Flocking and synchronization of particle models

Corrado Lattanzio

Sezione di Matematica per l'Ingegneria
Dipartimento di Matematica Pura ed Applicata
Università degli Studi dell'Aquila

Joint work with Seung-Yeal Ha, Bruno Rubino and Marshall Slemrod

Eleventh International Kinetic and Mean–field models in the Socio-Economic Sciences
Edinburgh, July 27–31 2009

Outline

Outline

The Two Notions

- Synchronization is an adjustment of rhythms of oscillating objects due to their weak interaction.
- Flocking represents the phenomenon in which self-propelled individuals using only limited environmental information and simple rules, organize into an ordered motion
  cf. Law of second thermodynamics is violated, a concentration phenomena in a phase-variable.
  “Birds of a feather flock together.”
  by William Turner (1545)

The Two Notions

- Synchronization is an adjustment of rhythms of oscillating objects due to their weak interaction.
- Flocking represents the phenomenon in which self-propelled individuals using only limited environmental information and simple rules, organize into an ordered motion
  cf. Law of second thermodynamics is violated, a concentration phenomena in a phase-variable.
  "Birds of a feather flock together."
  by William Turner (1545)

The Two Notions

- Synchronization is an adjustment of rhythms of oscillating objects due to their weak interaction.
- Flocking represents the phenomenon in which self-propelled individuals using only limited environmental information and simple rules, organize into an ordered motion
  cf. Law of second thermodynamics is violated, a concentration phenomena in a phase-variable.
  "Birds of a feather flock together."
  by William Turner (1545)

Outline

Results on Flocking Models

Partial list of previous works on flocking:

- Biology: Shaw '78, Aoki '82, Partridge '82, Parish-Viscido-Grunbaum '02,
- Computer Sciences: Reynolds '87,
- Physics: Vicsek et al. '95, Toner-Tu '98, Levine-Rappel '00,
- Applied analysis: Levine-Rappel '00, Topaz-Bertozzi '04, D'Orsogna '06, Cucker-Smale[1], Shen '07, Degond-Motsch '07, Carrillo et al. '09,

$$\frac{dx_i}{dt} = v_i,$$

$$\frac{dv_i}{dt} = \frac{\lambda}{N}\sum_{j=1}^{N}\psi(|x_j - x_i|)(v_j - v_i).$$

[1] IEEE Trans. Automat. Control (2007)

Results on Flocking Models

Partial list of previous works on flocking:

- Biology: Shaw '78, Aoki '82, Partridge '82, Parish-Viscido-Grunbaum '02,
- Computer Sciences: Reynolds '87,
- Physics: Vicsek et al. '95, Toner-Tu '98, Levine-Rappel '00,
- Applied analysis: Levine-Rappel '00, Topaz-Bertozzi '04, D'Orsogna '06, Cucker-Smale[1], Shen '07, Degond-Motsch '07, Carrillo et al. '09,

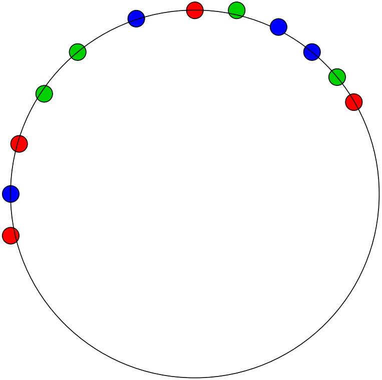
$$\frac{dx_i}{dt} = v_i,$$

$$\frac{dv_i}{dt} = \frac{\lambda}{N} \sum_{j=1}^{N} \psi(|x_j - x_i|)(v_j - v_i).$$

[1] IEEE Trans. Automat. Control (2007)

Phase Descriptions for Synchronization

Phase descriptions for synchronization: Winfree (1967), Kuramoto (1975), . . .



