ASYMPTOTICS FOR THE CRITICAL MASS KELLER-SEGEL MODEL

Adrien Blanchet – GREMAQ – Université de Toulouse

PDEs in kinetic theories: kinetic description of biological models

Joint works with J. A. Carrillo & E. Carlen

PLAN

1. INTRODUCTION

2. MAIN RESULTS

3. IDEAS OF THE PROOF

THE KELLER-SEGEL MODEL

THE KELLER-SEGEL SYSTEM

$$\begin{cases} \dfrac{\partial \rho}{\partial t} = \Delta \rho - \mathrm{div}(\rho \nabla c) & \text{in } (0,+\infty)\times\mathbb{R}^2 \\ \Delta c = -\rho & \text{in } (0,+\infty)\times\mathbb{R}^2 \,, \\ \rho(t=0) = \rho_0 \geq 0 & \text{in } \mathbb{R}^2 \,. \end{cases} \tag{KS}$$

Defining

$$G(|x|) := -\frac{1}{2\pi} \log|x| \,,$$

we can rewrite (KS) as

$$\frac{\partial \rho}{\partial t} = \Delta \rho - \mathrm{div}(\rho \nabla G * \rho) \quad \text{in } (0,+\infty)\times\mathbb{R}^2$$

Define the masse as

$$\int_{\mathbb{R}^2} \rho(x,t)\ \mathrm{d}x = \int_{\mathbb{R}^2} \rho_0(x)\ \mathrm{d}x =: M\ .$$

We compute

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^2} |x|^2 \rho(t,x)\,\mathrm{d}x = 4\,M \left(1 - \frac{M}{8\pi}\right)\ .$$

BLOW-UP CRITERION

If $M > 8\pi$ and

$$\int_{\mathbb{R}^2} |x|^2 \rho_0(x) < \infty$$

then the solutions to (KS) blowup in finite time.

THE FREE ENERGY FUNCTIONAL

$$\mathcal{F}_{\mathrm{PKS}}[\rho] = \int_{\mathbb{R}^2} \rho(x) \log \rho(x) \, \mathrm{d}x + \frac{1}{4\pi} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \rho(x) \log |x-y| \rho(y) \, \mathrm{d}x \, \mathrm{d}y \; .$$

If ρ is a smooth solution to (KS) then

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{F}_{\mathrm{PKS}}[\rho(t)] = - \int_{\mathbb{R}^2} \rho \, |\nabla \log \rho - \nabla c|^2 \, \mathrm{d}x \; .$$

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LOGARITHMIC HARDY-LITTLEWOOD-SOBOLEV'S INEQUALITY [CARLEN-LOSS, 1992]

Let $f \in L^1_+(\mathbb{R}^2)$ such that $f\log f$ and $f\log(1+|x|^2)$ are bounded in $L^1(\mathbb{R}^2)$. If $\int_{\mathbb{R}^2} f\,\mathrm{d}x = M$, then

$$\int_{\mathbb{R}^2} f\log f + \frac{2}{M}\iint_{\mathbb{R}^2\times\mathbb{R}^2} f(x)\log|x-y|f(y)\,\mathrm{d}x\,\mathrm{d}y \geq -C(M)\ . \qquad \text{(logHLS)}$$

Let $\lambda \geq 0$, the minimisers of (logHLS) are the translations of

$$\bar{\rho}_\lambda(x) := \frac{M}{\pi}\frac{\lambda}{(\lambda+|x|^2)^2}\ .$$

Global existence:

$$\left(1-\frac{M}{8\pi}\right)\int_{\mathbb{R}^2}\rho\log\rho \leq \mathcal{F}_{\mathrm{PKS}}[\rho_0] + C(M)\frac{M}{8\pi} < \infty \quad \text{if } M < 8\pi.$$

CLASSICAL RESULTS

KNOWN RESULTS [B., CARRILLO, DOLBEAULT, MASMOUDI, PERTHAME, ...]

Under the assumptions

$$\rho_0 \geq 0\,, \quad \rho_0 \in L^1(\mathbb{R}^2)\,, \quad \rho_0 \log \rho_0 \in L^1(\mathbb{R}^2) \quad \text{and} \quad |x|^2 \rho_0 \in L^1(\mathbb{R}^2)\,. \tag{H}$$

- If $M < 8\pi$, solutions to (KS) exist globally in time and converge to the self-similar profile.
- If $M = 8\pi$, solutions to (KS) exist globally in time and blowup as a Dirac mass of mass $8\,\pi$ centred at the centre of mass in infinite time.
- If $M > 8\pi$, solutions to (KS) blowup in finite time.

