# A Kinetic Flocking Model with Diffusion: Patterns and Stability

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# Outline

A flocking model with diffusion

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The main result

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Mathematical tools and architecture

- Global existence and uniqueness: micro-macro decomposition and a priori estimates
- Rate of convergence: linearization a non-standard hypocoercivity

# Outline

# Vision

Pattern formation in collective phenomena (flocking, mills etc.) are interesting properties of recent nonlinear models;

- Need of mathematical tools and techniques for describing such patterns and their stability.

This work is motivated by this need – for a specific model – and it introduces some fundamental techniques which might be useful for other models:

*A Kinetic Flocking Model with Diffusion* (with R. Duan and G. Toscani), preprint, 2009

# The Cucker and Smale model with noise

Evolution of $m$ particles (e.g. birds) with positions and velocities $(x_i, \xi_i) = (x_i(t), \xi_i(t))$at time $t$ in the phase space $\mathbb{R}^n \times \mathbb{R}^n$:

$$\begin{cases} dx_i = \xi_i dt, \\ \\ d\xi_i = \sum_{j=1}^{m} U(|x_j - x_i|)(\xi_j - \xi_i)dt + \sqrt{2\mu \sum_{j=1}^{m} U(|x_j - x_i|)} dW_i. \end{cases}$$

Here, $U$ denotes the distance potential (communication rate). Typical example (Cucker-Smale model)

$$U(x) = \frac{C_{n,\gamma}}{(1 + |x|^2)^\gamma}, \quad x \in \mathbb{R}^n.$$

Random noise term: $W_i = W_i(t)$ are $m$ independent Wiener processes with values in $\mathbb{R}^n$, and $\mu \geq 0$ is the noise strength.

# Mean-field limit

Set

$$U = \frac{\kappa}{m} U_0$$

for some function $U_0$ and some constant $\kappa > 0$ and let

$$f^{(m)}(t, x, \xi) = \frac{1}{m} \sum_{i=1}^{m} \delta(x - x_i(t)) \delta(\xi - \xi_i(t)).$$

Mean-field limit: there is a temporal measure $f(t) \in \mathcal{M}(\mathbb{R}^{2n})$ such that

$$f^{(m)} \to f(t) \ \text{ in } w^*\text{-}\mathcal{M}(\mathbb{R}^{2n}) \quad \text{as } m \to \infty.$$

and

$$\partial_t f + \xi \cdot \nabla_x f + \kappa U_0 * \rho_{\xi f} \cdot \nabla_\xi f = \kappa U_0 * \rho_f \nabla_\xi \cdot (\mu \nabla_\xi f + \xi f),$$

with

$$\rho_f(t, x) = \int_{\mathbb{R}^n} f(t, x, \xi) d\xi, \quad \rho_{\xi f}(t, x) = \int_{\mathbb{R}^n} \xi f(t, x, \xi) d\xi.$$

# Differences with the Cucker and Smale kinetic model

- We assume $U$ is continuous in $x$ with

$$U(x) = U(|x|) \geq 0, \quad \int_{\mathbb{R}^n} U(x)dx = 1.$$

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- Moreover, the equation

$$\partial_t f + \xi \cdot \nabla_x f + U * \rho_{\xi f} \cdot \nabla_\xi f = U * \rho_f \left[ \nabla_\xi \cdot (\xi f) + \Delta_\xi f \right],$$
$$f(0, x, \xi) = f_0(x, \xi),$$

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  contains a diffusive term (in velocity).
- No natural Lyapunov functional is available except when the potential $U$ is the Dirac delta function; the model reduces to the classical local nonlinear Fokker-Planck equation.

# A stationary pattern

A steady state of

$$\partial_t f + \xi \cdot \nabla_x f + U * \rho_{\xi f} \cdot \nabla_\xi f = U * \rho_f \nabla_\xi \cdot (\nabla_\xi f + \xi f),$$

is the global Maxwellian:

$$\mathbf{M} = \mathbf{M}(\xi) = \frac{1}{(2\pi)^{n/2}} \exp\left(-|\xi|^2/2\right),$$

- The solution has infinite mass (not physically meaningful, birds everywhere!), but the model is a good prototype for stability analysis!