Outline

Kuramoto's Phase Model

Consider nonlinear oscillators $\{x_i = e^{\sqrt{-1}\theta_i}\}$ rotating along S^1 with their natural frequency Ω_i which is randomly drawn from some probability distribution.

$$\frac{d\theta_i}{dt} = \Omega_i + \frac{\lambda}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i), \quad t > 0, \quad i = 1, \cdots, N,$$

subject to initial phase $\theta_i(0) = \theta_{i0}, \qquad i = 1, \cdots, N.$

Differentiate to relate Kuramoto's model to Cucker-Smale model:

$$\frac{d\theta_i}{dt} = \omega_i, \quad t > 0, \quad i = 1, \cdots, N,$$

$$\frac{d\omega_i}{dt} = \frac{\lambda}{N}\sum_{j=1}^{N}\cos(\theta_j - \theta_i)(\omega_j - \omega_i), \quad t > 0, \quad i = 1, \cdots, N,$$

subject to initial phase and frequency: $\theta_i(0) = \theta_{i0}, \ \omega_i(0) = \Omega_i.$

Kuramoto's Phase Model

Consider nonlinear oscillators $\{x_i = e^{\sqrt{-1}\theta_i}\}$ rotating along S^1 with their natural frequency Ω_i which is randomly drawn from some probability distribution.

$$\frac{d\theta_i}{dt} = \Omega_i + \frac{\lambda}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i), \quad t > 0, \quad i = 1, \cdots, N,$$

subject to initial phase $\theta_i(0) = \theta_{i0}, \qquad i = 1, \cdots, N.$
Differentiate to relate Kuramoto's model to Cucker-Smale model:

$$\frac{d\theta_i}{dt} = \omega_i, \quad t > 0, \quad i = 1, \cdots, N,$$

$$\frac{d\omega_i}{dt} = \frac{\lambda}{N}\sum_{j=1}^{N}\cos(\theta_j - \theta_i)(\omega_j - \omega_i), \quad t > 0, \quad i = 1, \cdots, N,$$

subject to initial phase and frequency: $\theta_i(0) = \theta_{i0}, \ \omega_i(0) = \Omega_i.$

Kuramoto's Phase Model

Consider nonlinear oscillators $\{x_i = e^{\sqrt{-1}\theta_i}\}$ rotating along S^1 with their natural frequency Ω_i which is randomly drawn from some probability distribution.

$$\frac{d\theta_i}{dt} = \Omega_i + \frac{\lambda}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i), \quad t > 0, \quad i = 1, \cdots, N,$$

subject to initial phase $\theta_i(0) = \theta_{i0}, \qquad i = 1, \cdots, N.$

Differentiate to relate Kuramoto's model to Cucker-Smale model:

$$\frac{d\theta_i}{dt} = \omega_i, \quad t > 0, \quad i = 1, \cdots, N,$$

$$\frac{d\omega_i}{dt} = \frac{\lambda}{N}\sum_{j=1}^{N}\cos(\theta_j - \theta_i)(\omega_j - \omega_i), \quad t > 0, \quad i = 1, \cdots, N,$$

subject to initial phase and frequency: $\theta_i(0) = \theta_{i0}, \ \omega_i(0) = \Omega_i.$

Equation for Fluctuations

We set

$$\langle \Omega \rangle := \frac{1}{N}\sum_{i=1}^{N} \Omega_i, \qquad \langle \theta \rangle := \frac{1}{N}\sum_{i=1}^{N} \theta_i, \quad \langle \omega \rangle := \frac{1}{N}\sum_{i=1}^{N} \omega_i,$$

$$\hat{\Omega}_i := \Omega_i - \langle \Omega \rangle, \quad \hat{\theta}_i := \theta_i - \langle \theta \rangle, \quad \hat{\omega}_i := \omega_i - \langle \omega \rangle.$$

Then it is easy to see that $\langle \theta(t) \rangle = \langle \theta_0 \rangle + t\langle \Omega \rangle, \quad \langle \omega(t) \rangle = \langle \Omega \rangle,$ and the fluctuations satisfy

$$\frac{d\hat{\theta}_i}{dt} = \hat{\Omega}_i + \frac{\lambda}{N}\sum_{j=1}^{N} \sin(\hat{\theta}_j - \hat{\theta}_i), \quad t > 0, \quad i = 1, \cdots, N,$$

$$\frac{d\hat{\omega}_i}{dt} = \frac{\lambda}{N}\sum_{j=1}^{N} \cos(\hat{\theta}_j - \hat{\theta}_i)(\hat{\omega}_j - \hat{\omega}_i), \quad t > 0, \quad i = 1, \cdots, N.$$