SETTING OF THIS TALK

In the sequel the mass $M = 8\pi$ and we do not assume $|x|^2 \rho_0 \in L^1(\mathbb{R}^2)$.

PLAN

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A SECOND LYAPUNOV FUNCTIONAL: FOR THE CFD EQUATION

THE "CRITICAL" NONLINEAR FOKKER-PLANCK EQUATION

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta\left(\sqrt{u}\right) + \dfrac{1}{\sqrt{2\lambda}}\,\mathrm{div}(x\,u) & t > 0\,,\ x \in \mathbb{R}^2\,, \\ u(0) = u_0 \geq 0 & x \in \mathbb{R}^2\,, \end{cases} \tag{1}$$

Define

FAST DIFFUSION FUNCTIONAL

$$\mathcal{H}_\lambda[u] := \int_{\mathbb{R}^2} \frac{\left(\sqrt{u} - \sqrt{\bar\rho_\lambda}\right)^2}{\sqrt{\bar\rho_\lambda}}\,\mathrm{d}x$$

It follows that for classical solutions u of (1) ,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_\lambda[u(t)] = -\int_{\mathbb{R}^2} u(t,x)\left|\nabla\left(\frac{1}{\sqrt{\bar\rho_\lambda}} - \frac{1}{\sqrt{u}}\right)\right|^2\,\mathrm{d}x \leq 0\,.$$

The $\bar\rho_\lambda$ are stationary solutions of (KS).

A SECOND LYAPUNOV FUNCTIONAL: FOR THE PKS SYSTEM 8π CASE

If ρ is the smooth solution to (KS) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_\lambda[\rho(t)] = -8\int_{\mathbb{R}^2} |\nabla(\rho^{1/4})|^2 \,\mathrm{d}x + \int_{\mathbb{R}^2} \rho^{3/2} \,\mathrm{d}x =: \mathcal{D}[\rho(t)]\ .$$

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GAGLIARDO-NIRENBERG-SOBOLEV INEQUALITY, [DEL PINO, DOLBEAULT]

For all functions f in $\mathbb{R}^2$ with a square integrable distributional gradient ∇f,

$$\pi\int_{\mathbb{R}^2} |f|^6\,\mathrm{d}x \le \int_{\mathbb{R}^2} |\nabla f|^2\,\mathrm{d}x \int_{\mathbb{R}^2} |f|^4\,\mathrm{d}x\ ,$$

and there is equality if and only if f is a multiple of a translate of $\bar{\rho}_\lambda^{1/4}$ for some $\lambda > 0$.

DISSIPATION OF $\mathcal{H}_\lambda$

For all solution ρ to (KS) of mass $M = 8\pi$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_\lambda[\rho] = \mathcal{D}[\rho(t)] \le 0\ ,$$

and moreover, there is equality if and only ρ is a translate of $\bar{\rho}_\lambda$ for some $\lambda > 0$.

CONCENTRATION CONTROL

CONCENTRATION CONTROL FOR $\mathcal{F}_{\mathrm{PKS}}$

Let ρ be any density with mass 8π, with $\rho\log\rho$ integrable and bounded in $L^1(\mathbb{R}^2, \log(e+|x|^2)\,\mathrm{d}x)$. Let $C_2 = 2\lambda^{-1/4}8^{3/4}\sqrt{\pi}$. Given $0<\varepsilon_0<8\pi$, there exists $0<\gamma_1\le 1$ depending only ε_0, such that if

$$\mathcal{H}_\lambda[\rho] < \left(\frac{8\,\pi-\varepsilon_0}{2\,C_2}\right)^2$$

then there exists a finite positive constant C_{CCF}, depending only on λ and ε_0, such that

$$\gamma_1\int_{\mathbb{R}^2}\rho\log_+\rho\,\mathrm{d}x \le \mathcal{F}_{\mathrm{PKS}}[\rho] + C_{\mathrm{CCF}}\,.$$

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$$\gamma_1 \int_{\mathbb{R}^2} \rho \log_+ \rho \,\mathrm{d}x \leq \mathcal{F}_{\mathrm{PKS}}[\rho] + C_{\mathrm{CCF}}\,.$$