## Goal: establish the technology for stability of equilibrium states

Consider the perturbation $u = u(t, x, \xi)$ and

$$f = \mathbf{M} + \sqrt{\mathbf{M}}u.$$

Goal: we want to show local stability, that means

$$u(t) \to 0, \quad t \to \infty,$$

for $u(0)$ small (!), with a controlled (algebraic) rate.

- Result not yet physically relevant;
- Mathematical tools and proof architecture of highest interest (still strongly bound to the particular structure of the global Maxwellian);
- Role of noise (diffusion) in stabilizing steady states.

# Properties of the perturbation

Then, $u$ satisfies

$$\begin{aligned} \partial_t u + \xi \cdot \nabla_x u + U * \rho_{\xi\sqrt{\mathbf{M}}u} \cdot \nabla_\xi u &= \mathbf{L}u + \Gamma(u,u), \\ u(0,x,\xi) &= u_0(x,\xi), \end{aligned}$$

where $u_0$ is in the form of

$$u_0 = \mathbf{M}^{-1/2}(f_0 - \mathbf{M}),$$

and the linear part $\mathbf{L}u$ and the nonlinear part $\Gamma(u,u)$ are respectively given by

$$\begin{aligned} \mathbf{L}u &= \underbrace{\Delta_\xi u + \frac{1}{4}(2n - |\xi|^2)u}_{:=\mathbf{L}_{FP}u} + \underbrace{U * \rho_{\xi\sqrt{\mathbf{M}}u} \cdot \xi\sqrt{\mathbf{M}}}_{:=\mathbf{A}u}, \\ \Gamma(u,u) &= U * \rho_{\sqrt{\mathbf{M}}u}[\Delta_\xi u + \frac{1}{4}(2n - |\xi|^2)u] + \frac{1}{2}U * \rho_{\xi\sqrt{\mathbf{M}}u} \cdot \xi u. \end{aligned}$$

# The main result

## Theorem

*Let $n \geq 3$ and $N \geq 2[n/2]+2$. Suppose that $f_0 \equiv \mathbf{M} + \sqrt{\mathbf{M}} u_0 \geq 0$, and $\|u_0\|_{H^N_{x,\xi}}$ is small enough. Then, the following holds.*

*(i) The Cauchy problem for $u$ admits a unique global solution $u(t,x,\xi)$, satisfying*

$$u \in C([0,\infty); H^N(\mathbb{R}^n \times \mathbb{R}^n)), \quad f \equiv \mathbf{M} + \sqrt{\mathbf{M}} u \geq 0,$$

*and*

$$\|u(t)\|_{H^N_{x,\xi}} \leq C \|u_0\|_{H^N_{x,\xi}},$$

*(ii) The obtained solution $u$ enjoys the time-decay estimate as follows. If $\|u_0\|_{Z_1}$ is bounded and $\|\xi u_0\|$ is small enough, then it holds that*

$$\|u(t)\|_{H^N_{x,\xi}} \leq C \left( \|u_0\|_{H^N_{x,\xi}} + \|u_0\|_{Z_1} \right) (1+t)^{-\frac{n}{4}}, \quad t \geq 0.$$

# Outline

# Global existence and uniqueness

The tools used for global existence and uniqueness:

- Contraction principle for local existence and uniqueness
- Micro-macro decomposition and reduction of the equations to macro quantities
- Micro-macro a priori estimates
- Energy dissipation method for global stability, existence, and uniqueness

