The Holomorphic Brauer Group of Complex Tori

O. Ben-Bassat

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Outline

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- Let M be a complex manifold. $\mathcal{O}_M$ are the holomorphic functions on M, and $\mathcal{O}_M^\times$ the non-vanishing holomorphic functions. An $\mathcal{O}_M^\times$-torsor is:
- a sheaf of sets T on M
- an action

$$\mathcal{O}_M^\times \times T \to T$$

such that for each $m \in M$ there is a U open containing m such that

- $T(U)$ is non empty and
- For each $t \in T(U)$ the map

$$\mathcal{O}^\times(U) \to T(U)$$

$$f \mapsto f \cdot t$$

is an isomorphism.

- The category of $\mathcal{O}_M^\times$-torsors is equivalent to the category of line bundles where the morphisms are taken to be only isomorphisms.

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A way to construct $\mathcal{O}_M^\times$-torsors

- Take $p : \mathcal{U} \to M$ a surjective local isomorphism, $f \in \mathcal{O}^\times(\mathcal{U} \times_M \mathcal{U})$.
- Define
$$(u_1, z_1), (u_2, z_2) \in \mathcal{U} \times \mathbb{C}^\times$$
as equivalent if $f(u_1, u_2)z_1 = z_2$ (wherever defined)
- If f satisfies a certain consistency condition then this is an equivalence relation and the (sheaf of sections of)
$$(\mathcal{U} \times \mathbb{C}^\times)/\sim \ \to \ M$$
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- Any $\mathcal{O}^\times$-torsor T on M such that p^*T is trivial is isomorphic to one of this form.

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Outline

Definition of Complex Tori, Line Bundles

- Let V be a complex vector space of complex dimension g.
- Let $\Lambda \subset V$ be a full lattice. $\Lambda \cong \mathbb{Z}^{2g}$
- The quotient is a complex torus.

$$X = V/\Lambda$$

- We can define line bundles on complex tori through group cocycles $Z^1(\Lambda, \mathcal{O}^\times(V))$
- These are nothing but the functions on $V \times_X V \cong \Lambda \times V$ described earlier. This construction shows that the first *group cohomology* group of Λ acting on sections of $\mathcal{O}^\times(V)$ by translation gives the isomorphism classes of line bundles.

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The Dual Complex Torus I

- Take X a complex torus.
- Consider $H^1(X, \mathcal{O}^\times)$.
- Elements of the group $H^1(X, \mathcal{O}^\times)$ are equivalence classes of line bundles on X.
- $$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^\times \to 1$$
- $$H^0(X, \mathcal{O}) \twoheadrightarrow H^0(X, \mathcal{O}^\times) \to H^1(X, \mathbb{Z}) \hookrightarrow H^1(X, \mathcal{O})$$
- What is the quotient?

$$H^1(X, \mathcal{O})/H^1(X, \mathbb{Z}) \subset H^1(X, \mathcal{O}^\times)$$

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- Hodge Decomposition gives an isomorphism

$$H^1(X, \mathcal{O}) \cong \overline{V}^\vee$$

- $$\overline{V}^\vee = \{\ell \in Hom_{\mathbb{R}}(V, \mathbb{C}) | \ell(iv) = -i\ell(v)\}$$

- $$dim_{\mathbb{C}}(\overline{V}^\vee) = dim_{\mathbb{C}}(V)$$

- There is a commutative diagram:

$$\begin{tikzcd}
H^1(X, \mathbb{Z}) \arrow[r, hook] \arrow[d, "\cong"] & H^1(X, \mathcal{O}) \arrow[d, "\cong"] \\
\mathrm{Hom}(\Lambda, \mathbb{Z}) \arrow[r, hook] & \overline{V}^\vee
\end{tikzcd}$$

- Can show image of $\mathrm{Hom}(\Lambda, \mathbb{Z})$ in $\overline{V}^\vee$ is also a full lattice so we get the *dual torus.*