Equation for Fluctuations

We set

$$\langle\Omega\rangle := \frac{1}{N}\sum_{i=1}^{N}\Omega_i, \qquad \langle\theta\rangle := \frac{1}{N}\sum_{i=1}^{N}\theta_i, \quad \langle\omega\rangle := \frac{1}{N}\sum_{i=1}^{N}\omega_i,$$

$$\hat{\Omega}_i := \Omega_i - \langle\Omega\rangle, \quad \hat{\theta}_i := \theta_i - \langle\theta\rangle, \quad \hat{\omega}_i := \omega_i - \langle\omega\rangle.$$

Then it is easy to see that $\langle\theta(t)\rangle = \langle\theta_0\rangle + t\langle\Omega\rangle, \quad \langle\omega(t)\rangle = \langle\Omega\rangle,$
and the fluctuations satisfy

$$\frac{d\hat{\theta}_i}{dt} = \hat{\Omega}_i + \frac{\lambda}{N}\sum_{j=1}^{N}\sin(\hat{\theta}_j - \hat{\theta}_i), \quad t > 0, \quad i = 1, \cdots, N,$$

$$\frac{d\hat{\omega}_i}{dt} = \frac{\lambda}{N}\sum_{j=1}^{N}\cos(\hat{\theta}_j - \hat{\theta}_i)(\hat{\omega}_j - \hat{\omega}_i), \quad t > 0, \quad i = 1, \cdots, N.$$

Equation for Fluctuations

We set

$$\langle \Omega \rangle := \frac{1}{N}\sum_{i=1}^{N} \Omega_i, \qquad \langle \theta \rangle := \frac{1}{N}\sum_{i=1}^{N} \theta_i, \quad \langle \omega \rangle := \frac{1}{N}\sum_{i=1}^{N} \omega_i,$$

$$\hat{\Omega}_i := \Omega_i - \langle \Omega \rangle, \quad \hat{\theta}_i := \theta_i - \langle \theta \rangle, \quad \hat{\omega}_i := \omega_i - \langle \omega \rangle.$$

Then it is easy to see that $\langle \theta(t) \rangle = \langle \theta_0 \rangle + t\langle \Omega \rangle, \quad \langle \omega(t) \rangle = \langle \Omega \rangle,$
and the fluctuations satisfy

$$\frac{d\hat{\theta}_i}{dt} = \hat{\Omega}_i + \frac{\lambda}{N}\sum_{j=1}^{N} \sin(\hat{\theta}_j - \hat{\theta}_i), \quad t > 0, \quad i = 1, \cdots, N,$$

$$\frac{d\hat{\omega}_i}{dt} = \frac{\lambda}{N}\sum_{j=1}^{N} \cos(\hat{\theta}_j - \hat{\theta}_i)(\hat{\omega}_j - \hat{\omega}_i), \quad t > 0, \quad i = 1, \cdots, N.$$

Outline

Dynamic of Standard Deviations

Denote with σ_θ, σ_ω and σ_Ω the standard deviations of θ_i, ω_i and Ω_i respectively:

$$\sigma_\theta^2 := \frac{1}{N}\sum_{i=1}^N |\hat{\theta}_i|^2, \qquad \sigma_\omega^2 := \frac{1}{N}\sum_{i=1}^N |\hat{\omega}_i|^2, \qquad \sigma_\Omega^2 := \frac{1}{N}\sum_{i=1}^N |\hat{\Omega}_i|^2.$$

The dynamic of these quantities is given by

$$\frac{d}{dt}\sum_{i=1}^N |\hat{\theta}_i|^2 = 2\sum_{i=1}^N \hat{\theta}_i\hat{\Omega}_i - \frac{\lambda}{N}\sum_{i,j}(\hat{\theta}_j - \hat{\theta}_i)\sin(\hat{\theta}_j - \hat{\theta}_i),$$

$$\frac{d}{dt}\sum_{i=1}^N |\hat{\omega}_i|^2 = -\frac{\lambda}{N}\sum_{i,j}\cos(\hat{\theta}_j - \hat{\theta}_i)|\hat{\omega}_j - \hat{\omega}_i|^2.$$

