INSTANTONS ON NONCOMMUTATIVE TORIC VARIETIES

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Toric geometry

- **Definition:** A complex variety X is a **toric variety of dimension** n if it densely contains a (complex) algebraic torus $T = (\mathbb{C}^\times)^n$ and the natural T-action on itself (by translations) extends to X
- **Examples:** $X = T, \quad \mathbb{C}^n, \quad \mathbb{P}^n$
- X described by combinatorial data encoded in **toric diagram**:
  - Vertices $f =$ fixed points of T-action on X, with T-invariant open chart $\cong \mathbb{C}^n$
  - Edges $e = T$-invariant projective lines $\mathbb{P}^1$ joining 2 fixed points f_1, f_2
  - "Gluing rules": Near $\mathbb{P}^1$, X looks like $\mathcal{O}_{\mathbb{P}^1}(-m_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(-m_{n-1})$ – normal bundle determines local geometry of X near edge
- Or equivalently in dual **toric fan**: Maximal (polyhedral) cones dual to vertices define toric open cover of X; $n-1$-cones dual to edges; Gluing rules along adjacent faces $\sigma \cap \tau$ of cones σ, τ

Motivation – Instanton counting

- Heuristic local enumeration of (generalized) noncommutative instantons on each $\mathbb{C}^n \subset X$ assemble to global quantities using gluing rules of toric geometry

  (Nekrasov '02, . . . ; Iqbal *et al.* '08)

- Determines instanton contributions to SUSY gauge theory partition functions on X; computes enumerative invariants of varieties **e.g.,** Seiberg–Witten and Donaldson invariants for $n = 2$, Donaldson–Thomas invariants for $n = 3$

- Find global notion of "noncommutative toric variety" and construction of instantons

Motivation – String geometry

- Chiral fermions on a "quantum curve" embed in string theory as intersecting D-branes with B-field; described by a holonomic D-module (Dijkgraaf, Hollands & Sulkowski '09)

- $\left\{ \ D\text{-modules} \ \right\} \Longleftrightarrow \left\{ \ \text{modules on a noncommutative variety} \ \right\}$

- **Example:** $\left\{ \ \text{right ideals of Weyl algebra } \mathbb{C}[z, \partial_z] \ \right\}$ $\Longleftrightarrow \left\{ \ \text{line bundles on a noncommutative } \mathbb{P}^2 \ \right\}$
  (Baranovsky, Ginzburg & Kuznetsov '02)

- Vector bundles on noncommutative $\mathbb{P}^2$
  $\longleftrightarrow$ instantons on noncommutative $\mathbb{R}^4$
  (Nekrasov & Schwarz '98; Kapustin, Kuznetsov & Orlov '01)

Classical instantons

- $SU(r)$ vector bundle $E \longrightarrow M$ ($M = S^4$ or locally $M = \mathbb{R}^4$ and finite Yang–Mills energy) with anti-self-dual connection:

$$*F = -F$$

  labelled by "topological charge" (instanton number)
  $c_2(E) = \frac{1}{8\pi^2} \int_M \mathrm{Tr}\,(F \wedge F) = k \in H^4(M, \mathbb{Z})$

- Rank r holomorphic vector bundles $E \longrightarrow \mathbb{P}^2$ with $c_2(E) = k$, trivial on $\mathbb{P}^1$ at ∞ (Hitchin–Kobayashi correspondence)

- ADHM data (linear algebra!)

- Rank r holomorphic vector bundles $E \longrightarrow \mathbb{P}^3$ with $c_2(E) = k$, trivial on $\mathbb{P}^1$ at ∞, $H^1(\mathbb{P}^3, E(-2)) = 0$, reality condition (Atiyah–Penrose–Ward twistor correspondence)

Cocycle twist quantization

(Majid '95)

- H commutative Hopf algebra
  $F : H \otimes H \longrightarrow \mathbb{C}$ convolution-invertible unital two-cocycle on H

- H_F – new Hopf algebra, $H = H_F$ as coalgebra, but with:

$$h \times_F g := F(h_{(1)}, g_{(1)})\,(h_{(2)}\, g_{(2)})\, F^{-1}(h_{(3)}, g_{(3)})$$