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$$X^\vee = \overline{V}^\vee/\mathrm{Hom}(\Lambda,\mathbb{Z}) = H^1(X,\mathcal{O})/H^1(X,\mathbb{Z})$$

The Poincaré Bundle

- Take X a complex torus, $X^\vee$ the dual complex torus.
- The Poincaré Bundle $\mathcal{P}$ is a line bundle on $X \times X^\vee$.
- Key Property:

$$[\mathcal{P}|_{X\times\{\alpha\}}] = \alpha \text{ for } \alpha \in X^\vee \subset H^1(X, \mathcal{O}^\times)$$

- In fact for any complex analytic space T,any line bundle on $X \times T$ topologically trivial on each fiber is (up to the tensor by line bundles trivial on each fiber) isomorphic to the pullback of $\mathcal{P}$ under a map

$$(1, f) : X \times T \to X \times X^\vee$$

for some morphism $f : T \to X^\vee$.

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Outline

- Let M be a complex manifold. Before describing $\mathcal{O}_M^\times$-gerbes we have to describe what will end up being the trivial gerbe $\mathcal{P}ic_M$.
- For every open set $U \subset M$, denote by $\mathcal{P}ic_M(U)$ the groupoid of line bundles (category of $\mathcal{O}_M^\times$ torsors) on U. This is a category whose objects are line bundles on U and whose morphisms are isomorphisms of line bundles.
- Notice that for an inclusion of open sets $U \subset V$ there is a restriction functor $\mathcal{P}ic_M(V) \to \mathcal{P}ic_M(U)$ and for any three open sets $U \subset V \subset W$ there is a natural equivalence of the restriction $\mathcal{P}ic_M(W) \to \mathcal{P}ic_M(U)$ with the composed restriction $\mathcal{P}ic_M(W) \to \mathcal{P}ic_M(V) \to \mathcal{P}ic_M(U)$. For four nested open sets the resulting natural equivalences satisfy a compatability condition.

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What is a Gerbe? A $\mathcal{O}^\times$ *gerbe* $\mathfrak{G}$ over a complex manifold M is a twisted form of $\mathcal{P}ic_M$. It consists of

- A "sheaf" of categories (stack) $\mathfrak{G}$ on M. (This includes a category $\mathfrak{G}(U)$ for each open set $U \subset M$ satisfying a bunch of gluing conditions as with $\mathcal{P}ic_M$)
- An action functor of the tensor category $\mathcal{P}ic_M$

$$\mathcal{P}ic_M \times \mathfrak{G} \to \mathfrak{G}$$

$$(\mathcal{L}, G) \mapsto \mathcal{L} \cdot G$$

such that for every $m \in M$ there is an open U containing m such that

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What is a Gerbe? A $\mathcal{O}^\times$ *gerbe* $\mathfrak{G}$ over a complex manifold M is a twisted form of $\mathcal{P}ic_M$. It consists of

- A "sheaf" of categories (stack) $\mathfrak{G}$ on M. (This includes a category $\mathfrak{G}(U)$ for each open set $U \subset M$ satisfying a bunch of gluing conditions as with $\mathcal{P}ic_M$)
- An action functor of the tensor category $\mathcal{P}ic_M$

$$\mathcal{P}ic_M \times \mathfrak{G} \longrightarrow \mathfrak{G}$$

$$(\mathcal{L}, G) \mapsto \mathcal{L} \cdot G$$

such that for every $m \in M$ there is an open U containing m such that

- $\mathfrak{G}(U)$ is non-empty and
- For each object G of $\mathfrak{G}(U)$, the functor

$$\mathcal{P}ic_M(U) \to \mathfrak{G}(U)$$

$$\mathcal{L} \mapsto \mathcal{L} \cdot G$$

is an equivalence of categories.