Dynamic of Standard Deviations

Denote with σ_θ, σ_ω and σ_Ω the standard deviations of θ_i, ω_i and Ω_i respectively:

$$\sigma_\theta^2 := \frac{1}{N}\sum_{i=1}^N |\hat{\theta}_i|^2, \qquad \sigma_\omega^2 := \frac{1}{N}\sum_{i=1}^N |\hat{\omega}_i|^2, \qquad \sigma_\Omega^2 := \frac{1}{N}\sum_{i=1}^N |\hat{\Omega}_i|^2.$$

The dynamic of these quantities is given by

$$\frac{d}{dt}\sum_{i=1}^N |\hat{\theta}_i|^2 = 2\sum_{i=1}^N \hat{\theta}_i\hat{\Omega}_i - \frac{\lambda}{N}\sum_{i,j}(\hat{\theta}_j - \hat{\theta}_i)\sin(\hat{\theta}_j - \hat{\theta}_i),$$

$$\frac{d}{dt}\sum_{i=1}^N |\hat{\omega}_i|^2 = -\frac{\lambda}{N}\sum_{i,j}\cos(\hat{\theta}_j - \hat{\theta}_i)|\hat{\omega}_j - \hat{\omega}_i|^2.$$

Dynamic of Standard Deviations

Under the a priori assumption

$$\sup_{0\leq t<T} \sigma_\theta(t) < \frac{\pi}{4},$$

σ_θ and σ_ω satisfy the system of differential inequalities

$$\left|\frac{d\sigma_\theta}{dt}\right| \leq \sigma_\omega, \quad \frac{d\sigma_\omega}{dt} \leq -\lambda\cos(2\sqrt{N}\sigma_\theta)\sigma_\omega.$$

Dynamic of Standard Deviations

Under the a priori assumption

$$\sup_{0 \le t < T} \sigma_\theta(t) < \frac{\pi}{4},$$

σ_θ and σ_ω satisfy the system of differential inequalities

$$\left| \frac{d\sigma_\theta}{dt} \right| \le \sigma_\omega, \quad \frac{d\sigma_\omega}{dt} \le -\lambda \cos(2\sqrt{N}\sigma_\theta)\sigma_\omega.$$

Lyapunov Approach

Define Lyapunov functionals[2]

$$\mathcal{H}^{\pm}[\sigma_\theta(t), \sigma_\omega(t)] := \sigma_\omega(t) \pm \lambda \int_{\sigma_\theta(0)}^{\sigma_\theta(t)} \cos(2\sqrt{N}s)ds.$$

Under the same a priori assumption, we have

$$(i)\ \mathcal{H}^{\pm}[\sigma_\theta(t), \sigma_\omega(t)] \le \mathcal{H}^{\pm}[\sigma_\theta(0), \sigma_\omega(0)],$$

$$(ii)\ \sigma_\omega(t) + \lambda \left| \int_{\sigma_\theta(0)}^{\sigma_\theta(t)} \cos(2\sqrt{N}s)ds \right| \le \sigma_\omega(0).$$

[2] [Ha–Liu, Commun. Math. Sci. 2009]

Lyapunov Approach

Define Lyapunov functionals[2]

$$\mathcal{H}^{\pm}[\sigma_\theta(t), \sigma_\omega(t)] := \sigma_\omega(t) \pm \lambda \int_{\sigma_\theta(0)}^{\sigma_\theta(t)} \cos(2\sqrt{N}s)ds.$$

Under the same a priori assumption, we have

$$(i)\ \mathcal{H}^{\pm}[\sigma_\theta(t), \sigma_\omega(t)] \le \mathcal{H}^{\pm}[\sigma_\theta(0), \sigma_\omega(0)],$$

$$(ii)\ \sigma_\omega(t) + \lambda \left| \int_{\sigma_\theta(0)}^{\sigma_\theta(t)} \cos(2\sqrt{N}s)ds \right| \le \sigma_\omega(0).$$

