On non-local interaction equations

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Background

Non-local Fokker-Planck type equation

ρ individual/particle density, mass $\int_{\mathbb{R}} \rho = 1$ conserved

$$\partial_t \rho = \partial_x (\rho \, \partial_x [a(\rho) + W * \rho + V])$$

“smooth”, even interaction potential $W(x) = W(-x)$

V external potential, a “diffusion” [a]

- inelastic material
- (swarming, flocking) collective behaviour (attractive, repulsive/attractive)
- chemotaxis

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[a] names in audience: Bertozzi, Blanchet, Carrillo

Non-local interaction equations

Non-local interaction equation

measure solutions, mass $\int_{\mathbb{R}} \rho = 1$ conserved

$$\partial_t \rho = \partial_x (\rho \, \partial_x [W * \rho + V])$$

“smooth”, even interaction potential $W(x) = W(-x)$[a]

1D: consider $u(z)$ pseudo-inverse of the distribution function $u(z) = \inf \left\{ x \in \mathbb{R}; \int_{-\infty}^{x} \rho \, dx > z \right\}$, for $z \in [0,1]$

$$\partial_t u(z) = \int_0^1 W'(u(\zeta) - u(z)) \, d\zeta + V'(u(z)) \,,$$

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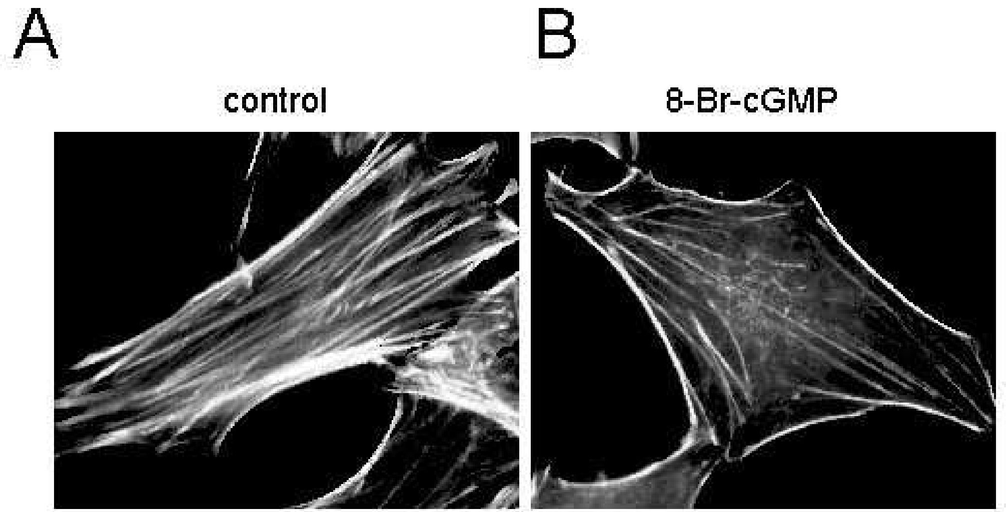
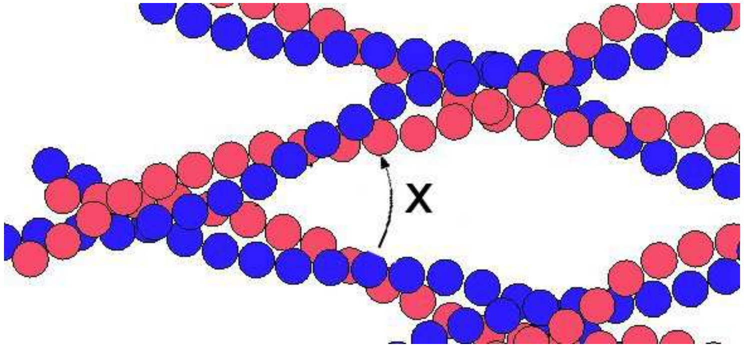
[a] [Burger, Di Francesco], [Carrillo, Di Francesco, Figalli, Laurent, Slepčev]

Non-local interaction equations

Actin filaments with or without cross-linking proteins





[a]

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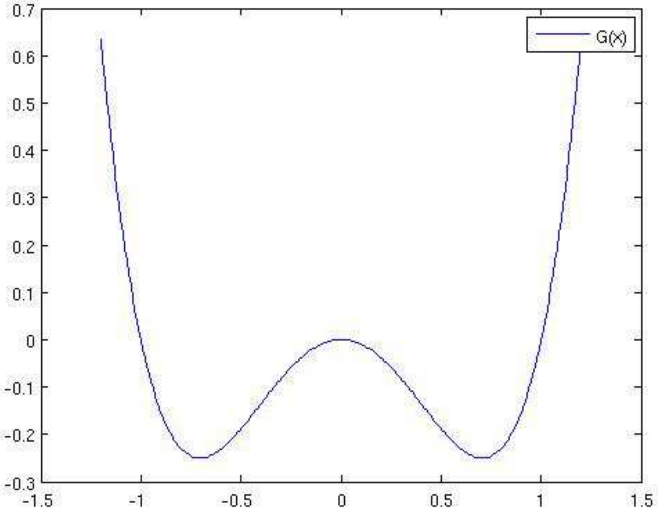
[a] [Kang, Perthame, Primi, Stevens, Velazquez]

A non-local repulsion-aggregation model

A double-well repulsion-aggregation potential, $V = 0$



double-well: local max at $x = 0$ and local min at $x = 2x_0$

$$\beta := - W''(0) > 0 , \qquad \alpha := W''(2x_0) > 0 .$$

A non-local repulsion-aggregation model

Conservation of (the centre of) mass, $V = 0$

$$\frac{d}{dt}\int_0^1 u(t,z)\,dz = \int_0^1\int_0^1 W'\left(u(\xi)-u(z)\right)\,d\xi\,dz = 0\,,$$