Outline

Deconstruction of Gerbes

- Let $\mathfrak{G}$ be an $\mathcal{O}^\times$ gerbe on M.
- Let $\mathcal{U} \to M$ be a surjective local isomorphism with $Pic(\mathcal{U} \times_M \mathcal{U})$ trivial. Suppose that we can chose a trivialization of the pullback of $\mathfrak{G}$ to $\mathcal{U}$.
- If we do so, there are two different ways of pulling back the trivialization to $\mathcal{U} \times_M \mathcal{U}$. They differ by an $\mathcal{O}^\times$ torsor $\mathcal{L} \to \mathcal{U} \times_M \mathcal{U}$.
- There are three pull-backs of $\mathcal{L}$ to $\mathcal{U} \times_M \mathcal{U} \times_M \mathcal{U}$,related by an isomorphism.
- This isomorphism satisfies a certain coherence relation on $\mathcal{U} \times_M \mathcal{U} \times_M \mathcal{U} \times_M \mathcal{U}$.
- If we pick a trivialization of $\mathcal{L}$ then the isomorphism of line bundles becomes simply an element of $\mathcal{O}^\times(\mathcal{U} \times_M \mathcal{U} \times_M \mathcal{U})$ satisfying an equation involving the product of the various pullbacks to $\mathcal{U} \times_M \mathcal{U} \times_M \mathcal{U} \times_M \mathcal{U}$.

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Construction of Gerbes

- Given the kind of data we had from the deconstruction: $\mathcal{U} \to M$, $\mathcal{L} \to \mathcal{U} \times_M \mathcal{U}$, etc., we can actually construct a gerbe in a canonical way.
- Let $B\mathbb{C}^\times$ be the groupoid of $\mathbb{C}^\times$ torsors. Then we can define a gerbe on M as a certain quotient of $\mathcal{U} \times B\mathbb{C}^\times$.
- Fibers $u_1 \times B\mathbb{C}^\times$ and $u_2 \times B\mathbb{C}^\times$ are identified using the equivalence of categories

  $$F_{u_1,u_2} : T \mapsto T \otimes \mathcal{L}|_{(u_1,u_2)}$$

  wherever this is defined
- Have natural isomorphisms

  $$F_{u_2,u_3} \circ F_{u_1,u_2} \to F_{u_1,u_3}$$

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Classification of Gerbes

- Thereby, a gerbe $\mathfrak{G}$ gives an element of the "Čech" cohomology of $\mathcal{U} \to M$ with coefficients in $\mathcal{O}^\times$. If $\mathcal{U}$ is just an open cover the gerbe gives an element in the usual Čech cohomology group $H^2(M, \mathcal{O}^\times)$.
- Two gerbes are *isomorphic* if they have functors between them on every open set commuting with the action of $\mathcal{P}ic$. The equivalence classes of gerbes form the group $H^2(M, \mathcal{O}^\times)$.
- $\mathcal{O}^\times$-gerbes have a natural (monoidal) product structure the product of $\mathfrak{G}$ and $\mathfrak{H}$ is denoted $\mathfrak{G} \otimes \mathfrak{H}$.
- Each gerbe has a topological class defined as its image under the map

  $$H^2(M, \mathcal{O}^\times) \to H^3(M, \mathbb{Z}).$$

  If the image is trivial, we call the gerbe *topologically trivial*

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Outline

- Let $X = V/\Lambda$ be a complex torus with projection map $p : V \to X$. If $\mathcal{S}$ is a sheaf of abelian groups that has no higher cohomology on the vector space V then there are natural isomorphisms

  $$H^i(X, \mathcal{S}) \cong H^i(\Lambda, p^{-1}\mathcal{S}(V)).$$

  The right hand side is the *group cohomology* of Λ acting on global sections of $p^{-1}\mathcal{S}$ by translation. We can apply this in the case $\mathcal{S} = \mathcal{O}$, $\mathcal{O}^\times$, and $\mathbb{Z}$. The group cohomology