- Simultaneously deforms all H-covariant constructions as functorial isomorphism of categories of left comodules:

$$\mathcal{Q}_F : {}^{H}\mathcal{M} \longrightarrow {}^{H_F}\mathcal{M}$$

Notation: $\Delta_L : A \longrightarrow H \otimes A$ left coaction of H on A,
$\Delta_L(a) := a^{(-1)} \otimes a^{(0)}$

Comodule twisting of algebras

- Trivial "flip" braiding on monoidal category ${}^{H}\mathcal{M}$:

$$\Psi \ : \ A \otimes B \ \longrightarrow \ B \otimes A \ , \qquad \Psi(a \otimes b) \ = \ b \otimes a$$

- Twist into new braiding on ${}^{H_F}\mathcal{M}$:

$$\Psi_F \ : \ A_F \otimes B_F \ \longrightarrow \ B_F \otimes A_F \ , \quad \Psi_F(a \otimes b) \ = \ F^{-2}\big(b^{(-1)}, a^{(-1)}\big) \ \big(b^{(0)} \otimes a^{(0)}\big)$$

- A — H-comodule algebra $\Longrightarrow$ $A_F \ = \ \mathcal{Q}_F(A)$ — H_F-comodule algebra with new product:

$$a \cdot b \ := \ F\big(a^{(-1)} \,,\, b^{(-1)}\big) \ \big(a^{(0)} \, b^{(0)}\big)$$

- A, B algebras in ${}^{H_F}\mathcal{M}$ $\Longrightarrow$ so is braided tensor product $A \underline{\otimes} B$ with:

$$(a \otimes b) \cdot (c \otimes d) \ = \ a \, \Psi_F(b \otimes c) \, d$$

Noncommutative algebraic torus $T_\theta = (\mathbb{C}^\times_\theta)^n$

- $H := \mathbb{C}(t_1, \dots, t_n) = A(T)$ generated by $t^p := t_1^{p_1} \cdots t_n^{p_n}$, $p \in \mathbb{Z}^n$ with:

$$\Delta(t^p) = t^p \otimes t^p \ , \qquad \epsilon(t^p) = 1 \ , \qquad S(t^p) = t^{-p}$$

- Cocycle: $F(t_i, t_j) = \exp\left(\frac{\mathrm{i}}{2}\,\theta_{ij}\right) =: q_{ij}, \ \theta_{ij} = -\theta_{ji} \in \mathbb{C}$
$H = H_F$ as Hopf algebras, but category of H-comodules twisted

- $\Delta : H \longrightarrow H \otimes H$ makes H into comodule algebra in ${}^H\mathcal{M}$, so cotwisted torus has:

$$t_i \cdot t_j = F(t_i, t_j)\ t_i\, t_j = F^2(t_i, t_j)\ t_j \cdot t_i = q_{ij}^2\ t_j \cdot t_i$$

Noncommutative torus $A(T_\theta)$ as object of ${}^{H_F}\mathcal{M}$

Quantization of toric varieties $X \longrightarrow X_\theta$

(Ingalls)

- Noncommutative affine toric varieties $\sigma \longmapsto A\big(U_\theta[\sigma]\big)$
  finitely-generated H_F-comodule subalgebras of $A(T_\theta)$

- **Example:** $A(\mathbb{C}^n_\theta) = \mathbb{C}_\theta[t_1, \dots, t_n]$, $t_i\, t_j = q_{ij}^2\, t_j\, t_i$
  "Algebraic Moyal plane":

$$t_i \longmapsto z_i = \log t_i \ , \qquad [z_i, z_j] = \mathrm{i}\,\theta_{ij}$$

  Generally modulo relations

- Gluing rules $\Longrightarrow$ algebra automorphisms in category ${}^{H_F}\mathcal{M}$

- Uses **same fan**, deforms coordinate algebra of each cone σ

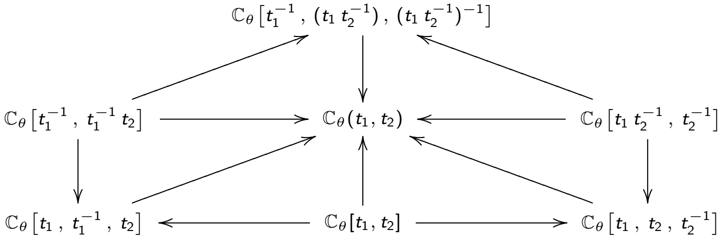
Noncommutative projective plane $\mathbb{P}^2_\theta$