Conservation of (the centre of) mass $\int_0^1 u_{in}(z)\,dz$

$$\int_0^1 u(z,t)\,dz = \int_0^1 u_{in}(z)\,dz = 0 \qquad t \geq 0\,,$$

A non-local repulsion-aggregation model

Steady states

e.g. trivial solution $\bar{u}(z) = 0 \;\Leftrightarrow\; \bar{\rho} = \delta_0$

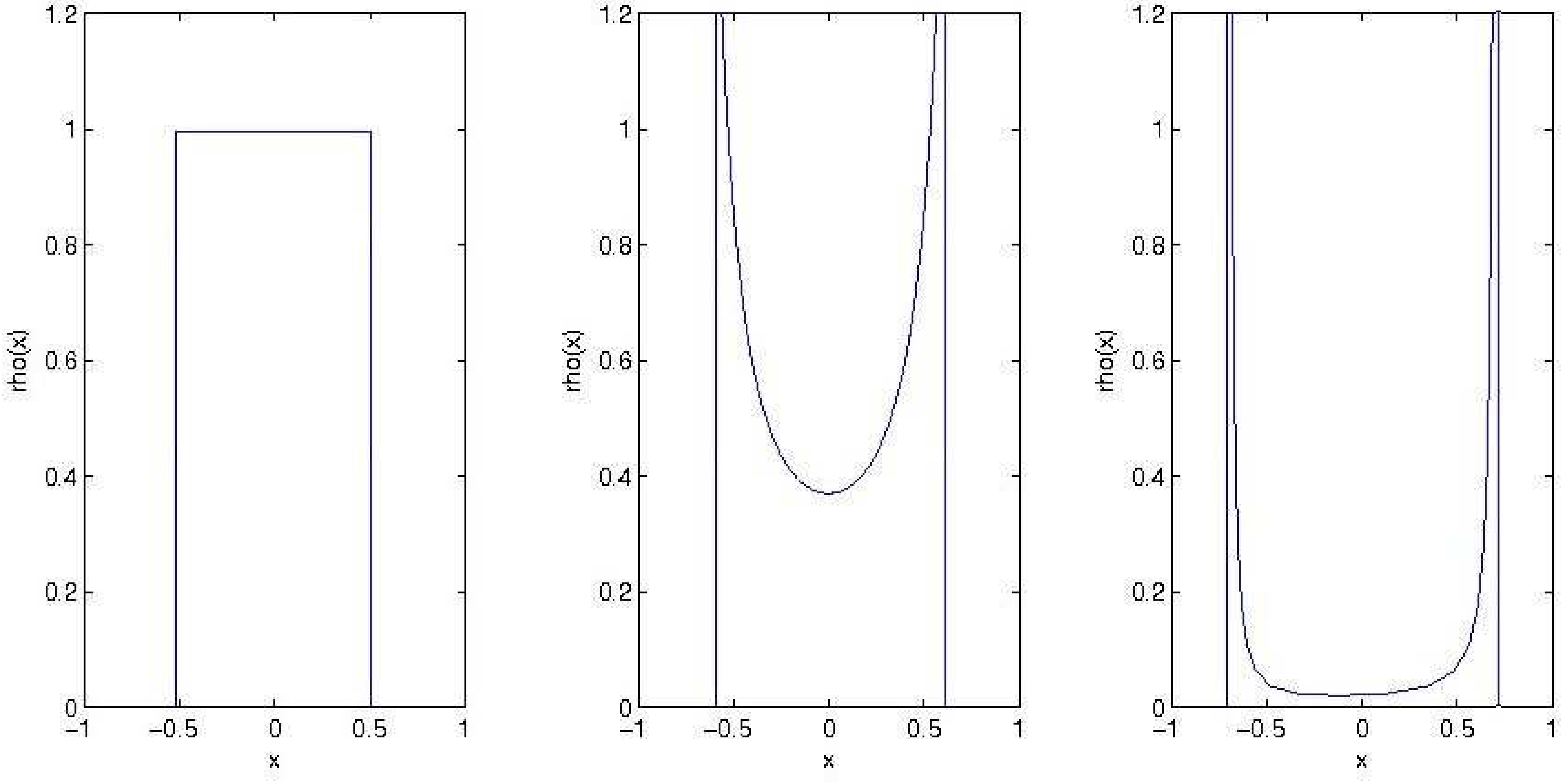
Family of monotone increasing two-valued steady states

$$\begin{aligned} \bar{u}(z, z_0) &= \begin{cases} -2(1-z_0)x_0 & z < z_0\,, \\ 2z_0x_0 & z > z_0\,. \end{cases} \\ \bar{\rho}(x, z_0) &= z_0\,\delta_{-2(1-z_0)x_0} + (1-z_0)\,\delta_{2z_0x_0} \end{aligned}$$