# Contraction principle and local existence and uniqueness

Define the approximate solution sequence $(f^m(t,x,\xi))_{m=0}^{\infty}$ by solutions to the Cauchy problem of the iterative equations as follows

$$\begin{cases} \partial_t f^{m+1} + \xi \cdot \nabla_x f^{m+1} + U * \rho_{\xi f^m} \cdot \nabla_\xi f^{m+1} \\ \qquad\qquad\qquad\qquad = U * \rho_{f^m} \nabla_\xi \cdot (\nabla_\xi f^{m+1} + \xi f^{m+1}), \\ f^{m+1} \equiv \mathbf{M} + \sqrt{\mathbf{M}} u^{m+1}, \\ f^{m+1}|_{t=0} = f_0 \equiv \mathbf{M} + \sqrt{\mathbf{M}} u_0, \end{cases}$$

or equivalently in terms of $u^m(t,x,\xi)$:

$$\begin{cases} \partial_t u^{m+1} + \xi \cdot \nabla_x u^{m+1} + U * b^{u^m} \cdot \nabla_\xi u^{m+1} \\ \qquad\qquad\qquad\qquad = \mathbf{L}_{FP} u^{m+1} + \Gamma(u^m, u^{m+1}) + \mathbf{A} u^m, \\ u^{m+1}|_{t=0} = u_0, \end{cases}$$

where $m \geq 0$, and $u^0 \equiv 0$ is set at the beginning of iteration

# Contraction principle and local existence and uniqueness

Define the solution space $X(0,T;M)$ by

$$X(0,T;M) = \left\{ \begin{array}{l} v \in C([0,T]; H^N(\mathbb{R}^n \times \mathbb{R}^n)) : \\ \sup_{0 \le t \le T} \|v(t)\|_{H^N_{x,\xi}} \le M, \quad \mathbf{M} + \sqrt{\mathbf{M}} v \ge 0 \end{array} \right\}.$$

**Theorem**

*Let $n, N$ satisfy as before. There are constants $T_* > 0$, $\epsilon_0$, $M_0$ such that if $u_0 \in H^N(\mathbb{R}^n \times \mathbb{R}^n)$ with $f_0 \equiv \mathbf{M} + \sqrt{\mathbf{M}} u_0 \ge 0$ and $\|u_0\|_{H^N_{x,\xi}} \le \epsilon_0$, then for each $m \ge 1$, $u^m$ is well-defined with*

$$u^m \in X(0, T_*; M_0).$$

*Furthermore, $(u^m)_{m \ge 0}$ is a Cauchy sequence in the Banach space $C([0,T_*]; H^{N-1}(\mathbb{R}^n \times \mathbb{R}^n))$, and the corresponding limit function denoted by $u$ belongs to $X(0, T_*; M_0)$, and $u$ is the unique solution to the Cauchy problem.*

# Global existence and uniqueness via energy dissipation

We claim that there are the equivalent energy $\mathcal{E}(u(t)) \sim \|u(t)\|^2_{H^N_{x,\xi}}$ and energy dissipation rate $\mathcal{D}(u(t)) \geq 0$ such that

$$\frac{d}{dt}\mathcal{E}(u(t)) + \lambda\mathcal{D}(u(t)) \leq 0,$$

holds for any $0 \leq t \leq T$ under a priori smallness assumption. Hence

$$\sup_{0\leq t\leq T}\left\{\|u(t)\|_{H^N_{x,\xi}} + \lambda\int_0^t \mathcal{D}(u(s))ds\right\} \leq \mathcal{E}(u_0) \leq C\|u_0\|^2_{H^N_{x,\xi}},$$

where $C$ is independent of $T$ and $u_0$.
The global existence and uniqueness of solutions to the Cauchy problem follows from the above uniform a priori estimate together with the local existence as well as the continuum argument.

# Micro-macro decomposition

$$\begin{cases} u(t,x,\xi) = \mathbf{P}u + \{\mathbf{I}-\mathbf{P}\}u, \\ \mathbf{P}u \equiv \{a^u + b^u\cdot\xi\}\sqrt{\mathbf{M}}, \\ a^u = \langle\sqrt{\mathbf{M}}, u\rangle, \quad b^u = \langle\xi\sqrt{\mathbf{M}}, u\rangle, \end{cases}$$

where $\mathbf{P}u$ is called the macroscopic component of $u$ while $\{\mathbf{I}-\mathbf{P}\}u$ is called the corresponding microscopic component.