[2] [Ha–Liu, Commun. Math. Sci. 2009]

Outline

Frequency Synchronization

Theorem (Frequency synchronization)

Suppose the standard deviations of initial configuration $\sigma_\theta(0)$ and $\sigma_\omega(0)$ satisfy

$$\sigma_\theta(0) < \frac{\pi}{16\sqrt{N}}, \qquad \sigma_\omega(0) < \frac{\lambda\pi}{16\sqrt{2N}}.$$

Then we have

$$(i) \quad \sigma_\omega(t) \le \sigma_\omega(0) e^{-\frac{\lambda t}{\sqrt{2}}}, \quad t \ge 0.$$

$$(ii) \quad \lim_{t\to\infty} |\omega_i(t) - \omega_j(t)| = 0, \qquad 1 \le i, j \le N.$$

Outline

The Classical Cucker-Smale Model

$$\frac{dx_i}{dt} = v_i,$$
$$\frac{dv_i}{dt} = \frac{\lambda}{N}\sum_{j=1}^{N} \psi(|x_j - x_i|)(v_j - v_i).$$

where ψ is a communication rate.

- Cucker-Smale communication rate: $\psi(|x_i - x_j|) = \frac{1}{(1+|x_i-x_j|^2)^\beta}$, $\beta \geq 0$.
- All-to-all communication: $\psi(|x_j - x_i|) = 1$.
- Finite-range communication: Visceck's model '95 $\psi(|x_j - x_i|) = \mathbf{1}_{|x_j - x_i| \leq r}$.
- Contact communication: Benedetto-Caglioti-Pulvirenti '97 (granular flow modeling) $\psi(|x_j - x_i|) = \delta(x_j - x_i)$.

The Classical Cucker-Smale Model

$$\frac{dx_i}{dt} = v_i,$$
$$\frac{dv_i}{dt} = \frac{\lambda}{N} \sum_{j=1}^{N} \psi(|x_j - x_i|)(v_j - v_i).$$

where ψ is a communication rate.

- Cucker-Smale communication rate: $\psi(|x_i - x_j|) = \frac{1}{(1+|x_i-x_j|^2)^\beta}$, $\beta \geq 0$.
- All-to-all communication: $\psi(|x_j - x_i|) = 1$.
- Finite-range communication: Visceck's model '95 $\psi(|x_j - x_i|) = \mathbf{1}_{|x_j - x_i| \leq r}$.
- Contact communication: Benedetto-Caglioti-Pulvirenti '97 (granular flow modeling) $\psi(|x_j - x_i|) = \delta(x_j - x_i)$.

The Classical Cucker-Smale Model

$$\frac{dx_i}{dt} = v_i,$$
$$\frac{dv_i}{dt} = \frac{\lambda}{N}\sum_{j=1}^{N}\psi(|x_j - x_i|)(v_j - v_i).$$

where ψ is a communication rate.

- Cucker-Smale communication rate: $\psi(|x_i - x_j|) = \frac{1}{(1+|x_i-x_j|^2)^\beta}, \ \beta \geq 0.$
- All-to-all communication: $\psi(|x_j - x_i|) = 1.$
- Finite-range communication: Visceck's model '95 $\psi(|x_j - x_i|) = \mathbf{1}_{|x_j - x_i| \leq r}.$
- Contact communication: Benedetto-Caglioti-Pulvirenti '97 (granular flow modeling) $\psi(|x_j - x_i|) = \delta(x_j - x_i).$

The Classical Cucker-Smale Model

$$\frac{dx_i}{dt} = v_i,$$
$$\frac{dv_i}{dt} = \frac{\lambda}{N} \sum_{j=1}^{N} \psi(|x_j - x_i|)(v_j - v_i).$$

where ψ is a communication rate.

