Well-posedness in a space of measures of kinetic models for collective motion

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What kind of behavior do we want to model?

- Large groups of animals show complex behavior: flocks, swarms...
- We are interested in behavior that is the collective result of their individual behavior (not, for example, due to them following a "leader".)
- We think of a number of individuals typically of the order seen in nature ($\approx$ between 100 and 10000.)

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Possible effects we want to include

- A tendency to stay close to others, but not too close.
- A tendency to imitate the velocity of nearby individuals.
- Properties depending only on one individual, such as:
  - A maximum possible velocity (they cannot go arbitrarily fast)
  - A preferred velocity.
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Mathematical models

One can consider at least two frameworks to model the previous behavior:

1. A system of $2N$ ordinary differential equations, a pair for each individual: one for position, one for velocity. (Actually, $2Nd$, where d is the space dimension we consider.)
2. A PDE which captures the evolution of the *density* of a large group of such individuals.

ODE models

ODEs

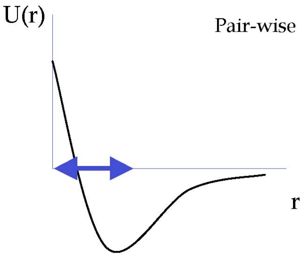
$$\dot{x}_i = v_i$$

$$\dot{v}_i = -\frac{1}{N}\sum_{j\neq i} \nabla U(x_i - x_j) \qquad \text{attraction / repulsion}$$

$$+ (\alpha - \beta |v_i|^2) v_i \qquad \text{self-propulsion}$$

$$+ \sum_{j\neq i} a_{ij} (v_j - v_i) \qquad \text{Cucker-Smale}$$

$$a_{ij} := \frac{1}{(1 + |x_i - x_j|^2)^\gamma}.$$



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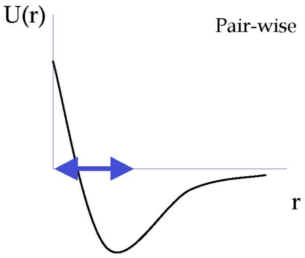

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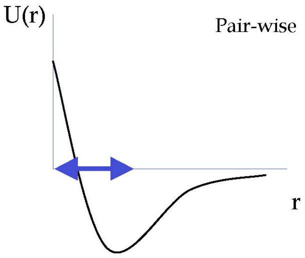

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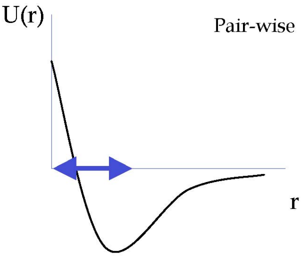

Kinetic models

$$f = f(t,x,v), \qquad \rho = \rho(t,x) = \int f(t,x,v)\,dv.$$

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Kinetic equation

$$\begin{aligned} 0 &= \partial_t f + v\nabla_x f \\ &\quad + \mathrm{div}_v\,((\nabla_x U * \rho)f) \\ &\quad + \mathrm{div}_v((\alpha - \beta\,|v|^2)v\,f) \\ &\quad - \mathrm{div}_v\,((H * f)f) \end{aligned}$$

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The W^1 distance between measures

Look at probability measures μ, ν on $\mathbb{R}^d$ as mass distributions. *A plan* to carry one to the other is a measure π on $\mathbb{R}^d \times \mathbb{R}^d$ such that (abusing notation a bit):

$\pi(x, y) \equiv$ how much mass from point x should go to point y.

Then, the W^1 distance between μ and ν is the *total cost* of carrying one to the other, using the best possible plan:

$$W^1(\mu, \nu) = \inf_{\pi} \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} \underbrace{|x - y|}_{\text{how far to go}} \quad \underbrace{\pi(x, y)}_{\text{how much mass to carry}} \right\}$$

Existence & continuous dependence

Theorem (Existence)

1. *For any compactly supported measure* $f_0 \in W^1(\mathbb{R}^{2d})$ *there exists a solution*

$$f \in \mathcal{C}([0, +\infty), W^1(\mathbb{R}^{2d}))$$

with initial condition f_0.

2. *For all* $t \geq 0$, f_t *has compact support.*

Theorem (Continuous dependence)

For any two solutions f, g *of the kinetic equation, there is* $r = r(t)$ *(depending only on the details of the equation, and the size of the support of* f *and* g *at* $t = 0$*) such that*

$$W^1(f_t, g_t) \leq r(t) W^1(f_0, g_0).$$

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Convergence of the particle method

Consider $\{(x_i(t), v_i(t)\}$ solution of the ODEs with N particles, f solution of the PDE. We define:

$$f_t^N := \sum_{i=1}^{N} m_i \, \delta_{(x_i(t),v_i(t))}.$$

Theorem (Mean-field limit)

If $$\lim_{N\to\infty} W_1(f_0^N, f_0) = 0,$$

then $$\lim_{N\to\infty} W_1(f_t^N, f_t) = 0, \qquad t \geq 0.$$

Hydrodynamic solutions

Assume *hydrodynamic solutions* of the kinetic eq. with potential + self-propulsion exist:

$$f(t,x,v) = \rho(t,x)\,\delta(v - u(t,x)).$$

Then, ρ and u should satisfy:

$$\partial_t \rho + \mathrm{div}_x(\rho u) = 0,$$
$$\partial_t u + (u \cdot \nabla) u = u(\alpha - \beta\,|u|^2) - \nabla U * \rho.$$

Theorem (Hydrodynamic solutions)

1. *Hydrodynamic solutions are unique.*
2. *Solutions with near-monokinetic initial conditions are close to solutions with monokinetic initial conditions.*

Things to be done

Study the *qualitative behavior* of these models:

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- Stability of these solutions?
- Asymptotic behavior for reasonably general initial conditions?
- Applications?

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The end

Thanks!