CONCENTRATION CONTROL FOR $\mathcal{D}$

Let ρ be a any density in $L^{3/2}(\mathbb{R}^2)$ with mass 8π. For any $\gamma_2 \in (0, 4\,\pi)$, if

$$\mathcal{H}_\lambda[\rho] \leq \left(\frac{\gamma_2}{2C_2}\right)^2 \;,$$

then there exist a finite positive constant C_{CCD}, depending only on λ and γ_2, such that:

$$\gamma_2 \int_{\mathbb{R}^2} |\nabla(\rho^{1/4})|^2 \,\mathrm{d}x \leq \pi\mathcal{D}[\rho] + C_{\mathrm{CCD}}\,.$$

GLOBAL EXISTENCE AND LARGE TIME BEHAVIOUR

Given any density ρ_0 with total mass 8π such that there exists $\lambda > 0$ with

$$\mathcal{H}_\lambda[\rho_0] < C_{\mathrm{I}} \ ,$$

where $C_{\mathrm{I}} = (\gamma_2/2C_2)^2$. Then there exists $\rho \in \mathcal{AC}^0([0,T],\mathcal{P}_2(\mathbb{R}^2))$, with $\rho(t) \in L^1(\mathbb{R}^2)$ for all $t \geq 0$ being a global-in-time weak solution of (KS). Moreover, the solutions constructed satisfy

$$\mathcal{F}_{\mathrm{PKS}}[\rho(t)] \leq \mathcal{F}_{\mathrm{PKS}}[\rho_0] \ ,$$

and

$$\mathcal{H}_\lambda[\rho(t)] + \int_0^t \mathcal{D}[\rho(t)] \, \mathrm{d}t \leq \mathcal{H}_\lambda[\rho_0] \ .$$

Furthermore,

$$\lim_{t\to\infty} \|\rho(t) - \bar\rho_\lambda\|_{L^1(\mathbb{R}^2)} = 0 \ .$$

And the system satisfies the hypercontractivity property *i.e.* for any $t^* > 0$, the constructed solution ρ is bounded in $L^\infty(t^*,\infty,L^p(\mathbb{R}^2))$, for any $p \in [1,\infty)$.

Basin on attraction: If $\lambda \neq \mu$ then

$$\mathcal{W}_2(\bar\rho_\mu,\bar\rho_\lambda) = \frac{1}{2}\int_{\mathbb{R}^2} \left|\frac{\lambda}{\mu}x - x\right|^2 \bar\rho_\mu = +\infty \ .$$

Talagrand's inequality:

$$\mathcal{W}_2^2(\rho,\bar\rho_\lambda) \leq 2\sqrt{2\lambda}\,\mathcal{H}_\lambda[\rho] \ .$$

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THE CONCENTRATION CONTROLLED INEQUALITY

CONCENTRATION CONTROL BY $\mathcal{H}_\lambda$

For any density ρ with mass 8π and any $\beta > 0$. Define $A_\beta = \{x \,:\, \rho(x) \geq \beta\}$. There exists C_1 and C_2 such that

$$\int_{A_\beta} \rho \;\, \mathrm{d}x \leq \frac{C_1}{\beta} + C_2 \sqrt{\mathcal{H}_\lambda[\rho]}\ ,$$

with $C_1 := 64\pi/\lambda$ and $C_2 := C_{\mathrm{CKLP}}/\sqrt{\lambda}$.

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Idea of the proof of the concentration controlled inequality (I): We split the function ρ is two parts: Given $\beta > 0$, define $\rho_\beta(x) = \min\{\rho(x)\;,\;\beta\}$. For β large enough, $\rho - \rho_\beta$ is such that:

$$\int_{\mathbb{R}^2} \rho - \rho_\beta \leq \frac{C_1}{\beta} + C_2\,\sqrt{\mathcal{H}_\lambda[\rho]} \leq \frac{C_1}{\beta} + \frac{8\pi - \varepsilon_0}{2} < 8\,\pi - \varepsilon_0\;.$$

We apply the classical method to $\rho - \rho_\beta$ of mass less than 8π.

Idea of the proof of the concentration controlled inequality (II): Let $f := \rho^{1/4}$. We split f in two parts by defining $f_\beta := \min\{f, \beta^{1/4}\}$ and $h_\beta := f - f_\beta$. We apply the Gagliardo-Nirenberg-Sobolev inequality to control h_β.