- Cucker-Smale communication rate: $\psi(|x_i - x_j|) = \frac{1}{(1+|x_i-x_j|^2)^\beta}, \ \beta \geq 0.$
- All-to-all communication: $\psi(|x_j - x_i|) = 1.$
- Finite-range communication: Visceck's model '95 $\psi(|x_j - x_i|) = \mathbf{1}_{|x_j - x_i| \leq r}.$
- Contact communication: Benedetto-Caglioti-Pulvirenti '97 (granular flow modeling) $\psi(|x_j - x_i|) = \delta(x_j - x_i).$

The Classical Cucker-Smale Model

$$\frac{dx_i}{dt} = v_i,$$
$$\frac{dv_i}{dt} = \frac{\lambda}{N}\sum_{j=1}^{N} \psi(|x_j - x_i|)(v_j - v_i).$$

where ψ is a communication rate.

- Cucker-Smale communication rate: $\psi(|x_i - x_j|) = \frac{1}{(1+|x_i-x_j|^2)^\beta}, \ \beta \geq 0.$
- All-to-all communication: $\psi(|x_j - x_i|) = 1.$
- Finite-range communication: Visceck's model '95 $\psi(|x_j - x_i|) = \mathbf{1}_{|x_j - x_i| \leq r}.$
- Contact communication: Benedetto-Caglioti-Pulvirenti '97 (granular flow modeling) $\psi(|x_j - x_i|) = \delta(x_j - x_i).$

Outline

1-d Mechanical Derivation

Connection[3] between Cucker-Smale and singularly perturbed oscillator chain system in one dimension:

$$\frac{dx_i}{dt} = v_i,$$
$$\varepsilon \frac{dv_i}{dt} = \frac{\lambda}{N} \sum_{j=1}^{N} \varphi'(x_j - x_i) - v_i.$$

Differentiate with respect to time to obtain

$$\varepsilon \frac{d^2 v_i}{dt^2} = \frac{\lambda}{N} \sum_{j=1}^{N} \varphi''(x_j - x_i)(v_j - v_i) - \frac{dv_i}{dt}.$$

At the limit $\varepsilon \downarrow 0$ we recover 1-d Cucker-Smale model.

[3] Ha, Slemrod, J. Dynam. Differential Equations (2009)

1-d Mechanical Derivation

Connection[3] between Cucker-Smale and singularly perturbed oscillator chain system in one dimension:

$$\frac{dx_i}{dt} = v_i,$$

$$\varepsilon \frac{dv_i}{dt} = \frac{\lambda}{N} \sum_{j=1}^{N} \varphi'(x_j - x_i) - v_i.$$

Differentiate with respect to time to obtain

$$\varepsilon \frac{d^2 v_i}{dt^2} = \frac{\lambda}{N} \sum_{j=1}^{N} \varphi''(x_j - x_i)(v_j - v_i) - \frac{dv_i}{dt}.$$

At the limit $\varepsilon \downarrow 0$ we recover 1-d Cucker-Smale model.

[3] Ha, Slemrod, J. Dynam. Differential Equations (2009)

1-d Mechanical Derivation

Connection[3] between Cucker-Smale and singularly perturbed oscillator chain system in one dimension:

$$\frac{dx_i}{dt} = v_i,$$
$$\varepsilon\frac{dv_i}{dt} = \frac{\lambda}{N}\sum_{j=1}^{N}\varphi'(x_j - x_i) - v_i.$$

Differentiate with respect to time to obtain

$$\varepsilon\frac{d^2 v_i}{dt^2} = \frac{\lambda}{N}\sum_{j=1}^{N}\varphi''(x_j - x_i)(v_j - v_i) - \frac{dv_i}{dt}.$$

At the limit $\varepsilon \downarrow 0$ we recover 1-d Cucker-Smale model.