# Macro equations

The macro components $a^u$ and $b^u$ satisfy:

$$\begin{aligned}
&\partial_t a^u + \nabla_x \cdot b^u = 0 \text{ (balance law of mass)},\\
&\partial_t b_i^u + \partial_i a^u - (U * b_i^u - b_i^u) + U * a^u b_i^u - U * b_i^u a^u\\
&\qquad\qquad + \sum_{j=1}^n \partial_j A_{ij}(\{\mathbf{I}-\mathbf{P}\}u) = 0 \text{ (balance law of momentum)},\\
&\partial_t A_{ii}(\{\mathbf{I}-\mathbf{P}\}u) + \partial_i b_i^u - U * b_i^u b_i^u = A_{ii}(l + r),\\
&\partial_t A_{ij}(\{\mathbf{I}-\mathbf{P}\}u) + \partial_i b_j^u + \partial_j b_i^u - U * b_i^u b_j^u - U * b_j^u b_i^u\\
&\qquad = A_{ij}(l + r),\ i \neq j \text{ (evolution of second-order moments of } \{\mathbf{I}-\mathbf{P}\}u),
\end{aligned}$$

for

$$\begin{aligned}
l &= -\xi \cdot \nabla_x \{\mathbf{I}-\mathbf{P}\}u + \mathbf{L}_{FP}\{\mathbf{I}-\mathbf{P}\}u,\\
r &= U * a^u \mathbf{L}_{FP}\{\mathbf{I}-\mathbf{P}\}u + \frac{1}{2} U * b^u \cdot \xi\{\mathbf{I}-\mathbf{P}\}u - U * b^u \cdot \nabla_\xi \{\mathbf{I}-\mathbf{P}\}u,
\end{aligned}$$

and the moment function $A = (A_{ij}(\cdot))_{n\times n}$ is

$$A_{ij}(u) = \int_{\mathbb{R}^n} (\xi_i \xi_j - 1)\sqrt{\mathbf{M}} u \, d\xi.$$

# A priori estimates: micro case

- Zero-order:

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2+\lambda\|\{\mathbf{I}-\mathbf{P}\}u\|_\nu^2+\frac{1}{2}\|T_\Delta b^u\|_U^2$$
$$\le C\|(a^u,b^u)\|_{L^2_x\cap L^\infty_x}(\|\{\mathbf{I}-\mathbf{P}\}u\|_\nu^2+\|T_\Delta b^u\|_U^2)$$
$$+C\|(a^u,b^u)\|_{L^2_x}\|b^u\|^2_{L^\infty_x},$$

- Space derivatives:

$$\frac{1}{2}\frac{d}{dt}\sum_{1\le|\alpha|\le N}\|\partial_x^\alpha u(t)\|^2+\lambda\sum_{1\le|\alpha|\le N}\left(\|\partial_x^\alpha\{\mathbf{I}-\mathbf{P}\}u\|_\nu^2+\|T_\Delta\partial_x^\alpha b^u\|_U^2\right)$$
$$\le C\|\nabla_x(a^u,b^u)\|_{H_x^{N-1}}\left(\sum_{|\alpha|\le N}\|\partial_x^\alpha\{\mathbf{I}-\mathbf{P}\}u\|_\nu^2+\|\nabla_x(a^u,b^u)\|^2_{H_x^{N-1}}\right)$$
$$+C\|\nabla_x b^u\|_{H_x^{N-1}}\sum_{1\le|\alpha|\le N-1}\|\partial_x^\alpha\nabla_\xi\{\mathbf{I}-\mathbf{P}\}u\|^2,$$