  $$H^i(\Lambda, p^{-1}\mathcal{S}(V)) = Z^i(\Lambda, p^{-1}\mathcal{S}(V))/B^i(\Lambda, p^{-1}\mathcal{S}(V))$$

  is the homology of a certain chain complex with chains $C^i(\Lambda, p^{-1}\mathcal{S}(V))$ consisting of maps $\Lambda^i \to p^{-1}\mathcal{S}(V)$.
- For example if $f_\lambda \in C^1(\Lambda, p^{-1}\mathcal{S}(V))$ then $(\delta f)_{\lambda_1,\lambda_2} \in C^2(\Lambda, p^{-1}\mathcal{S}(V))$ is given by

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In order to get a handle on the group $H^2(X, \mathcal{O}^\times)$ we use the exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^\times \to 1.$$

Therefore we get the long exact sequence

$$\cdots \to H^1(X, \mathcal{O}^\times) \to H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}) \to$$
$$\to H^2(X, \mathcal{O}^\times) \to H^3(X, \mathbb{Z}) \to H^3(X, \mathcal{O}) \to \cdots.$$

Via skew-symmetrization

$$H^3(X, \mathbb{Z}) \cong \mathrm{Alt}^3(\Lambda, \mathbb{Z}),$$

the image of $H^2(X, \mathcal{O}^\times)$ is $(H^{(1,2)}(X) \oplus H^{(2,1)}(X)) \cap H^3(X, \mathbb{Z})$ which corresponds to

$$\{E \in \mathrm{Alt}^3(\Lambda, \mathbb{Z}) | E(x, y, z) = E(ix, iy, z) + E(ix, y, iz) + E(x, iy, iz)\}.$$

The rank of this group is an invariant of the torus.

Outline

- Just as in the classical Appell-Humbert Theorem for line bundles, we can come up with explicit elements of $Z^2(\Lambda, \mathcal{O}^\times(V))$ (and hence gerbes) corresponding to any specific element of $H^2(X, \mathcal{O}^\times)$.
- Given $B \in \wedge^2 \overline{V}^\vee \cong H^2(X, \mathcal{O})$ and E as above, the appropriate element of $Z^2(\Lambda, \mathcal{O}^\times(V))$ turns out to be

$$(\lambda_1, \lambda_2) \mapsto \exp\left(\frac{1}{2}B(\lambda_1, \lambda_2) + H_{\lambda_1,\lambda_2} + \beta_{\lambda_1,\lambda_2}\right).$$

  Here $H_{\lambda_1,\lambda_2} : V \to \mathbb{C}$ is linear in v and additive in E, while $\beta_{\lambda_1,\lambda_2}$ are constant in v and additive in E
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$$\begin{aligned} H_{\lambda_1,\lambda_2}(v) = \frac{1}{8}\Big(E(v,\lambda_1,\lambda_2) + \frac{1}{2}E(iv, i\lambda_1, \lambda_2) + \frac{1}{2}E(iv, \lambda_1, i\lambda_2)\Big) \\ + \frac{i}{8}\Big(\frac{1}{2}E(v, i\lambda_1, \lambda_2) + \frac{1}{2}E(v, \lambda_1, i\lambda_2) - E(iv, \lambda_1, \lambda_2)\Big) \end{aligned} \quad (1)$$

- The class of this gerbe under the map

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Outline

- In order to understand how gerbes pullback under different translation and multiplication maps of the torus, it helps to think in general about gerbes on the product of four spaces. We have a theorem of the *hyper-cube* analogous to the theorem of the cube for line bundles.
- Let $f_i : Y_i \to S$ be a proper surjective maps with section of complex analytic spaces satisfying $f_{i*}\mathcal{O} = \mathcal{O}$ for $i = 1, \ldots, 4$. Let $\mathfrak{G} \to Y_1 \times_S Y_2 \times_S Y_3 \times_S Y_4$ be an $\mathcal{O}^\times$ gerbe.
- Using the sections of each Y_i it makes sense to restrict the gerbe to any three of the four spaces.
- The we have the following theorem [B]

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The gerbe $\mathfrak{G}$ is trivial if and only if it is trivial when restricted to any three of the four spaces.

