Nonlinear Cross Diffusion Models for Crowds

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Motivation

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One-Dimensional Model

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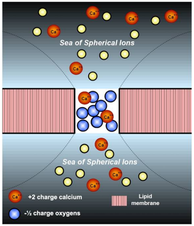
Analysis

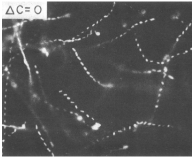
Motivation

aim: modelling of diffusing particles with size exclusion



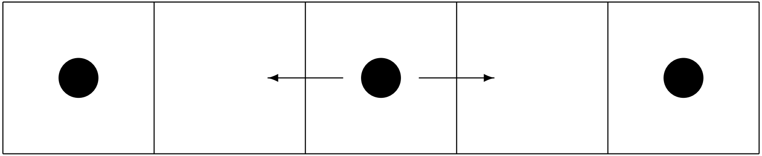
application:

- human crowds
- swarming
- ion channels
- chemotaxis



Derivation of Model

one-dimensional hopping model:



Probabilities

$r(x,t) = P(\text{red particle at position } x \text{ at time } t)$
$b(x,t) = P(\text{blue particle at position } x \text{ at time } t)$

Derivation of Model

- **transition rate:**
  $\Pi_+^{r/b}(x,t) =$
  $P(\text{jump of } r/b \text{ from position } x \text{ to } x+h \text{ in } (t, t+\Delta t))\cdot$
  $\frac{1}{\Delta t} \cdot P(x+h \text{ is empty})$

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  $\frac{1}{\Delta t} \cdot P(x+h \text{ is empty})$
- **probability** that a red particle is located at position x at time $t+\Delta t$:
  $r(x,t+\Delta t) = r(x,t)(1-\Pi_+(x,t)-\Pi_-(x,t))$
  $+r(x+h,t)\Pi_-(x+h,t)+r(x-h,t)\Pi_+(x-h,t)$

Derivation of Model

Resulting Model

$$\frac{\partial r}{\partial t} = \frac{\partial}{\partial x}((1-m)\frac{\partial r}{\partial x} + r\frac{\partial m}{\partial x} - \mu_r r(1-m)\frac{\partial V}{\partial x})$$
$$\frac{\partial b}{\partial t} = \frac{\partial}{\partial x}(D((1-m)\frac{\partial b}{\partial x} + b\frac{\partial m}{\partial x} + \mu_b b(1-m)\frac{\partial V}{\partial x}))$$

mass density m(x,t) $= r(x,t) + b(x,t)$,

diffusion coefficient **D**,

potential V(x) (e.g. prefered walking direction),

scaled **mobility** constants μ_r, μ_b

Multidimensional Model

$$\partial_t r = \nabla \cdot ((1-m)\nabla r + r\nabla m - \mu_r r(1-m)\nabla V)$$
$$\partial_t b = \nabla \cdot (D((1-m)\nabla b + b\nabla m + \mu_b b(1-m)\nabla V))$$

we apply **von-Neumann boundary conditions** on all boundaries:

$$((1-m)\nabla r + r\nabla m - \mu_r r(1-m)\nabla V) \cdot n = 0$$

$$((1-m)\nabla b + b\nabla m + \mu_b b(1-m)\nabla V) \cdot n = 0$$

Problems in the Analysis

System of equations

$$\partial_t r = \nabla \cdot ((1-m)\nabla r + r\nabla m - \mu_r r(1-m)\nabla V)$$
$$\partial_t b = \nabla \cdot (D((1-m)\nabla b + b\nabla m + \mu_b b(1-m)\nabla V))$$

nonlinear cross diffusion $\Rightarrow$ few literature available

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- existence and uniqueness of a solution?
- no a-priori estimates
- no maximum principle (only $0 \leq r, b, m \leq 1$)

Entropy

Entropy

$E(r, b) =$

$\int (r\log(r) + b\log(b) + (1-m)\log(1-m) - \mu_r rV + \mu_b bV)\, dx$

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- behaviour in time?

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- behaviour in time?
- $\frac{\partial E}{\partial t} = -\int r(1-m)|\nabla \xi_1|^2 + Db(1-m)|\nabla \xi_2|^2 dx$

  $\xi_1 = \xi_1(r,b,V) \qquad \xi_2 = \xi_2(r,b,V)$

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- $\frac{\partial E}{\partial t} = -\int r(1-m)|\nabla\xi_1|^2 + Db(1-m)|\nabla\xi_2|^2 dx$

  $\xi_1 = \xi_1(r, b, V) \qquad \xi_2 = \xi_2(r, b, V)$
- E is decreasing during the process
- E is minimal in equilibrium state

Minima of Entropy

$E =$
$\int \left(r \log(r) + b \log(b) + (1-m)\log(1-m) - \mu_r rV + \mu_b bV \right) dx$