mass distribution parameter $z_0 \in (0,1)$

A non-local repulsion-aggregation model

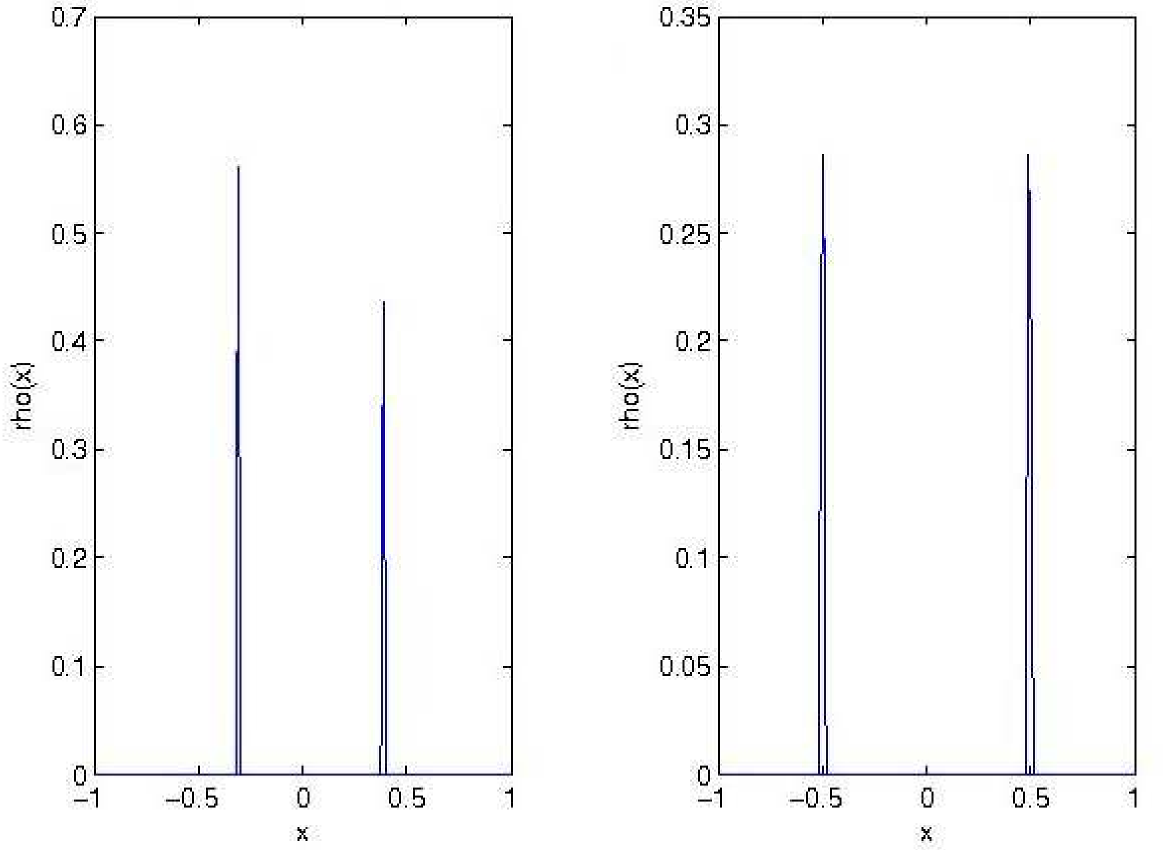
Numerics towards two-valued states



mass conserving scheme for $\partial_t u(z) = \int_0^1 W'(u(\zeta) - u(z))\,d\zeta$

A non-local repulsion-aggregation model

Non-uniqueness of two-valued states



Nevertheless, both should be stable.

A non-local repulsion-aggregation model

Linearised nonlocal operator

Consider mass preserving perturbations $v(z)$: $\int_0^1 v(z)\,dz = 0$.

$$F'(\bar{u}(z_0))(v(z)) = \begin{cases} \lambda_1\, v(z) - (\alpha+\beta) \int_0^{z_0} v(z)\,dz & z < z_0\,, \\ \lambda_2\, v(z) + (\alpha+\beta) \int_0^{z_0} v(z)\,dz & z > z_0\,. \end{cases}$$

where λ_1 and λ_2 denote

$$\lambda_1 := z_0\beta - (1-z_0)\alpha\,, \qquad \lambda_2 := (1-z_0)\beta - z_0\alpha\,.$$

λ_1 and λ_2 are convex combinations of β and $-\alpha$.

A non-local repulsion-aggregation model

Linear stability: Ansatz $v(z) = e^{\lambda t}\varphi(z)$

Eigenproblem $\lambda\varphi = F'(\bar{u}(z_0))(\varphi)$ with $\int_0^1 \varphi(z)\,dz = 0$

$$\begin{cases} (\lambda_1 - \lambda)\,\varphi(z) = +(\alpha+\beta)\int_0^{z_0}\varphi(z)\,dz & z < z_0\,, \\ (\lambda_2 - \lambda)\,\varphi(z) = -(\alpha+\beta)\int_0^{z_0}\varphi(z)\,dz & z > z_0\,. \end{cases}$$

$\lambda_1 \neq \lambda \neq \lambda_2$: Then, φ is piecewise constant and

$$\lambda = -\alpha < 0\,, \qquad v(z) = \begin{cases} -\frac{1-z_0}{z_0}v_r & z < z_0\,, \\ v_r & z > z_0\,. \end{cases}$$

Stability w.r.t. “local shifts” due to confinement

A non-local repulsion-aggregation model

Linear stability: Ansatz $v(z) = e^{\lambda t}\varphi(z)$

Eigenproblem $\lambda\varphi = F'(\bar{u}(z_0))(\varphi)$ with $\int_0^1 \varphi(z)\,dz = 0$

$$\begin{cases} (\lambda_1 - \lambda)\,\varphi(z) = +(\alpha+\beta)\int_0^{z_0}\varphi(z)\,dz & z < z_0\,, \\ (\lambda_2 - \lambda)\,\varphi(z) = -(\alpha+\beta)\int_0^{z_0}\varphi(z)\,dz & z > z_0\,. \end{cases}$$

$\lambda_1 = \lambda = \lambda_2$, $z_0 = \frac{1}{2}$: Then,

$$\lambda = \frac{\beta-\alpha}{2}\,, \qquad v(z) = \begin{cases} v_l(z) : \int_0^{1/2} v_l\,dz = 0 & z < \frac{1}{2}\,, \\ v_r(z) : \int_{1/2}^{1} v_r\,dz = 0 & z > \frac{1}{2}\,. \end{cases}$$

Stability w.r.t. “local smoothing” provided repulsion is controlled by aggregation.

A non-local repulsion-aggregation model

Linear stability: Ansatz $v(z) = e^{\lambda t}\varphi(z)$

Eigenproblem $\lambda\varphi = F'(\bar{u}(z_0))(\varphi)$ with $\int_0^1 \varphi(z)\,dz = 0$

$$\begin{cases} (\lambda_1 - \lambda)\,\varphi(z) = +(\alpha+\beta)\int_0^{z_0}\varphi(z)\,dz & z < z_0\,, \\ (\lambda_2 - \lambda)\,\varphi(z) = -(\alpha+\beta)\int_0^{z_0}\varphi(z)\,dz & z > z_0\,. \end{cases}$$

