕ (result)

By comparing with $w(x) = C(1+x)$ and $w(x) = x^k \phi$ (with $k < 1$ and ϕ a truncation so that $w(L) = 0$), we can prove

$$C_k x^k \leq \phi(x) \leq C(1+x)$$

when $\gamma > 0$.

Thanks!