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The proof uses two tools.

- First, the Künneth decomposition for cohomology with $\mathbb{Z}$ and $\mathcal{O}$ coefficients on each fiber. This is enough to prove the theorem when S is a point. The relevant Künneth decomposition for compact, reduced complex analytic spaces looks like

  $$H^2(A \times B, \mathcal{O}) = H^2(A, \mathcal{O}) \oplus H^1(A, \mathcal{O}) \otimes H^1(B, \mathcal{O}) \oplus H^2(B, \mathcal{O})$$

  and similarly for $H^3(A \times B, \mathbb{Z})$ (modulo torsion)
- Second, in order to compare the situation on the fibers of the maps to S with various tubular neighborhoods of the fibers, one uses that the push-forwards $R^j f_* \mathbb{Z}$ under a proper surjective analytic map f of complex analytic spaces are all *constructible sheaves*, and hence their cohomology behaves nicely, i.e. global sections are locally constant and higher cohomology is locally trivial.

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- As an example consequence if X is a complex torus, $\mathfrak{G}$ an $\mathcal{O}^\times$ gerbe on X and $(x_1, x_2, x_3, x_4) \in X^4$ then

  $$(x_1+x_2+x_3+x_4)^*\mathfrak{G} \otimes \prod_{1\leq i<j\leq 4} (x_i+x_j)^*\mathfrak{G} \cong \prod_{1\leq i\leq 4} x_i^*\mathfrak{G} \otimes \prod_{1\leq i<j<k\leq 4} (x_i+x_j+x_k)^*\mathfrak{G}$$

  since it holds when restricted to any three of the spaces.
- Similarly, for the isogeny $n : X \to X$ of multiplication by n on a complex torus, one has

  $$n^*\mathfrak{G} \cong \left(\mathfrak{G}^{\left(\frac{n^2+n^3}{2}\right)}\right) \otimes \left((-1)^*\mathfrak{G}^{\left(\frac{n^2-n^3}{2}\right)}\right).$$

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- Let X be a complex torus. Recall the short exact sequence of sheaves on X
$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^{\times} \to 1.$$
Therefore we get the long exact sequence
$$\cdots \to H^1(X, \mathcal{O}^{\times}) \to H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}) \to$$
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- There is a commutative diagram
$$\begin{tikzcd}
H^2(X, \mathbb{Z}) \arrow[r] \arrow[d, "\cong"] & H^2(X, \mathcal{O}) \arrow[d, "\cong"] \\
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But contrary to the case of line bundles, here we have a non-seperated space (non-Hausdorff in the classical topology) $\wedge^2 \overline{V}^{\vee} / \mathrm{Alt}^2(\Lambda, \mathbb{Z})$.

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  But contrary to the case of line bundles, here we have a non-seperated space (non-Hausdorff in the classical topology) $\wedge^2\overline{V}^\vee/\mathrm{Alt}^2(\Lambda,\mathbb{Z})$.

- For example, if $X = E \times E$ is the square of an elliptic curve without complex multiplication or X is a simple torus with indefinite quaternionic endomorphism algebra then the rank of the Neron-Severi group of X is 3 and so the image of $\mathrm{Alt}^2(\Lambda, \mathbb{Z}) \cong \mathbb{Z}^6$ in $\wedge^2 \overline{V}^\vee \cong \mathbb{C}$ has rank 3.
- Therefore we consider the *stack* $[\wedge^2 \overline{V}^\vee / \mathrm{Alt}^2(\Lambda, \mathbb{Z})]$.
- The key property of this (quotient) stack is that maps

$$T \to [\wedge^2 \overline{V}^\vee / \mathrm{Alt}^2(\Lambda, \mathbb{Z})]$$

are the groupoid whose objects consist of pairs : (1) a principal $\mathrm{Alt}^2(\Lambda, \mathbb{Z})$ bundle $P \to T$ and (2) an $\mathrm{Alt}^2(\Lambda, \mathbb{Z})$ equivariant map $P \to \wedge^2 \overline{V}^\vee$.