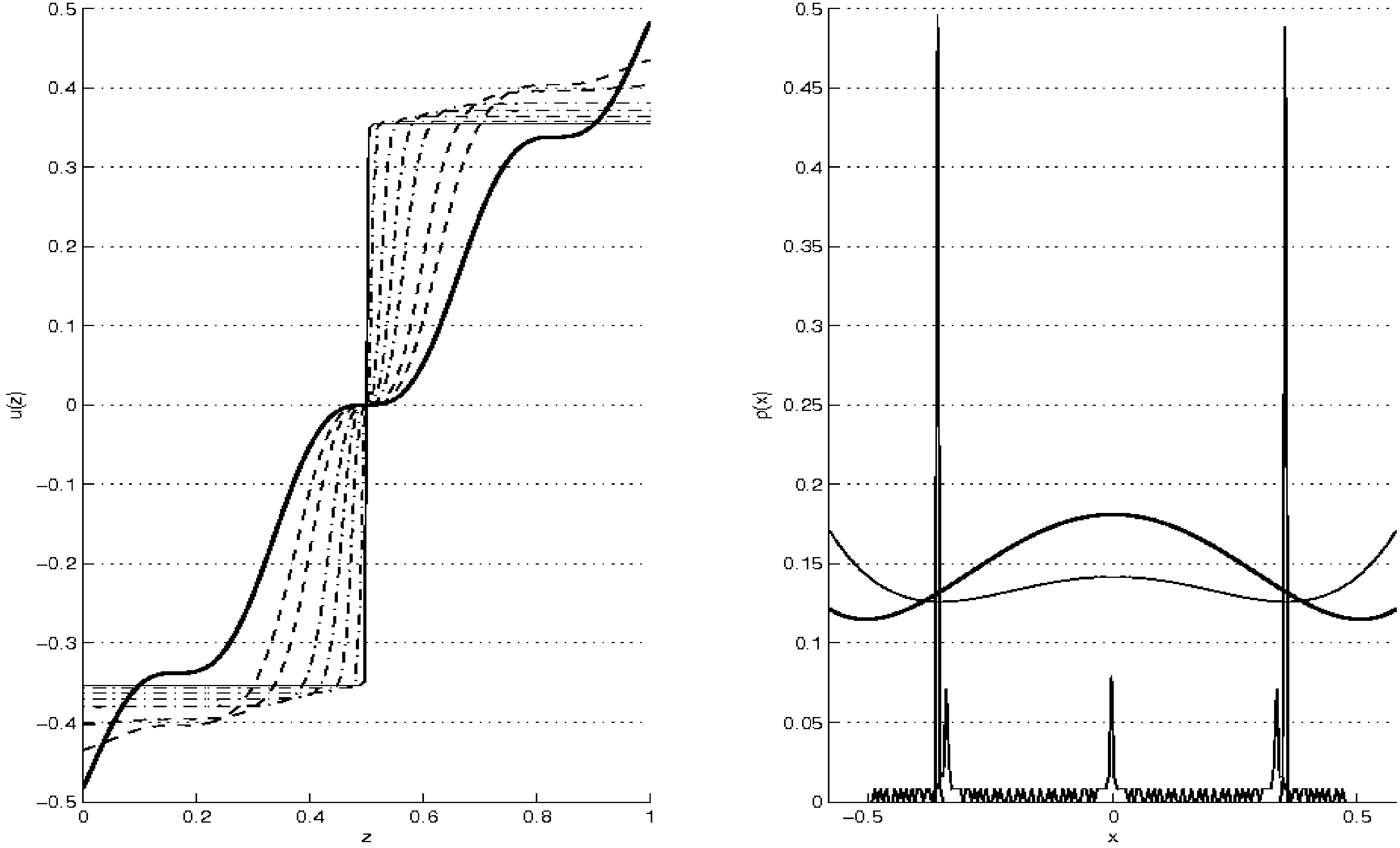
similar $\lambda_1 = \lambda$ or $\lambda_2 = \lambda$, $z_0 \neq \frac{1}{2}$ (symmetry $z_0 \to (1 - z_0)$)

Summary: Given $\beta - \alpha < 0$ there exists an open interval of parameters z_0 with linearly stable steady states $\bar{u}(z_0)$:

$$\max_{i=1,2}\{\lambda_i(z_0)\} < 0 \qquad \forall z_0 \in (1 - z_0^*, z_0^*)\,, \qquad z_0^* := \frac{\alpha}{\alpha+\beta} > \frac{1}{2}\,.$$