- Maximal cones: $U_\theta[\sigma_i] \;\cong\; \mathbb{C}^2_\theta,\;\; i \;=\; 1,2,3$

- Edges: $U_\theta[\sigma_i \cap \sigma_{i+1}] \;\cong\;$ noncommutative projective line $\mathbb{P}^1_\theta$:

$$w_1\, w_2 \;=\; q^2\; w_2\, w_1\;,\;\; w_1\, w_2^{-1} \;=\; q^{-2}\; w_2^{-1}\, w_1\;, \qquad q \;:=\; q_{12}$$

- Gluing morphisms:



Homogeneous coordinate algebras

(Auroux, Katzarkov & Orlov '08)

$$w_1\, w_2 \;=\; q^2\; w_2\, w_1 \;, \qquad w_1\, w_3 \;=\; w_3\, w_1 \;, \qquad w_2\, w_3 \;=\; w_3\, w_2$$

- $A \;=\; \mathbb{C}_\theta[w_1, w_2, w_3]$ **graded** algebra object in ${}^{H_F}\mathcal{M}$
- Degree 0 left Ore localization $A[w_i^{-1}]_0 \;\cong\; A\big(U_\theta[\sigma_i]\big)$
- $A \;\longrightarrow\; A_{\ell_\theta} \;:=\; A/A\cdot w_3$ noncommutative line $\ell_\theta \;\cong\; \mathbb{P}^1_\theta \;\hookrightarrow\; \mathbb{P}^2_\theta$ "at infinity" (given by $w_3 \;=\; 0$)
- Finitely-generated graded right A-modules M
  $\Longleftrightarrow$ "coherent sheaves" on $\mathbb{P}^2_\theta$
  Projective $\Longleftrightarrow$ "bundles"
  Torsion-free (no finite-dim. submodules) $\Longleftrightarrow$ embeds in a bundle

Instanton moduli spaces $\mathrm{M}_\theta(r,k)$

- Framed modules: A-module $M \implies A_{\ell_\theta}$-module
  $M_{\ell_\theta} = M/M \cdot w_3$; trivialize M_{ℓ_θ} = free A_{ℓ_θ}-module

- $\mathrm{M}_\theta(r,k)$ = isomorphism classes of framed torsion-free A-modules with fixed trivialization $M_{\ell_\theta} \cong A_{\ell_\theta}^{\oplus r}$, $\dim_{\mathbb{C}} \mathrm{Ext}^1(A, M(-1)) = k$

- Invariants:

  $\mathrm{rank}(M)$ = max. no. non-zero direct summands of M = r

  $$\chi(M) = \sum_{p\geq 0} (-1)^p \dim_{\mathbb{C}} \mathrm{Ext}^p(A,M) = r-k$$

  Euler characteristic

Noncommutative ADHM construction

- Matrices $B_1, B_2 \in \mathsf{Mat}_{k\times k}(\mathbb{C})$, $I \in \mathsf{Mat}_{k\times r}(\mathbb{C})$, $J \in \mathsf{Mat}_{r\times k}(\mathbb{C})$ satisfying:

  1. Noncommutative complex ADHM equation:

  $$[B_1, B_2]_\theta + I\,J = 0$$

  $[B_1, B_2]_\theta := B_1\,B_2 - q^{-2}\,B_2\,B_1$ braided commutator

  2. Stability: $V \subsetneq \mathbb{C}^k,\ B_i(V) \subset V,\ \mathrm{im}(I) \subset V \implies V = 0$

- **Theorem:** $\mathsf{M}_\theta(r,k) = \left\{ B_1\,,\, B_2\,,\, I\,,\, J \right\}$ modulo free proper action of $GL(k,\mathbb{C})$:

$$B_i \longmapsto g\,B_i\,g^{-1}\,, \quad I \longmapsto g\,I\,, \quad J \longmapsto J\,g^{-1}\,; \qquad g \in GL(k,\mathbb{C})$$