# A priori estimates: micro case

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$$\le C\|(a^u,b^u)\|_{L_x^2\cap L_x^\infty}(\|\{\mathbf{I}-\mathbf{P}\}u\|_\nu^2 + \|T_\Delta b^u\|_U^2)$$
$$+C\|(a^u,b^u)\|_{L_x^2}\|b^u\|_{L_x^\infty}^2,$$

- Space derivatives:

$$\frac{1}{2}\frac{d}{dt}\sum_{1\le|\alpha|\le N}\|\partial_x^\alpha u(t)\|^2 + \lambda\sum_{1\le|\alpha|\le N}\left(\|\partial_x^\alpha\{\mathbf{I}-\mathbf{P}\}u\|_\nu^2 + \|T_\Delta\partial_x^\alpha b^u\|_U^2\right)$$
$$\le C\|\nabla_x(a^u,b^u)\|_{H_x^{N-1}}\left(\sum_{|\alpha|\le N}\|\partial_x^\alpha\{\mathbf{I}-\mathbf{P}\}u\|_\nu^2 + \|\nabla_x(a^u,b^u)\|_{H_x^{N-1}}^2\right)$$
$$+C\|\nabla_x b^u\|_{H_x^{N-1}}\sum_{1\le|\alpha|\le N-1}\|\partial_x^\alpha\nabla_\xi\{\mathbf{I}-\mathbf{P}\}u\|^2,$$

# A priori estimates: micro case

- Mixed space-velocity derivatives: Let $1 \leq k \leq N$. It holds that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\sum_{\substack{|\beta|=k\\|\alpha|+|\beta|\leq N}}\|\partial_x^\alpha\partial_\xi^\beta\{\mathbf{I}-\mathbf{P}\}u\|^2+\lambda\sum_{\substack{|\beta|=k\\|\alpha|+|\beta|\leq N}}\|\partial_x^\alpha\partial_\xi^\beta\{\mathbf{I}-\mathbf{P}\}u\|_\nu^2\\
&\leq C\|(a^u,b^u)\|_{H_x^N}\left(\sum_{|\alpha|+|\beta|\leq N}\|\partial_x^\alpha\partial_\xi^\beta\{\mathbf{I}-\mathbf{P}\}u\|_\nu^2+\|\nabla_x(a^u,b^u)\|_{H_x^{N-1}}^2\right)\\
&\quad+C\left(\sum_{|\alpha|\leq N-k+1}\|\partial_x^\alpha\{\mathbf{I}-\mathbf{P}\}u\|_\nu^2+\|\nabla_x(a^u,b^u)\|_{H_x^{N-k}}^2\right)\\
&\quad+C\chi_{\{2\leq k\leq N\}}\sum_{\substack{1\leq|\beta|\leq k-1\\|\alpha|+|\beta|\leq N}}\|\partial_x^\alpha\partial_\xi^\beta\{\mathbf{I}-\mathbf{P}\}u\|_\nu^2,
\end{aligned}
$$

# A priori estimates: macro case

There exists a temporal free energy $\mathcal{E}^n_{free}(u(t))$ in the form of

$$
\begin{aligned}
\mathcal{E}^n_{free}(u(t)) \quad = \quad & 3\sum_{|\alpha|\leq N-1}\sum_{j}\sum_{i\neq j}\int_{\mathbb{R}^n} A_{ii}(\partial_x^\alpha \partial_j\{\mathbf{I}-\mathbf{P}\}u)\partial_x^\alpha b_j^u\,dx \\
& -3\sum_{|\alpha|\leq N-1}\sum_{ij}\int_{\mathbb{R}^n} A_{ij}(\partial_x^\alpha \partial_i\{\mathbf{I}-\mathbf{P}\}u)\partial_x^\alpha b_j^u\,dx \\
& +\sum_{|\alpha|\leq N-1}\int_{\mathbb{R}^n}\partial_x^\alpha \nabla_x a^u\cdot\partial_x^\alpha b^u\,dx,
\end{aligned}
$$