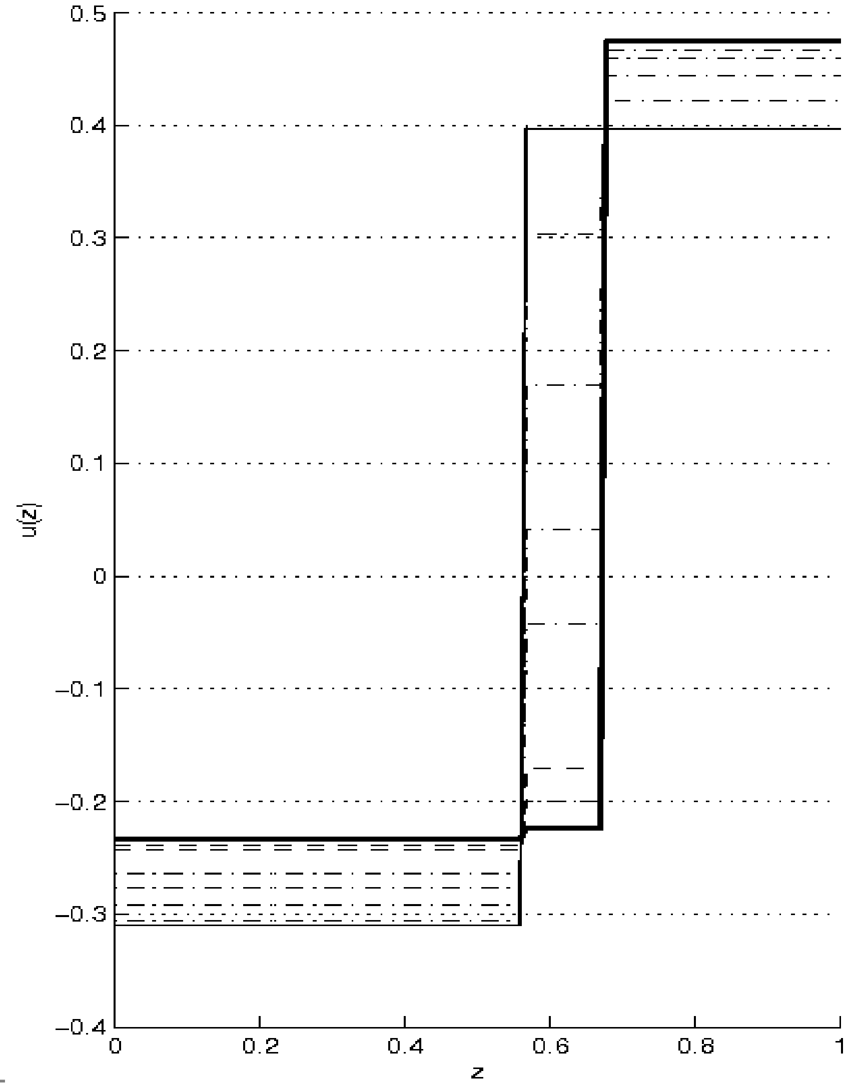
Non-local repulsion-aggregation

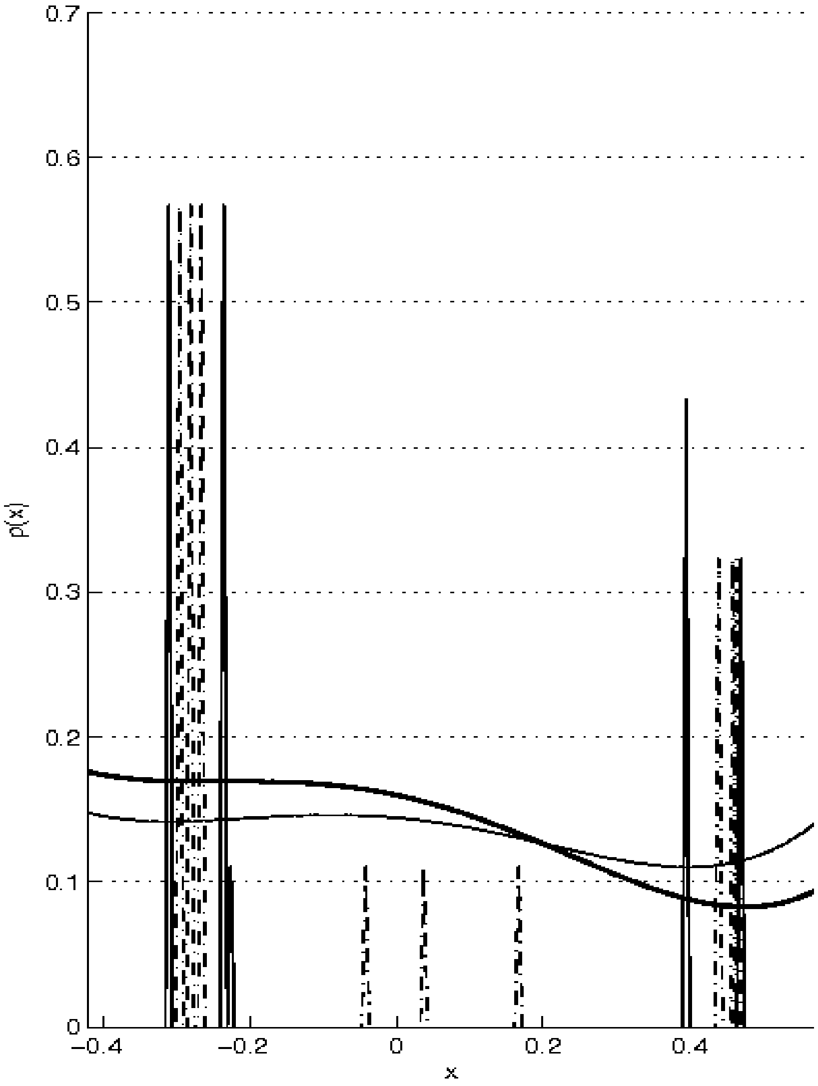
Three-valued states



Non-local repulsion-aggregation

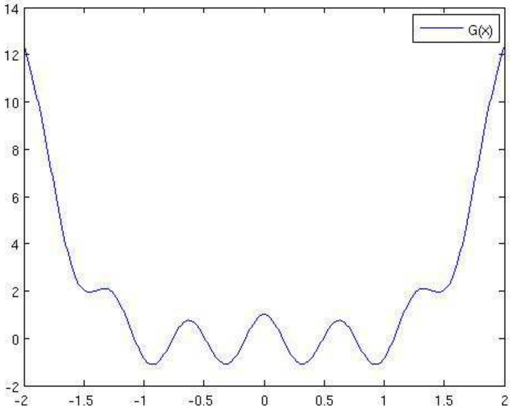
Three-valued states

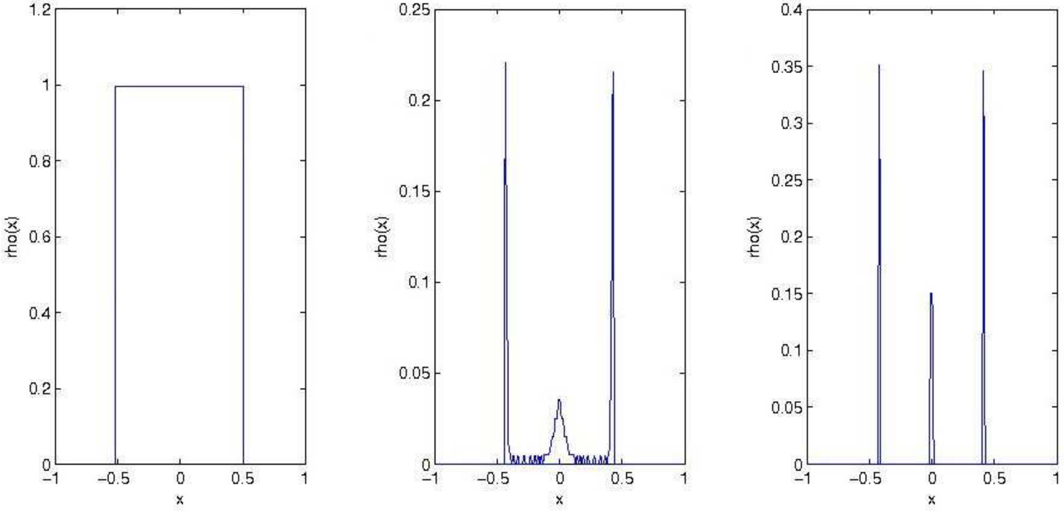




Non-local repulsion-aggregation

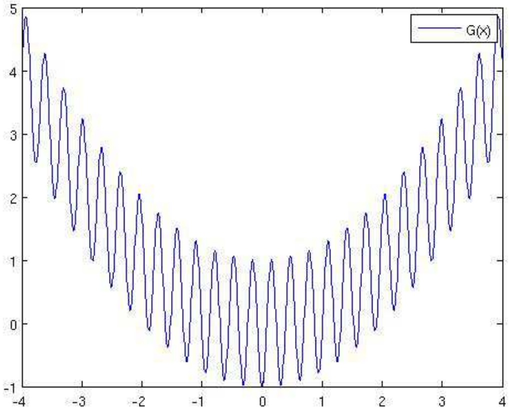
Three-valued states

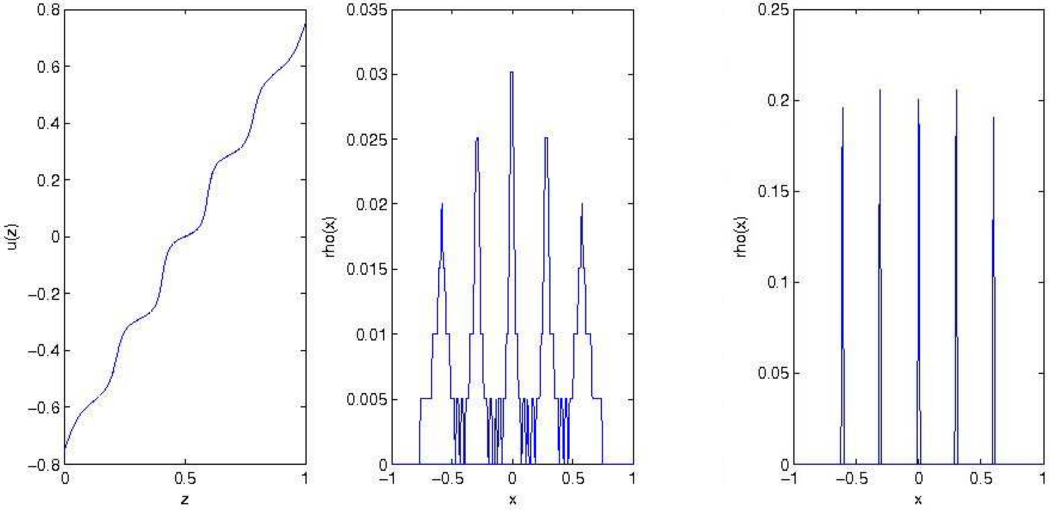




Non-local repulsion-aggregation

Non-uniqueness of two-valued states





Non-local interaction equation

Structure of steady states

- W analytic $\Rightarrow$ the steady states are “discrete“ sums of Diracs $\bar{\rho} = \sum_{i \in \mathbb{N}} \rho_i \delta_{u_i}$
- $W \in C^2 \Rightarrow$ accumulating Diracs have no spectral gap

A sum of Diracs $\bar{\rho} = \sum_{i=1}^{n} \bar{\rho}_i \delta_{\bar{u}_i}$, $\bar{\rho}_i > 0$ is steady state iff

$$(\bar{\rho}_j) \in Ker\left(\left(W'(\bar{u}_j - \bar{u}_i)\right)\right),$$

Proof:

$\partial_t \bar{u} = \int_0^1 W'\left(\bar{u}(t,\xi) - \bar{u}(t,z)\right) d\xi = \sum_j \bar{\rho}_j W'\left(\bar{u}_j - \bar{u}_i\right)$ on $z \in I_i$

Non-local interaction equation

Three Dirac steady states for double-well

A positive, normalised vector of masses

$$(\rho_1, \rho_2, \rho_3) = \frac{(-W'(u_3 - u_2), W'(u_3 - u_1), -W'(u_2 - u_1))}{-W'(u_3 - u_2) + W'(u_3 - u_1) - W'(u_2 - u_1)}$$

solves

$$\begin{pmatrix} -W'(u_3 - u_2) \\ W'(u_3 - u_1) \\ -W'(u_2 - u_1) \end{pmatrix} \times \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix} = 0\,,$$

for the double-well if and only if choose (u_i) such that $0 < u_2 - u_1, u_3 - u_2 < x_1$ and $u_3 - u_1 > x_1$

Local stability analysis

Linear stability steady state $\bar\rho = \sum_{i=1}^n \bar\rho_i \delta_{\bar u_i}$

- linear stability under small “local shifts” provided

$$\sum_j \bar\rho_j W''(\bar u_j - \bar u_i) > 0\,, \quad \forall i = 1, \ldots, n$$

- linearly stable under “local smoothing“ of the $\bar u_i$, if the matrix

$$M = \operatorname{diag}\left(\left(W''(u_j - u_i)\right) \cdot (\rho_j)\right) - \left(\rho_i W''(u_j - u_i)\right)$$