Lagrange Functional

$L(r, b, \lambda_r, \lambda_b) = E(r, b) + \lambda_r(\int r - m_r) + \lambda_b(\int b - m_b)$

m_r: mass of red particles $\qquad$ m_b: mass of blue particles

$\frac{\partial L}{\partial r} = \frac{\partial E}{\partial r} + \lambda_r = 0$
$\frac{\partial L}{\partial b} = \frac{\partial E}{\partial b} + \lambda_b = 0$ $\qquad \Rightarrow$ **equilibrium solutions**

Minima of Entropy

$E =$
$\int (r \log(r) + b \log(b) + (1-m)\log(1-m) - \mu_r r V + \mu_b b V)\, dx$

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- $m \equiv 1 \Rightarrow r_{\mathrm{eq}} = k_1 \exp(\mu_r V), \qquad b_{\mathrm{eq}} = k_2 \exp(-\mu_b V)$

Minima of Entropy

$E =$
$\int (r\log(r) + b\log(b) + (1-m)\log(1-m) - \mu_r rV + \mu_b bV)\,dx$

Lagrange Functional

$L(r, b, \lambda_r, \lambda_b) = E(r, b) + \lambda_r(\int r - m_r) + \lambda_b(\int b - m_b)$

m_r: mass of red particles m_b: mass of blue particles

$$\frac{\partial L}{\partial r} = \frac{\partial E}{\partial r} + \lambda_r = 0$$
$$\frac{\partial L}{\partial b} = \frac{\partial E}{\partial b} + \lambda_b = 0$$

$\Rightarrow$ **equilibrium solutions**

- $m \equiv 1 \Rightarrow r_{eq} = k_1 \exp(\mu_r V)$, $b_{eq} = k_2 \exp(-\mu_b V)$
- $b \equiv 0 \Rightarrow r_t = \nabla \cdot (\nabla r - r(1-r)\nabla V) \Rightarrow r_{eq} = \frac{k_1 \exp(\mu_r V)}{k_1 \exp(\mu_r V) + 1}$

Solutions for Equilibrium

$$\partial_t r = \nabla \cdot ((1-m)\nabla r + r\nabla m - \mu_r r(1-m)\nabla V)$$
$$\partial_t b = \nabla \cdot (D((1-m)\nabla b + b\nabla m + \mu_b b(1-m)\nabla V))$$

Equilibrium Solutions

$$r_{\text{eq}}(x,t) = \frac{k_1 \exp(\mu_r V(x))}{k_1 \exp(\mu_r V(x)) + k_2 \exp(-\mu_b V(x)) + 1}$$
$$b_{\text{eq}}(x,t) = \frac{k_2 \exp(-\mu_b V(x))}{k_1 \exp(\mu_r V(x)) + k_2 \exp(-\mu_b V(x)) + 1}$$

- k_1 and k_2 are constants that can be determined from total mass, $k_1 = \exp(-\lambda_r)$, $k_2 = \exp(-\lambda_b)$

Special Case $V(x) = 0$

system of equations with $\mathbf{V}(\mathbf{x}) = \mathbf{0}$:

$$\partial_t r = \nabla \cdot ((1-m)\nabla r + r\nabla m)$$
$$\partial_t b = \nabla \cdot (D((1-m)\nabla b + b\nabla m))$$

equilibrium solutions: constants $\bar{n}, \bar{p}$

First Order Linearization

$\partial u_t = (1-\bar{p})\Delta u + \bar{n}\Delta v$
$\partial v_t = D((1-\bar{n})\Delta v + \bar{p}\Delta u)$

for $w = u + \alpha v$, we obtain the heat equation

$$\partial_t w = k\Delta w \qquad k \geq 0$$

$\Rightarrow$ stability around equilibria

Special Case $V(x) = 0$

$$\partial_t r = \nabla \cdot ((1-m)\nabla r + r\nabla m)$$
$$\partial_t b = \nabla \cdot (D((1-m)\nabla b + b\nabla m))$$

Transformation

$u = F(r,b) = G\left(\frac{1-r-b}{(r-b-\alpha)^\alpha}\right) \qquad \alpha = \frac{D-1}{D+1}$

we can show that u satisfies a maximum-principle.

special cases:

$D = 0$: $\quad u = G((1-r-b)(r-b-1))$

$D = 1$: $\quad u = G(1-m)$

Transformation to Slotboom-Variables

$$\partial_t r = \nabla \cdot ((1-m)\nabla r + r\nabla m - \mu_r r(1-m)\nabla V)$$
$$\partial_t b = \nabla \cdot (D((1-m)\nabla b + b\nabla m + \mu_b b(1-m)\nabla V))$$

The system is not easy to solve. Hence we try to transform the variables to obtain a simplified system. In classical semiconductor theory, the variables after the transformation are called 'Slotboom-Variables'.

- transformation function $F_1(r, b) = \log r - \log(1-m)$
- transformation $u = F_1^{-1}(F_1(r, b) - \mu_r V)$
  $v = F_2^{-1}(F_2(r, b) + \mu_b V)$

Open Questions

- existence for the general case?
- uniqueness?
- longtime behaviour for the general case?

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Thank you for your Attention!