such that

$$
\begin{aligned}
& \frac{d}{dt}\mathcal{E}^n_{free}(u(t)) + \lambda\|\nabla_x(a^u,b^u)\|^2_{H_x^{N-1}} \\
& \leq C\sum_{|\alpha|\leq N}(\|T_\Delta\partial_x^\alpha b^u\|_U^2 + \|\partial_x^\alpha\{\mathbf{I}-\mathbf{P}\}u\|^2) \\
& \quad +C\|(a^u,b^u)\|^2_{H_x^N}(\|\nabla_x(a^u,b^u)\|^2_{H_x^{N-1}} + \sum_{|\alpha|\leq N}\|\partial_x^\alpha\{\mathbf{I}-\mathbf{P}\}u\|^2)
\end{aligned}
$$

holds for $t\geq 0$, where $\lambda>0$ and $C$ are constants depending only on $n$. Moreover, it holds that

$$
|\mathcal{E}^n_{free}(u(t))| \leq C\|u(t)\|^2_{L^2_\xi(H_x^N)}.
$$

# Energy dissipation

We define

$$
\begin{aligned}
&\mathcal{E}(u(t)) \\
= \quad & M\|u(t)\|^2_{L^2_\xi(H^N_x)} + \mathcal{E}^n_{free}(u(t)) + K \sum_{1\le k\le N} C_k \sum_{|\beta|=k|\alpha|+|\beta|\le N} \|\partial^\alpha_x \partial^\beta_\xi \{\mathbf{I}-\mathbf{P}\}u\|^2,
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{D}(u(t)) \quad = \quad & \sum_{|\alpha|+|\beta|\le N} \|\partial^\alpha_x \partial^\beta_\xi \{\mathbf{I}-\mathbf{P}\}u(t)\|^2_\nu + \sum_{|\alpha|\le N} \| T_\Delta \partial^\alpha_x b^u(t)\|^2_U \\
& + \|\nabla_x (a^u, b^u)(t)\|^2_{H^{N-1}_x}.
\end{aligned}
$$

We have eventually

$$
\frac{d}{dt}\mathcal{E}(u(t)) + \lambda \mathcal{D}(u(t)) \le 0,
$$

under a priori smallness assumption.

# Outline

# Rate of convergence

The tools used for global existence and uniqueness:

- Linearization (spectral analysis + energy dissipation = hypocoercivity)
- Extension to the fully nonlinear problem

# Non-standard hypocoercivity and no exponential decay expected

Define the projector $\mathbf{P}_0$ by

$$\mathbf{P}_0 u = a^u \sqrt{\mathbf{M}}, \quad a^u \equiv \langle \sqrt{\mathbf{M}}, u \rangle.$$

Notice that it is straightforward to make estimates on $\mathbf{A}$ as

$$\left| \int_{\mathbb{R}^n} \langle \mathbf{A} u, u \rangle dx \right| \quad \leq \quad C_n \iint_{\mathbb{R}^n \times \mathbb{R}^n} |\{\mathbf{I} - \mathbf{P}_0\} u|^2 dx d\xi,$$

where $C_n = \langle |\xi|^2, \mathbf{M} \rangle$ depends only on $n$, and it holds that

$$- \int_{\mathbb{R}^n} \langle \mathbf{L}_{FP} u, u \rangle dx \geq \lambda_0 \| \{\mathbf{I} - \mathbf{P}_0\} u \|_\nu^2.$$

Since it is not clear whether $\lambda_0$ is strictly larger than $C_n$ for this time, then it is nontrivial to get the similar coercivity estimate on $\mathbf{L} = \mathbf{L}_{FP} + \mathbf{A}$.