has a positive spectrum on the hyperspace $\{(w_i) : \sum_{i=1}^n w_i = 0\}$

Local stability analysis

Local stability without exchange of mass

A steady-state $\bar\rho = \sum_{i=1}^{n} \bar\rho_i \delta_{\bar u_i}$

stable w.r.t. “local shifts and smoothing” is locally stable

$$\|u(0) - \bar u\|_\infty \quad \Rightarrow \quad \|u(t) - \bar u\|_\infty \le C e^{-\kappa(\|W'''\|_\infty, n)t}$$

Proof: Consider the vector $w := \left(\|v\|_\infty, \int_{I_1} v, \cdots \int_{I_n} v\right)^T$, then

$$\frac{d}{dt} w = \begin{pmatrix} -W''(u_j - u_i)\cdot(\rho_j) & O(1) \\ 0 & -M \end{pmatrix} w + O(\|w\|^2),$$

Stability in higher dimensions via atomisation, see [CDFLS]

A non-local repulsion-aggregation model

Bifurcation for $z_0 > \frac{1}{2}$

Formal expansion if $\varepsilon := \lambda_1 \ll 1$

$$u(z) = \bar{u}(z, z_0 + \varepsilon a) + \varepsilon v(z)\,, \quad \int_0^1 v(z)\,dz = 0\,.$$

$W'''(0) = 0$, denote $W'''(2x_0) = \gamma$, $z_0^\varepsilon := z_0 + \varepsilon a$

$$\begin{cases} \varepsilon \left[-(\alpha+\beta) \int_0^{z_0} v\,dz \right] + O(\varepsilon^2) & z < z_0^\varepsilon \\ \varepsilon \left[(\beta-\alpha) v + (\alpha+\beta) \int_0^{z_0} v\,dz \right] + O(\varepsilon^2) & z > z_0^\varepsilon \end{cases}$$

$$O(\varepsilon): \quad \int_0^{z_0} v\,dz = \varepsilon V \qquad \text{and} \qquad v(z) = \varepsilon \tilde{v}(z)\,,\ z > z_0^\varepsilon\,,$$

A non-local repulsion-aggregation model

Bifurcation for $z_0 > \frac{1}{2}$

Integrate over $(0, z_0^\varepsilon)$ and $(z_0^\varepsilon, 1)$

$$V = -\frac{\gamma}{2}\frac{1-z_0}{2}\int_0^{z_0} v^2\,dz + O(\varepsilon)\,.$$

Reinsert

$$\begin{cases} \varepsilon^2\left[(1+\frac{a\alpha}{z_0})v - \frac{\gamma}{2}(1-z_0)v^2 + \frac{\gamma}{2}\frac{1-z_0}{z_0}\int_0^{z_0} v^2\,dz\right] + O(\varepsilon^3) & z < z_0^\varepsilon \\ \varepsilon^2\left[\frac{1-2z_0}{z_0}\alpha\tilde{v} - \frac{\gamma}{2}\frac{1-2z_0}{z_0}\int_0^{z_0} v^2\,dz\right] + O(\varepsilon^3) & z > z_0^\varepsilon \end{cases}$$