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Using group cocycles we define a gerbe $\mathfrak{P}$ on $X \times [H^2(X,\mathcal{O})/H^2(X,\mathbb{Z})]$. We have the following [B].

Theorem

Let T be a complex analytic space. Consider the functor

$$\textit{Holomorphic Maps}(T, [H^2(X,\mathcal{O})/H^2(X,\mathbb{Z})]) \to (\textit{Gerbes on } X \times T)$$

$$f \mapsto (1,f)^*\mathfrak{P}.$$

We then have

- *For each point $t \in T$ the gerbe $(1,f)^*\mathfrak{P}|_{X\times\{t\}}$ has class in $H^2(X,\mathcal{O}^\times)$ agreeing with the element coming from $f(t)$.*
- *Also, $[H^2(X,\mathcal{O})/H^2(X,\mathbb{Z})]$ is a fine moduli stack for gerbes on X: Any gerbe on $X \times T$ topologically trivial over every point t is a pullback up to equivalence and product by gerbes trivial on each fiber.*

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- The key to the proof is to show that the gerbe defines a holomorphic map from the universal cover U_T of T to $\wedge^2 \overline{V}^\vee = H^2(X, \mathcal{O})$.

- Let $\mathfrak{G}$ be a gerbe on $X \times T$, topologically trivial on each fiber.
- Let U_T be the universal cover of T. Let $\rho : X \times T \to T$ and $\tilde{\rho} : X \times U_T \to U_T$ be the projection maps.
- The pull-back of $\mathfrak{G}$ to $X \times U_T$ defines a class in $H^0(U_T, R^2\tilde{\rho}_*\mathcal{O}^\times)$ by restricting to the fibers.
- Since U_T is simply connected, there exists a lift of this class to $H^0(U_T, R^2\tilde{\rho}_*\mathcal{O}) = Hol(U_T, \wedge^2\overline{V}^\vee)$.
- One can chose a compatible group homomorphism $\pi_1(T) \to \mathrm{Alt}^2(\Lambda, \mathbb{Z})$ and thereby get a map

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Open Question

- Can we find a nice "linear" category of theta line bundles of a gerbe analogous to the vector space of theta functions of a line bundle?
- Some Progress: There is a notion of sheaves of $\mathcal{O}_{\mathfrak{G}}$ - modules on a gerbe $\mathfrak{G}$, these from an abelian category $Mod(\mathcal{O}_{\mathfrak{G}})$. The group H of translations of a torus X that fix a gerbe $\mathfrak{G} \to X$ act on this category in a gerby way (Frenkel and Zhu call this a gerbal representation.) This is like the projective representation of the group of translations which fix a line bundle $\mathcal{L} \to X$ acting on $H^0(X, \mathcal{L})$.
- To be more explicit, let $\mathfrak{G}$ be a gerbe with topological class E. Then for every $v \in H$ there is an equivalence F_v of $Mod(\mathcal{O}_{\mathfrak{G}})$ with itself.
- There are natural isomorphisms $N_{v,w} : F_{v+w} \to F_v \circ F_w$
- Given $v_1, v_2, v_3 \in H$ there are two ways of going from $F_{v_1+v_2+v_3}$ to $F_{v_1} \circ F_{v_2} \circ F_{v_3}$.
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- Given $v_1, v_2, v_3 \in H$ there are two ways of going from $F_{v_1+v_2+v_3}$ to $F_{v_1} \circ F_{v_2} \circ F_{v_3}$.
- They differ by multiplication with the complex number $\exp(E(v_1, v_2, v_3))$.

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