[3] Ha, Slemrod, J. Dynam. Differential Equations (2009)

Multi-d Mechanical Derivation

Analogous multidimensional derivation:

$$\frac{dx_i}{dt} = v_i,$$

$$\varepsilon \frac{dv_i}{dt} = \frac{\lambda}{N} \sum_{j=1}^{N} \nabla_x \varphi(|x_j - x_i|) - v_i.$$

Differentiate with respect to time to obtain ($r_{ij} = |x_j - x_i|$)

$$\varepsilon \frac{d^2 v_i}{dt^2} = \frac{\lambda}{N} \sum_{j=1}^{N} \Big[\frac{\varphi'(r_{ij})}{r_{ij}} (v_j - v_i) + \Big(\frac{\varphi''(r_{ij})}{r_{ij}^2} - \frac{\varphi'(r_{ij})}{r_{ij}^3} \Big) ((x_j - x_i) \cdot (v_j - v_i))(x_j - x_i) \Big] - \frac{dv_i}{dt}$$

Multi-d Mechanical Derivation

Analogous multidimensional derivation:

$$\frac{dx_i}{dt} = v_i,$$
$$\varepsilon \frac{dv_i}{dt} = \frac{\lambda}{N} \sum_{j=1}^{N} \nabla_x \varphi(|x_j - x_i|) - v_i.$$

Differentiate with respect to time to obtain ($r_{ij} = |x_j - x_i|$)

$$\varepsilon \frac{d^2 v_i}{dt^2} = \frac{\lambda}{N} \sum_{j=1}^{N} \Big[\frac{\varphi'(r_{ij})}{r_{ij}} (v_j - v_i) + \Big(\frac{\varphi''(r_{ij})}{r_{ij}^2} - \frac{\varphi'(r_{ij})}{r_{ij}^3} \Big) ((x_j - x_i) \cdot (v_j - v_i))(x_j - x_i) \Big] - \frac{dv_i}{dt}$$

New Multi-d Model

At the limit $\varepsilon \downarrow 0$ we obtain a new multidimensional model:

$$\frac{dx_i}{dt} = v_i,$$

$$\frac{dv_i}{dt} = \frac{\lambda}{N}\sum_{j=1}^{N}\Big[\psi_1(r_{ij})(v_j - v_i) + \psi_2(r_{ij})((x_j - x_i)\cdot(v_j - v_i))(x_j - x_i)\Big],$$

where $r_{ij} = |x_j - x_i|$ and

$$\psi_1(r) = \frac{\varphi'(r)}{r} \geq 0, \qquad \psi_2(r) = \frac{1}{r}\Big(\frac{\varphi'(r)}{r}\Big)' = \frac{1}{r}\psi_1'(r) \geq 0.$$

Outline

Momentum and Energy Dissipation Estimates

Lemma

The total momentum and energy satisfy

$$(i)\ \frac{d}{dt}\sum_{i=1}^{N} v_i = 0.$$

$$(ii)\ \frac{d}{dt}\sum_{i=1}^{N} |v_i|^2 = -\frac{\lambda}{N}\sum_{1\le i,j\le N}\psi_1(|x_j - x_i|)|v_j - v_i|^2 - \frac{\lambda}{N}\sum_{1\le i,j\le N}\psi_2(|x_j - x_i|)|(x_j - x_i)\cdot(v_j - v_i)|^2.$$

Equation for Fluctuations

$$\langle x\rangle := \frac{1}{N}\sum_{i=1}^{N} x_i,\ \langle v\rangle := \frac{1}{N}\sum_{i=1}^{N} v_i, \quad \bar{x}_i := x_i - \langle x\rangle,\ \bar{v}_i := v_i - \langle v\rangle.$$

Dynamics for the averaged quantities and fluctuations are completely decoupled:

$\langle v(t)\rangle = \langle v(0)\rangle$, $\langle x(t)\rangle = \langle x(0)\rangle + t\langle v(0)\rangle$, and

$$\frac{d\bar{x}_i}{dt} = \bar{v}_i,$$

$$\frac{d\bar{v}_i}{dt} = \frac{\lambda}{N}\sum_{j=1}^{N} \psi_1(|\bar{x}_j - \bar{x}_i|)(\bar{v}_j - \bar{v}_i)$$

$$+ \frac{\lambda}{N}\sum_{j=1}^{N} \psi_2(|\bar{x}_j - \bar{x}_i|)((\bar{x}_j - \bar{x}_i)\cdot(\bar{v}_j - \bar{v}_i))(\bar{x}_j - \bar{x}_i).$$