A non-local repulsion-aggregation model

Bifurcation for $z_0 > \frac{1}{2}$

v can assume at most two different values for $z < z_0$.

$$v(z) = \begin{cases} v_1 := -\frac{z_0 - z_1}{z_1} v_2 & 0 < z < z_1 \\ v_2 & z_1 < z < z_0 \end{cases} .$$

for a constant $v_2 \neq 0$ and $\int_0^{z_0} v \, dz = 0$.

$$v_1 = -\frac{2}{\gamma} \frac{z_0 + a\alpha}{(1 - z_0) z_0} \frac{z_0 - z_1}{2z_1 - z_0} , \qquad v_2 = \frac{2}{\gamma} \frac{z_0 + a\alpha}{(1 - z_0) z_0} \frac{z_1}{2z_1 - z_0} .$$

$$\tilde{v} = \frac{\gamma}{2\alpha} \frac{z_0 (z_0 - z_1)}{z_1} v_2^2 , \qquad V = \frac{\gamma}{2\alpha} \frac{(1 - z_0) z_0 (z_0 - z_1)}{z_1} v_2^2 .$$

A non-local repulsion-aggregation model

Three-valued steady states

Take two steps heights $0 < u_1, u_2 < 2x_0$ and $u_1 + u_2 > 2x_0$, denote $W'(u_1) = g_3$, $W'(u_1 + u_2) = -g_2$, and $W'(u_2) = g_1$.

Then, there are values $0 < z_1, z_2 < 1$ and $z_1 + z_2 < 1$ and

$$\bar{u}(z, u_1, u_2) = \begin{cases} u_l & 0 < z < z_1\,, \\ u_l + u_1 & z_1 < z < z_1 + z_2\,, \\ u_l + u_1 + u_2 & z_1 + z_2 < z < 1\,, \end{cases}$$

is a steady state with zero mass.

$u_1 = 2x_0$ or $u_2 = 2x_0$ yield the two-valued steady states

A non-local repulsion-aggregation model

Linear stability

Denote

$W''(0) = h_1, W''(u_1) = h_2, W''(u_1 + u_2) = h_3, W''(u_2) = h_4$

Linearised non-local operator $F'(\bar{u})(v)$

$$\begin{cases} \lambda_1 \, v(z) - h_1 \int_0^{z_1} v - h_2 \int_{z_1}^{z_1+z_2} v - h_3 \int_{z_1+z_2}^{1} v & z \in (0, z_1) \\ \lambda_2 \, v(z) - h_2 \int_0^{z_1} v - h_1 \int_{z_1}^{z_1+z_2} v - h_4 \int_{z_1+z_2}^{1} v & z \in (z_1, z_1 + z_2) \\ \lambda_3 \, v(z) - h_3 \int_0^{z_1} v - h_4 \int_{z_1}^{z_1+z_2} v - h_1 \int_{z_1+z_2}^{1} v & z \in (z_1 + z_2, 1) \end{cases}$$

where

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} h_1 & h_2 & h_3 \\ h_2 & h_1 & h_4 \\ h_3 & h_4 & h_1 \end{pmatrix} \cdot \begin{pmatrix} z_1 \\ z_2 \\ 1 - z_1 - z_2 \end{pmatrix}.$$

A non-local repulsion-aggregation model

Linear stability: ansatz $v(z) = e^{\lambda t}\varphi(z)$

Eigenproblem $\lambda\varphi = F'(\bar{u}(z_0))(\varphi)$ with $\int_0^1 \varphi(z)\,dz = 0$

$$\begin{cases} (\lambda_1 - \lambda)\,v(z) - (h_1 - h_3)\int_0^{z_1} v\,dz - (h_2 - h_3)\int_{z_1}^{z_1+z_2} v\,dz = 0 \\ (\lambda_2 - \lambda)\,v(z) - (h_2 - h_4)\int_0^{z_1} v\,dz - (h_1 - h_4)\int_{z_1}^{z_1+z_2} v\,dz = 0 \\ (\lambda_3 - \lambda)\,v(z) - (h_3 - h_1)\int_0^{z_1} v\,dz - (h_4 - h_1)\int_{z_1}^{z_1+z_2} v\,dz = 0 \end{cases}$$

A non-local repulsion-aggregation model

Linear stability: ansatz $v(z) = e^{\lambda t}\varphi(z)$

Eigenproblem $\lambda\varphi = F'(\bar{u}(z_0))(\varphi)$ with $\int_0^1 \varphi(z)\,dz = 0$

1) $\lambda \neq \lambda_i$, $i = 1, 2, 3$: v is piecewise constant. Eigenvalues

$$\mu_{1,2} = \ \frac{\tau}{2} \pm \sqrt{\frac{\tau^2}{4} - \Delta}\,,$$

$$\begin{cases} \tau = (h_2 + h_3)z_1 + (h_2 + h_4)z_2 + (h_3 + h_4)z_3\,, \\ \Delta = h_2h_3z_1 + h_2h_4z_2 + h_3h_4z_3 \end{cases}$$

$$\mu_3 = \ h_3z_1 + h_4z_2 + h_1z_3\,,$$

(in)stability of $\mu_{1,2,3}$ is not obvious.