DISPLACEMENT CONVEXITY OF $\mathcal{H}_\lambda[u]$

Push-forward: T transports μ onto ν and denote $T\#\mu = \nu$ if

$$\int_{\mathbb{R}^2} \zeta[T(x)]\,\mathrm{d}\mu(x) = \int_{\mathbb{R}^2} \zeta(x)\,\mathrm{d}\nu(x) \qquad \forall \zeta \in \mathcal{C}_0^b(\mathbb{R}^2)\,.$$

WASSERSTEIN DISTANCE

The Wassertsein distance between μ and ν can be defined by

$$\mathcal{W}_2^2(\mu,\nu) := \inf_{T\,:\,\nu=T\#\mu} \int_{\mathbb{R}^2} |x - T(x)|^2\;\mathrm{d}\mu(x)\,.$$

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In the Wasserstein metric the solution to the system (KS) is a gradient flow of the free energy:

$$\rho_t = -\nabla_W \mathcal{F}_{\mathrm{PKS}}[\rho(t)]\,.$$

We regularise (KS) with $G_\epsilon := \gamma_\epsilon * G * \gamma_\epsilon$, γ being a mollifier sequence.

THE JORDAN-KINDERLEHRER-OTTO (JKO) SCHEME

Given a time step τ, we define the solution by the minimising scheme:

$$\rho_\tau^{k+1} \in \mathrm{argmin}_{\rho\in\mathcal{K}} \left[\frac{\mathcal{W}_2^2(\rho,\rho_\tau^k)}{2\tau} + \mathcal{F}_{\mathrm{PKS}}^\epsilon[\rho]\right]\,.$$

DISPLACEMENT CONVEXITY OF $\mathcal{H}_\lambda[u]$

DISPLACEMENT CONVEXITY

Let u_0 and u_1 be two densities, and let ψ be such that $\nabla\psi\# u_0 = u_1$. Define $\psi_t(x) = (1-t)\frac{|x|^2}{2} + t\psi(x)$ and $u_t = \nabla\psi_t\# u_0$. $\mathcal{H}_\lambda$ is γ-displacement convex if for all $0 < t < 1$,

$$(1-t)\mathcal{H}_\lambda[u_0] + t\mathcal{H}_\lambda[u_1] - \mathcal{H}_\lambda[u_t] \geq \gamma t(1-t)\mathcal{W}_2^2(u_0,u_1)\ .$$

DISPLACEMENT CONVEXITY OF $\mathcal{H}_\lambda[u]$

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The displacement convexity of $\mathcal{H}_\lambda$ is <u>formally</u> obvious from the fact that

$$\mathcal{H}_\lambda[u] = \int_{\mathbb{R}^2} \left(-2\sqrt{u(x)} + \sqrt{\frac{1}{2\lambda}\frac{|x|^2}{2}u(x)}\right)\,\mathrm{d}x + C\ .$$

where $-\sqrt{u(x)}$ and $|x|^2 u(x)$ are displacement convex. Actually, we have to introduce

$$\mathcal{H}_{\lambda,\delta}[u] = \int_{\mathbb{R}^2} \frac{\left(\sqrt{u+\delta} - \sqrt{\bar\rho_\lambda + \delta}\right)^2}{\sqrt{\bar\rho_\lambda + \delta}}\,\mathrm{d}x$$

which is γ_δ-displacement convex with $\gamma_\delta \leq 0$.

CONSEQUENCE OF THE DISPLACEMENT CONVEXITY OF $\mathcal{H}_\lambda[u]$

Using

ABOVE THE TANGENT FORMULATION

$$\mathcal{H}_\lambda[u_0] \leq \mathcal{H}_\lambda[u_1] - \frac{1}{2}\int_{\mathbb{R}^2}\left[\sqrt{\frac{1}{2\lambda}}\,x + \frac{\nabla u_0}{u_0^{3/2}}\right]\cdot(\nabla\psi(x) - x)\,u_0\;\mathrm{d}x\;.$$

we obtain

CONVEXITY ESTIMATES

For solution ot the JKO scheme we have

$$\mathcal{H}_\lambda[\rho_\tau^{k+1}] \leq \mathcal{H}_\lambda[\rho_\tau^{k}] - \tau\,\mathcal{D}[\rho_\tau^{k+1}] + \tau\,\mathcal{A}\,\|\gamma\|_{4/3}\;,$$

and

$$\mathcal{H}_\lambda[\rho_\tau^{k+1}] \leq \mathcal{H}_\lambda[\rho_\tau^{k}] - \tau\,\mathcal{D}[\rho_\tau^{k+1}] + \tau\sqrt{\epsilon}\,\mathcal{A}\,\|\rho\|_{4/3}\;. \tag{2}$$

where $\mathcal{A} := 32\pi(2\lambda)^{-1/2}C_{\mathrm{HLS}}\,\||x|\,\gamma\|_{4/3}$.