# Non-standard hypocoercivity

However, we have the coercivity inequality

$$-\int_{\mathbb{R}^n} \langle \mathbf{L}u, u\rangle dx \quad \geq \quad \lambda_0 \|\{\mathbf{I} - \mathbf{P}\}u\|_\nu^2 + \frac{1}{2}\|T_\Delta b^u\|_U^2,$$

where

$$\mathbf{P}u \equiv \{a^u + b^u \cdot \xi\}\sqrt{\mathbf{M}}.$$

# Linearization

Let us consider the Cauchy problem of the linearized equation with a nonhomogeneous source:

$$\begin{cases} \partial_t u = \mathbf{B}u + h, \quad t > 0, x \in \mathbb{R}^n, \\ u\big|_{t=0} = u_0, \quad x \in \mathbb{R}^n, \end{cases}$$

$h = h(t, x, \xi)$ and $u_0 = u_0(x, \xi)$ are given, and the linear operator $\mathbf{B}$ is defined by

$$\mathbf{B} \;\; = \;\; -\xi \cdot \nabla_x + \mathbf{L}, \quad \mathbf{L} = \mathbf{L}_{FP} + \mathbf{A}.$$

Formally, the solution to the Cauchy problem can be written as the Duhamel formula

$$u(t) = e^{\mathbf{B}t} u_0 + \int_0^t e^{\mathbf{B}(t-s)} h(s) ds,$$

# Linearization

For $1 \le q \le 2$ and $m \ge 0$, set the rate index $\sigma_{q,m}$ by

$$\sigma_{q,m} = \frac{n}{2}\left(\frac{1}{q} - \frac{1}{2}\right) + \frac{m}{2}.$$

## Theorem

*Let $1 \le q \le 2$ and $n \ge 1$.*

*(i) For any $\alpha, \alpha'$ with $\alpha' \le \alpha$, and for any $u_0$ satisfying $\partial_x^\alpha u_0 \in L^2_{x,\xi}$ and $\partial_x^{\alpha'} u_0 \in Z_q$, one has*

$$\|\partial_x^\alpha e^{\mathbf{B}t} u_0\| \le C(1+t)^{-\sigma_{q,m}}(\|\partial_x^{\alpha'} u_0\|_{Z_q} + \|\partial_x^\alpha u_0\|).$$

*(ii) Similarly, for any $\alpha, \alpha'$ with $\alpha' \le \alpha$, and further technical conditions on $h$, one has*

$$\left\|\partial_x^\alpha \int_0^t e^{\mathbf{B}(t-s)} h(s) ds\right\|^2$$

$$\le C \int_0^t (1+t-s)^{-2\sigma_{q,m}}(\|\nu^{-1/2}\partial_x^{\alpha'} h(s)\|_{Z_q}^2 + \|\nu^{-1/2}\partial_x^\alpha h(s)\|^2) ds,$$

*for $t \ge 0$ with $m = |\alpha - \alpha'|$, where $C$ is a positive constant depending only on $n, m, q$.*

# Spectral analysis and energy dissipation

There is a free energy functional

$$\begin{aligned}\mathcal{E}^{l}_{free}(\widehat{u}(t,k)) \quad = \quad & 3\sum_{j}\sum_{i\neq j}\frac{\mathrm{i}k_j}{1+|k|^2}\left(A_{ii}(\{\mathbf{I}-\mathbf{P}\}\widehat{u}),\widehat{b^u_j}\right)\\ & -3\sum_{ij}\frac{\mathrm{i}k_i}{1+|k|^2}\left(A_{ij}(\{\mathbf{I}-\mathbf{P}\}\widehat{u}),\widehat{b^u_j}\right)\\ & -\frac{\mathrm{i}k}{1+|k|^2}\cdot\left(\widehat{b^u},\widehat{a^u}\right)\end{aligned}$$

such that

$$\begin{aligned}&\frac{\partial}{\partial t}\mathrm{Re}\ \mathcal{E}^{l}_{free}(\widehat{u}(t,k))+\frac{|k|^2}{4(1+|k|^2)}(|\widehat{a^u}|^2+|\widehat{b^u}|^2)\\ &\leq\frac{1-\mathrm{Re}\ \widehat{U}}{1+|k|^2}|\widehat{b^u}|^2+\frac{C}{1+|k|^2}\|\nu^{-1/2}\widehat{h}\|^2_{L^2_\xi}+C\|\{\mathbf{I}-\mathbf{P}\}\widehat{u}\|^2_{L^2_\xi}\end{aligned}$$

holds for $t\geq 0$ and $k\in\mathbb{R}^n$. Moreover,

$$|\mathcal{E}^{l}_{free}(\widehat{u}(t,k))|\leq C\|\widehat{u}(t,k)\|^2_{L^2_\xi},$$

for $t\geq 0$ and $k\in\mathbb{R}^n$.