Noncommutative Klein quadric $\mathbb{G}\mathrm{r}_\theta(2;4) \hookrightarrow \mathbb{P}^5_\Theta$

(Lauve '06)

- Braided exterior algebra $\bigwedge_\theta^2 \mathbb{C}^4$ in ${}^{H_F}\mathcal{M}$ given by cocycle twist
- Spanned by minors Λ^J, $J = (j_1\, j_2)$, $1 \leq j_1, j_2 \leq 4$:

$$\Lambda^J\,\Lambda^K = q^2_{j_1 k_1}\, q^2_{j_1 k_2}\, q^2_{j_2 k_1}\, q^2_{j_2 k_2}\; \Lambda^K\,\Lambda^J$$

- Existence of embedding $\mathbb{G}\mathrm{r}_\theta(2;4) \hookrightarrow \mathbb{P}^5_\Theta \cong \mathbb{P}(\bigwedge_\theta^2 \mathbb{C}^4)$:

$$\Theta^{JK} = \theta^{j_1 k_1} + \theta^{j_1 k_2} + \theta^{j_2 k_1} + \theta^{j_2 k_2}$$

- Noncommutative Plücker relation:

$$\Lambda^{(12)}\,\Lambda^{(34)} - q^2_{23}\; \Lambda^{(13)}\,\Lambda^{(24)} + q^2_{24}\, q^2_{34}\; \Lambda^{(14)}\,\Lambda^{(23)} = 0$$

Noncommutative sphere S^4_θ

- $A(S^4_\theta) \; := \; \mathbb{R}$-algebra generated by $*$-involution on $A(\mathbb{G}\mathrm{r}_\theta(2;4))$:

  $\Lambda^{(13)\,\dagger} \;=\; q\,\Lambda^{(24)}\,,\; \Lambda^{(14)\,\dagger} \;=\; -q^{-1}\,\Lambda^{(23)}\,,\; \Lambda^{(12)\,\dagger} \;=\; \Lambda^{(12)}\,,\; \Lambda^{(34)\,\dagger} \;=\; \Lambda^{(34)}$

  with $q_{12} \;=\; q_{21}^{-1} \;=:\; q \;\in\; \mathbb{R}$, $q_{ij} \;=\; 1$ otherwise

- Open affine subvariety $\mathbb{R}^4_\theta \;\subset\; S^4_\theta$ given by degree 0 right Ore localization $A(\mathbb{G}\mathrm{r}_\theta(2;4))[\Lambda^{(34)\,-1}]_0 \;\cong\; \mathbb{C}[\xi_1,\bar\xi_1,\xi_2,\bar\xi_2]$ with:

$$\begin{aligned}
\xi_1\,\bar\xi_1 \;&=\; q^2\;\bar\xi_1\,\xi_1 \quad , \quad & \xi_2\,\bar\xi_2 \;&=\; q^{-2}\;\bar\xi_2\,\xi_2 \\
\xi_1\,\xi_2 \;&=\; q^2\;\xi_2\,\xi_1 \quad , \quad & \bar\xi_1\,\bar\xi_2 \;&=\; q^{-2}\;\bar\xi_2\,\bar\xi_1 \\
\xi_1\,\bar\xi_2 \;&=\; \bar\xi_2\,\xi_1 \quad , \quad & \xi_2\,\bar\xi_1 \;&=\; \bar\xi_1\,\xi_2 \\
\xi_1^\dagger \;&=\; q^{-1}\;\bar\xi_1 \quad , \quad & \xi_2^\dagger \;&=\; -q^{-1}\;\bar\xi_2
\end{aligned}$$

Noncommutative twistor transform

- Twistor correspondence:

$$
\begin{array}{ccc}
 & A\big(\mathbb{F}\mathrm{l}_\theta(1,2;4)\big) & \\
\overset{p_1}{\nearrow} & & \overset{p_2}{\nwarrow} \\
A\big(\mathbb{P}^3_\theta\big) & & A\big(\mathbb{G}\mathrm{r}_\theta(2;4)\big)
\end{array}
$$