CONSEQUENCES OF THE DISPLACEMENT CONVEXITY OF $\mathcal{H}_\lambda[u]$

ONE STEP THEOREM

Let ρ_τ^k satisfy the bound

$$\mathcal{H}_\lambda[\rho_\tau^k] < C_{\mathrm{I}} \; . \tag{3}$$

Define $Q_k > 0$, $\tau_0 > 0$ by

$$Q_k := C_{\mathrm{I}} - \mathcal{H}_\lambda[\rho_\tau^k] \quad \text{and} \quad \tau_0 \, \mathcal{A} \|\gamma\|_{4/3} = \frac{Q_k}{2} \; .$$

Finally, given Q_k and $0 < \tau \le \tau_0$, and also any positive integer k, let ϵ_k be given by

$$\tau^{1/3} \sqrt{\epsilon_k} \, \mathcal{A} \left[\frac{2\pi \, \gamma_2}{C_2^2} + 8\tau \frac{C_{\mathrm{CCD}}}{\gamma_2} \right]^{1/3} := Q_k \, 2^{-k} \tau^2 \; .$$

If $\tau \le \min(\tau_0, 1)$ then ρ_τ^{k+1} satisfy the bound (3) and

$$\mathcal{H}_\lambda[\rho_\tau^{k+1}] \le \mathcal{H}_\lambda[\rho_\tau^k] - \tau \mathcal{D}[\rho_\tau^k] + Q_k \, 2^{-k} \tau^2 \; .$$

PROOF OF THE ONE-STEP THEOREM

$$\mathcal{H}_\lambda[\rho] \leq \mathcal{H}_\lambda[\rho_0] - \tau\mathcal{D}[\rho] + \frac{Q}{2} \leq C_{\mathrm{I}} - Q - \tau\mathcal{D}[\rho] + \frac{Q}{2} \leq C_{\mathrm{I}} - \tau\mathcal{D}[\rho]\,. \tag{4}$$

By the GNS inequality, $\mathcal{D}[\rho] \geq 0$ so that (4) implies that ρ also satisfies (3). On the other hand, since $\mathcal{H}_\lambda[\rho]$ cannot be negative it implies

$$\mathcal{D}[\rho] \leq \frac{1}{\tau} C_{\mathrm{I}}\,. \tag{5}$$

We can thus apply the concentration controlled inequality:

$$\int_{\mathbb{R}^2} \left|\nabla(\rho^{1/4})\right|^2 \,\mathrm{d}x \leq \frac{1}{\gamma_2}\mathcal{D}[\rho] + \frac{C_{\mathrm{CCD}}}{\gamma_2} \leq \frac{1}{\tau}\frac{C_{\mathrm{I}}}{\gamma_2} + \frac{C_{\mathrm{CCD}}}{\gamma_2}\,.$$

By the Sobolev inequality, we have for any $p \in [1, \infty)$

$$\int_{\mathbb{R}^2} \rho^p \,\mathrm{d}x \leq C_{\mathrm{S}} \int_{\mathbb{R}^2} \left|\nabla(\rho^{1/4})\right|^2 \,\mathrm{d}x \leq \frac{1}{\tau} \underbrace{C_{\mathrm{S}}\left[\frac{C_{\mathrm{I}}}{\gamma_2} + \tau_0\frac{C_{\mathrm{CCD}}}{\gamma_2}\right]}_{\tilde{C}},$$

Now using this bound in (2), we obtain

$$\mathcal{H}_\lambda[\rho] - \mathcal{H}_\lambda[\rho_0] \leq -\tau\mathcal{D}[\rho] + \underbrace{\sqrt{\epsilon}\,\mathcal{A}\,\tilde{C}}_{:=\frac{Q}{2-k}} - \kappa_\lambda W_2^2(\rho_0, \rho).$$

Using the one step theorem we have

$$\mathcal{H}_\lambda[\rho_\tau^k] \leq \mathcal{H}_\lambda[\rho_\tau^{k-1}] + \frac{Q_k}{2^{-k}} - \tau \mathcal{D}[\rho_\tau^k]\,. \tag{6}$$