# Spectral analysis and energy dissipation

It holds also that

$$\frac{1}{2}\frac{\partial}{\partial t}\|\widehat{u}(t,k)\|_{L^2_\xi} + \lambda|\{\mathbf{I}-\mathbf{P}\}\widehat{u}|_\nu^2 + (1-\mathrm{Re}\ \widehat{U})|\widehat{b^u}|^2 \leq C\|\nu^{-1/2}\widehat{h}(t,k)\|_{L^2_\xi}^2,$$

for any $t \geq 0$ and $k \in \mathbb{R}^n$.
There is an energy $\mathcal{E}_M^l(\widehat{u}(t,k))$ with

$$\|\widehat{u}(t,k)\|_{L^2_\xi}^2 \sim \mathcal{E}_M^l(\widehat{u}(t,k)) = M\|\widehat{u}(t,k)\|_{L^2_\xi}^2 + \mathrm{Re}\ \mathcal{E}_{free}^l(\widehat{u}(t,k))$$

such that

$$\frac{\partial}{\partial t}\mathcal{E}_M^l(\widehat{u}(t,k)) + \frac{\lambda|k|^2}{1+|k|^2}\mathcal{E}_M^l(\widehat{u}(t,k)) + \lambda(1-\mathrm{Re}\ \widehat{U})|\widehat{b^u}|^2 \leq C\|\nu^{-1/2}\widehat{h}(t,k)\|_{L^2_\xi}^2,$$

by using the Gronwall inequality, it gives

$$\mathcal{E}_M^l(\widehat{u}(t,k)) \leq e^{-\frac{\lambda|k|^2}{1+|k|^2}t}\mathcal{E}_M^l(\widehat{u_0}(k)) + C\int_0^t e^{-\frac{\lambda|k|^2}{1+|k|^2}(t-s)}\|\nu^{-1/2}\widehat{h}(s,k)\|_{L^2_\xi}^2\,ds.$$

# Extension to the fully nonlinear equation

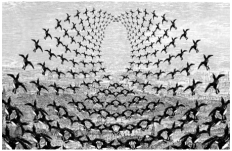
In the nonlinear case one has $h = \Gamma(u, u)$

$$\left\| \partial_x^\alpha \int_0^t e^{\mathbf{B}(t-s)} h(s) ds \right\|^2$$
$$\leq C \int_0^t (1 + t - s)^{-2\sigma_{q,m}} (\|\nu^{-1/2} \partial_x^{\alpha'} h(s)\|_{Z_q}^2 + \|\nu^{-1/2} \partial_x^\alpha h(s)\|^2) ds,$$

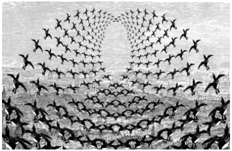
and the decay rate estimate follows by proper estimates of the (algebraic) decay of $\Gamma(u, u)$ (again by energy dissipation methods).

# Conclusion

- Summary: We presented a modified kinetic equation inspired by the Cucker and Smale model with an additional random noise



# Conclusion



- Summary: We presented a modified kinetic equation inspired by the Cucker and Smale model with an additional random noise
- We studied the stability of the global Maxwellian and the algebraic rate of convergence

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- **References**:

  - *Asymptotic flocking dynamics for the kinetic Cucker-Smale model* (with J. A. Carrillo, J. Rosado, and G. Toscani), preprint 2009.
  - *A kinetic flocking model with diffusion* (with R. Duan and G. Toscani), preprint 2009