- $A(\mathbb{P}^3_\theta) \ = \ $ "noncommutative twistor algebra";
  $\mathbb{F}\mathrm{l}_\theta(1,2;4) \ = \ $ noncommutative partial flag variety:

$$A\big(\mathbb{P}^3_\theta\big) \,\underline{\otimes}\, A\big(\mathbb{G}\mathrm{r}_\theta(2;4)\big) \ \longrightarrow \ A\big(\mathbb{F}\mathrm{l}_\theta(1,2;4)\big)$$

- Construct instantons on S^4_θ using twistor transform:

$$\Big\{ \ A(\mathbb{P}^3_\theta)\text{-modules} \ \Big\} \quad \longrightarrow \quad \Big\{ \ A(\mathbb{G}\mathrm{r}_\theta(2;4))\text{-modules} \ \Big\}$$

$$M \ \longmapsto \ {p_2}^* \, p_{1*}(M) \qquad , \qquad p_{1*}(M) \ = \ \Big[M \otimes_{A(\mathbb{P}^3_\theta)} A\big(\mathbb{F}\mathrm{l}_\theta(1,2;4)\big) \Big]_{\mathrm{diag}}$$

Self-conjugate instanton modules

- Quaternion structure on $A(\mathbb{P}^3_\theta)$:
  $\mathcal{J}(w_1, w_2, w_3, w_4) \; = \; (w_2, -w_1, w_4, -w_3)$ induces
  $M \; \longmapsto \; M^\dagger \; := \; \mathcal{J}^\bullet(M)^\vee$ on $A(\mathbb{P}^3_\theta)$-modules

- On ADHM data, $(B_1, B_2, I, J) \; \longmapsto \; (-B_2^\dagger, B_1^\dagger, -J^\dagger, I^\dagger)$, obey
  noncommutative real ADHM equation:

$$[B_1, B_1^\dagger]_\theta + q^{-2} \, [B_2, B_2^\dagger]_{-\theta} + I \, I^\dagger - J^\dagger \, J \; = \; 0$$

- **Theorem:** $\Big\{$ torsion-free $M \; \cong \; M^\dagger, \;\; M_{\ell_\theta} \; \cong \; A_{\ell_\theta}^{\oplus r}$,
  $\mathrm{Ext}^1(A(\mathbb{P}^3_\theta), M(-2)) \; = \; 0 \; \Big\} \; \Longleftrightarrow \; \Big\{$ real ADHM data $\Big\} \Big/ \, U(k)$

- Restriction to $\mathbb{P}^2_\theta \; \Longrightarrow$ anti-self-dual connections on a canonical "instanton bundle"

Construction of noncommutative instantons

- Twistor transform of self-conjugate $A(\mathbb{P}^3_\theta)$-module M gives module over $A(\mathrm{Gr}_\theta(2;4))$. Restricting to $\mathbb{R}^4_\theta$ gives right $A(\mathbb{R}^4_\theta)$-module:

$$\mathcal{N} \;=\; \ker \mathcal{D} \;, \qquad \mathcal{D} \;=\; \begin{pmatrix} B_1 - q^{-1}\,\xi_1 & B_2 - q\,\xi_2 & I \\ -B_2^\dagger - q^{-1}\,\bar{\xi}_2 & B_1^\dagger - q\,\bar{\xi}_1 & -J^\dagger \end{pmatrix}$$

- By stability:
  - $\mathcal{D} \;=\;$ surjective morphism of free $A(\mathbb{R}^4_\theta)$-modules
  - $\Delta \;=\; \mathcal{D}\,\mathcal{D}^\dagger \;=\;$ isomorphism
  - $\mathcal{N} \;=\;$ finitely-generated and projective of rank r with projection $P \;=\; 1 - \mathcal{D}^\dagger\,\Delta^{-1}\,\mathcal{D}$, $\;P^2 \;=\; P \;=\; P^\dagger$

- Using canonically defined differential structure $\Omega^\bullet(\mathbb{R}^4_\theta)$ gives instanton connection:

$$\nabla \;:=\; P \circ \mathrm{d}$$

with curvature $F_\nabla \;=\; \nabla^2 \;=\; P\,(\mathrm{d}P)^2$