Adding them, we deduce for all $k \in \mathbb{N}$

$$\mathcal{H}_\lambda[\rho_\tau^k] \leq \mathcal{H}_\lambda[\rho_0] + \sum_{m=1}^{k} \frac{Q_m}{2^{-m}} - \tau \sum_{m=1}^{k} \mathcal{D}[\rho_\tau^m] \leq \mathcal{H}_\lambda[\rho_0] + C_{\mathrm{I}} - \tau \sum_{m=1}^{k} \mathcal{D}[\rho_\tau^m]\,. \tag{7}$$

We thus obtain as before, a bound on $\sum_{m=1}^{k} \mathcal{D}[\rho_\tau^m]$ and then for any $p \in [1, \infty)$

$$\tau \sum_{m=1}^{k} \int_{\mathbb{R}^2} \left|\nabla[(\rho_\tau^k)^{1/4}]\right|^2 \,\mathrm{d}x + \tau \sum_{m=1}^{k} \int_{\mathbb{R}^2} (\rho_\tau^k)^p \,\mathrm{d}x \leq C_3.$$

PROOF OF THE MAIN THEOREM: ENTROPY SOLUTION

Let us fix a $\tilde{T} > 0$ and take $\tau_l = \tilde{T}/l$ with $l \in \mathbb{N}$.
We define the piecewise constant interpolant

$$\tilde{\rho}_\tau(t) = \rho_\tau^{k+1} \quad \text{for any } t \in (k\,\tau, (k+1)\,\tau]\ .$$

Using the previous estimate there exists a positive constant C_T such that

$$\int_0^T \int_{\mathbb{R}^2} \tilde{\rho}_l^{3/2}(t,x)\ \mathrm{d}x\ \ \mathrm{d}t + \int_0^T \int_{\mathbb{R}^2} \left|\nabla(\tilde{\rho}_l)^{1/4}\right|^2\ \mathrm{d}x\ \ \mathrm{d}t \le C_T\ . \tag{8}$$

As a consequence, there exists ρ such that for any $p \in [1,\infty)$

$$\{\rho_l\}_{l\in\mathbb{N}} \to \rho \quad \text{in } L^p((0,T)\times\mathbb{R}^2) \quad \text{and} \quad \{\nabla(\tilde{\rho}_l)^{1/4}\}_{l\in\mathbb{N}} \rightharpoonup \nabla\rho^{1/4} \quad \text{in } L^2((0,T)\times\mathbb{R}^2)$$

We deduce that

$$\lim_{l\to\infty} \int_0^T \int_{\mathbb{R}^2} \tilde{\rho}_l^{3/2}(t,x)\ \mathrm{d}x\ \ \mathrm{d}t = \int_0^T \int_{\mathbb{R}^2} \rho^{3/2}(t,x)\ \mathrm{d}x\ \ \mathrm{d}t\ ,$$

and by lower-semicontinuity

$$\int_0^T \int_{\mathbb{R}^2} \left|\nabla\rho^{1/4}\right|^2\ \mathrm{d}x\ \ \mathrm{d}t \le \liminf_{l\to\infty} \int_0^T \int_{\mathbb{R}^2} \left|\nabla(\tilde{\rho}_l)^{1/4}\right|^2\ \mathrm{d}x\ \ \mathrm{d}t\ ,$$

PROOF OF THE MAIN THEOREM: ASYMPTOTICS

Let $\{t_n\}_n$ be a non-decrasing diverging sequence and define $\rho_n(t) = \rho(t + t_n)$. As

$$\int_0^\infty \mathcal{D}[\rho(t)]\,\mathrm{d}t < \infty\,,$$

we deduce

$$\lim_{n\to\infty} \int_0^1 \mathcal{D}[\rho_n(t)]\,\mathrm{d}t = 0\,.$$

Passing to the limit using the above compactness, there exists ρ_∞ such that

$$\int_0^1 \int_{\mathbb{R}^2} \left(8 \left| \nabla(\rho_\infty^{1/4}) \right|^2 - \rho_\infty^{3/2} \right)\,\mathrm{d}x\,\mathrm{d}t = 0\,.$$

As a consequence there exists $\bar\lambda$ such that $\rho_\infty = \bar\rho_{\bar\lambda}$. Finally, we conclude that $\bar\lambda = \lambda$ is the unique λ such that $\mathcal{H}_\lambda[\rho_\infty] < \infty$.

Thank you for your attention