A non-local repulsion-aggregation model

Linear stability: ansatz $v(z) = e^{\lambda t}\varphi(z)$

Eigenproblem $\lambda\varphi = F'(\bar{u}(z_0))(\varphi)$ with $\int_0^1 \varphi(z)\, dz = 0$

1) $\lambda \neq \lambda_i$, $i = 1, 2, 3$:
Relating back to two-valued steady states
$u_2 = 2x_0$ such that $g_1 = 0 = z_1$

$$\mu_1 = -\alpha\,, \qquad \mu_2 = \beta(1 - z_2) - \alpha z_2\,,$$

with only μ_1 satisfying $v_2 z_2 + v_3(1 - z_2) = 0$

A non-local repulsion-aggregation model

Linear stability: ansatz $v(z) = e^{\lambda t}\varphi(z)$

Eigenproblem $\lambda\varphi = F'(\bar{u}(z_0))(\varphi)$ with $\int_0^1 \varphi(z)\,dz = 0$

2) $\lambda = \lambda_1$ and $\lambda \neq \lambda_j$, $j = 2, 3$

$$(h_1 - h_3)\int_0^{z_1} v\,dz = (h_3 - h_2)\int_{z_1}^{z_1+z_2} v\,dz$$

while v is piecewise constant on $(z_1, z_1 + z_2)$ and $(z_1 + z_2, 1)$.

$$v = \begin{cases} v_1(z) & z \in (0, z_1) \\ v_2 & z \in (z_1, z_1 + z_2) \\ v_3 & z \in (z_1 + z_2, 1) \end{cases}$$

A non-local repulsion-aggregation model

Linear stability: ansatz $v(z) = e^{\lambda t}\varphi(z)$

Eigenproblem $\lambda\varphi = F'(\bar{u}(z_0))(\varphi)$ with $\int_0^1 \varphi(z)\,dz = 0$

2) Signs of the eigenvalues λ_i not obvious.

Relating back: $u_2 = 2x_0$ with $z_1 = 0$ and $h_1 = \beta$ and $h_4 = -\alpha$

$$\lambda_1 = h_2 z_2 + h_3(1-z_2)\,,\ \lambda_2 = \beta z_2 - \alpha(1-z_2)\,,\ \lambda_3 = \alpha z_2 - \beta(1-z_2)\,,$$

λ_1 is spurious since $z_1 = 0$.

A non-local repulsion-aggregation model

Linear stability: ansatz $v(z) = e^{\lambda t}\varphi(z)$

Eigenproblem $\lambda\varphi = F'(\bar{u}(z_0))(\varphi)$ with $\int_0^1 \varphi(z)\,dz = 0$

For small perturbations $u_2 - 2x_0 = -\delta < 0$

$$\lambda_2 = \beta z_2 - \alpha(1 - z_2) + (h_2\frac{\alpha}{g} - W'''(2x_0)(1 - z_2) + \frac{\alpha^2}{g})\delta + O(\delta^2)$$

At bifurcation $z_2 = \frac{\alpha}{\alpha+\beta}$

$$(h_2\frac{\alpha}{g} - W'''(2x_0)\frac{\beta}{\alpha+\beta} + \frac{\alpha^2}{g})$$

has sign depending on W

Singular repulsion

An explicit example

formal: local repulsion → Dirac $\Longrightarrow$ quadratic diffusion

Singular repulsive potential $W(x) = x^2 - |x|$

Steady state: $\bar{\rho} = \mathbb{I}_{[-\frac{1}{2}.\frac{1}{2}]}$

$$\begin{aligned} 0 = W' * \rho &= \int_{\mathbb{R}} 2(x-y)d\rho(y) - \int_{\mathbb{R}} \mathrm{sign}(x-y)d\rho(y) \\ &= 2x - \int_{-\infty}^{x} d\rho(y) + \int_{x}^{-\infty} d\rho(y) = 2x + 1 - 2\int_{-\infty}^{x} d\rho(y) \end{aligned}$$

Singular repulsion

An explicit example

evenly smoothed modulus $|x|_\varepsilon$ on the interval $(-\varepsilon, \varepsilon)$ for $\varepsilon > 0$

$$W_\varepsilon(x) = x^2 - |x|_\varepsilon, \quad W_\varepsilon'(x) = 2x - \operatorname{sign}_\varepsilon(x), \quad W_\varepsilon''(x) = 2 - 2\delta_\varepsilon(0)$$

where we assume

$$\operatorname{sign}_\varepsilon(0) = 0 \quad \text{and} \quad \operatorname{sign}_\varepsilon(\pm\varepsilon) = \pm 1 \qquad \delta_\varepsilon(0) \approx \frac{1}{\varepsilon}\,.$$

Singular repulsion

An explicit example

Construct stable steady states with n Dirac masses:

$\bar{u} = \sum_{i=1}^{n} u_i \mathbb{I}_{I_i}$ with $|I_i| = \rho_i$ and $\max_i\{(u_{i+1} - u_i)\} > \varepsilon$

$$0 = \sum_{j=1}^{n} \rho_j W_\varepsilon'(u_j - u_i) = -2u_i + \sum_{j<i} \rho_j - \sum_{j>i} \rho_j$$

using $\sum_{j=1}^{n} \rho_j = 1$ and $\sum_{j=1}^{n} u_j \rho_j = 0$

Obtain multitude of steady states

$$(u_{i+1} - u_i) = \frac{\rho_i + \rho_{i+1}}{2} > \varepsilon \quad \Rightarrow \varepsilon < \frac{1}{n}$$

stability: $m_i = \sum_{j=1}^{n} \rho_j W_\varepsilon''(u_j - u_i) = 2 - \frac{\rho_i}{\varepsilon} > 0 \Rightarrow \varepsilon > \frac{\rho_i}{2}$

Singular repulsion

Weak limit

calculate $u_1 = -\frac{1}{2} + \frac{\rho_1}{2} \to -\frac{1}{2}$ and $u_n = \frac{1}{2} - \frac{\rho_n}{2} \to \frac{1}{2}$ as $\rho_i < \frac{1}{n}$ for $n \to \infty$.

$$\int_{\mathbb{R}} \varphi(x)\, d\bar{\rho}(x) = \sum_{i=1}^{n} \varphi(u_i)\, \rho_i = \sum_{i=1}^{n} \int_{u_i - \frac{\rho_i}{2}}^{u_i + \frac{\rho_i}{2}} \varphi(u_i)\, dx$$

$$= \int_{u_1 - \frac{\rho_1}{2}}^{u_n + \frac{\rho_n}{2}} \sum_{i=1}^{n} \varphi(u_i) \mathbb{I}_{[u_i - \frac{\rho_i}{2}, u_i + \frac{\rho_i}{2}]}\, dx$$

$$\to \int_{-\frac{1}{2}}^{\frac{1}{2}} \varphi(x)\, dx = \int_{\mathbb{R}} \mathbb{I}_{[-\frac{1}{2}, \frac{1}{2}]} \varphi(x)\ dx\,,$$

Theorem: $W_\varepsilon \to W = -|x| \quad \Rightarrow \quad \bar{\rho}_\varepsilon \rightharpoonup \bar{\rho}$

Non-local interaction equations

Conclusions

- double-well does NOT imply that only two Dirac steady state are stable
- singular repuslsion of interaction potential at 0 acts like diffusion

THANK YOU!