Equation for Fluctuations

$$\langle x\rangle := \frac{1}{N}\sum_{i=1}^{N} x_i,\ \langle v\rangle := \frac{1}{N}\sum_{i=1}^{N} v_i, \quad \bar{x}_i := x_i - \langle x\rangle,\ \bar{v}_i := v_i - \langle v\rangle.$$

Dynamics for the averaged quantities and fluctuations are completely decoupled:

$\langle v(t)\rangle = \langle v(0)\rangle,\ \langle x(t)\rangle = \langle x(0)\rangle + t\langle v(0)\rangle$, and

$$\frac{d\bar{x}_i}{dt} = \bar{v}_i,$$

$$\frac{d\bar{v}_i}{dt} = \frac{\lambda}{N}\sum_{j=1}^{N}\psi_1(|\bar{x}_j-\bar{x}_i|)(\bar{v}_j-\bar{v}_i)$$

$$+\frac{\lambda}{N}\sum_{j=1}^{N}\psi_2(|\bar{x}_j-\bar{x}_i|)((\bar{x}_j-\bar{x}_i)\cdot(\bar{v}_j-\bar{v}_i))(\bar{x}_j-\bar{x}_i).$$

Flocking Estimate

Dissipation estimate for the system of fluctuations $\rightsquigarrow$ flocking estimate for original system

Theorem

Our new multidimensional model admits flocking in time-asymptotic limit. More precisely, for any solution (x_i, v_i) *with a finite initial energy for perturbation, we have*

$$(i)\Big(\sum_{i=1}^{N} |\bar{v}_i(t)|^2\Big) \le e^{-2\lambda R_1 t}\Big(\sum_{i=1}^{N} |\bar{v}_i(0)|^2\Big), \quad t > 0.$$

$$(ii)\ \max_{1\le i\le N} |\bar{x}_i(t)| \le \max_{1\le i\le N} |\bar{x}_{i0}| + \frac{1}{\lambda R_1}\Big(\sum_{i=1}^{N} |\bar{v}_i(0)|^2\Big)^{\frac{1}{2}},$$

provided $\psi_1 > 0,\ \psi_1' > 0,\ \lim_{r\to 0^+} \psi_1(r) = R_1 > 0.$

Flocking Estimate

Dissipation estimate for the system of fluctuations $\rightsquigarrow$ flocking estimate for original system

Theorem

Our new multidimensional model admits flocking in time-asymptotic limit. More precisely, for any solution (x_i, v_i) *with a finite initial energy for perturbation, we have*

$$(i)\left(\sum_{i=1}^{N}|\bar{v}_i(t)|^2\right) \le e^{-2\lambda R_1 t}\left(\sum_{i=1}^{N}|\bar{v}_i(0)|^2\right), \quad t > 0.$$

$$(ii)\ \max_{1\le i\le N}|\bar{x}_i(t)| \le \max_{1\le i\le N}|\bar{x}_{i0}| + \frac{1}{\lambda R_1}\left(\sum_{i=1}^{N}|\bar{v}_i(0)|^2\right)^{\frac{1}{2}},$$

provided $\psi_1 > 0,\ \psi_1' > 0,\ \lim_{r\to 0^+}\psi_1(r) = R_1 > 0.$

Approximate Flocking for Singularly Perturbed Mechanical System

Corollary

Let $\varepsilon > 0$ be sufficiently small. Then for any small positive time t_1 and a given $T_* > t_1$, there exists a positive constant C such that perturbed solution $(x_i^\varepsilon, v_i^\varepsilon)$ satisfy

$$(i)\ |v_i^\varepsilon(t) - v_j^\varepsilon(t)| < 2\varepsilon + Ce^{-\lambda R_1 t}, \qquad t \in [t_1, T_*].$$

$$(ii)\ |x_i^\varepsilon(t) - x_j^\varepsilon(t)| \leq 2\varepsilon + C.$$

Here $(x_i^0(0), v_i^0(0))$ is the initial datum corresponding to unperturbed solution $(x_i^0(t), v_i^0(t))$, and the constant C depends only on the initial datum of the unperturbed solution.

THANK YOU!!!