

Higher dimensional algebraic geometry: Classification, minimal models, Fano varieties and stacks

WORKSHOP REPORT

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The two main objectives of this workshop were to put into focus rapid recent developments in the higher dimensional minimal model program (HD MMP), and to consolidate advances in the geometry of Fano and Calabi-Yau 3-folds that have arisen over the last decade. In addition, the meeting helped plan how to exploit new results, methods and problems arising from these developments.

The last few years have seen dramatic advances in HD MMP, with the proof of the existence of minimal models under appropriate conditions, and the prospect within a few years of having a complete generalisation of the minimal model program and the classification of varieties in all dimensions, comparable to the known foundational framework of the theory of surfaces and 3-folds. At the same time, other topics in algebraic geometry are enjoying prominent advances; these include birational geometry (especially the geometry of Fano varieties), the geometry of moduli spaces, resolution of singularities, orbifold cohomology, etc.

These topics feature in several international programs and conferences, including the 2007-08 Warwick EPSRC algebraic geometry symposium and the MSRI half-year program on Algebraic geometry Jan-May 2009. Some of the problem areas we covered are fast moving, but we also considered a number of more established questions.

Detailed report

The MMP for 3-folds was essentially complete by the time of Mori's 1990 Fields Medal. This extended the theory of surfaces in a rather complete way, and led to many very interesting problems, such as the detailed study of Fano 3-folds or the study of divisorial contractions and flips, the elementary birational transformations of Mori theory of 3-folds. The extension to higher dimension was already under way at this time at the hands of Shokurov, and it has developed in rather technical steps since then, notably at the 1991 Utah summer school (when Shokurov's program for 3-fold log flips was worked out by Koll'ar, Corti and others) and the 2002 Newton Institute HDG seminar (that digested and propagated Shokurov's strategy for dimension ≥ 4 , thus preparing the ground for the recent substantial breakthrough of Hacon, McKernan and others). At the ICMS workshop, Birkar described joint work in progress with Shokurov concerning several 3 or 4 dimensional conjectures on boundedness of Fano varieties, ACC, termination, and related problems. Musta{\c t}a explained how the description of singularities of pairs via spaces of arcs allows an approach to singularities of pairs by induction on the dimension. Corti introduced recent work of Lazi'c on an approach to proving finite generation of the canonical ring by induction on the dimension without using minimal model theory. Cascini described an emerging program aiming to make progress on HD MMP over a field of characteristic p .

The birational geometry of ruled surfaces and nonsingular Fano 3-folds are among our more established interests. There has been a lot of work on nonsingular Fano 3-folds since Iskovskikh and Manin's treatment of the quartic 3-fold, and the subsequent work of Iskovskikh's school (including Pukhlikov and Cheltsov in the UK). Mori's work of the 1980s put Fano 3-folds with mild singularities at the centre of the 3-fold minimal model program. More recent work (that won separate LMS prizes for Corti and Reid) was concerned with birational rigidity of Fano 3-folds with terminal singularities. The subject is closely related to the MMP for 3-folds, and it is certain to be deeply affected over the next few years by the current breakthroughs. Fano varieties relate to many other topics of current interest, for example, the question of Kähler-Einstein or constant scalar curvature metrics studied by Donaldson, or the geometry of moduli spaces (in many contexts), or the new topic of toric geometry, tropical geometry and moduli of phylogenetic trees. Pukhlikov discussed the linear and quadratic methods of proving birational (super)rigidity of Fano varieties and Mori fibre spaces. Prokhorov discussed singular Fano 3-folds of large Fano index and degree, completing work of Suzuki. Iskovskikh described his recent result on finite subgroups of the Cremona group of the plane over a perfect field. Brown explained how to use Kustin-Miller unprojection (the original special case of Type I unprojection of Papadakis-Reid) to construct Fano 3-folds embedded anticanonically in codimension 4 in weighted projective space and related Sarkisov links.

It is known that there are finitely many families of Fano 3-folds in the Mori category, and one suspects that the number is a few thousand. Several hundred of these families can be studied directly by graded ring methods. By contrast, the number of families of Calabi-Yau 3-folds is estimated to be in the billions (or infinite). These varieties relate to many kinds of calculations in different areas of math and physics, including toric geometry, subvarieties of homogeneous spaces, geometry of moduli spaces, singularity theory, conformal field theory, mirror symmetry, commutative and computer algebra, etc. Bryan discussed a conjectural relationship between the Donaldson-Thomas invariants of a three dimensional Calabi-Yau orbifold and the Donaldson-Thomas invariants of its crepant resolution. Verbitsky described his recent results on the nef divisor classes on hyper-Kähler manifolds of complex dimension $2n$ (a natural generalization of Calabi-Yau manifolds). Hwang lectured on his recent proof that the base of a holomorphic Lagrangian fibration is \mathbb{P}^n , that uses many ideas of Mori theory.

Golyshev has recently showed that the classification of non-singular Fano 3-folds is closely related to the classification of certain third order differential operators on \mathbb{G}_m with strong arithmetic properties. These operators encode information about the quantum cohomology and Gromov-Witten theory of Fano manifolds. Recent progress by Corti, Coates, Iritani and Tseng on the quantum cohomology of Fano orbifolds suggests that this general framework will extend to Fano orbifolds. As well as being of great interest in its own right - it brings tools from mirror symmetry to bear upon the enumerative geometry of Fano orbifolds - this work could open the way to a detailed classification of the finitely many \mathbb{Q} -Fano 3-folds in terms of Gromov-Witten theory. Coates described his recent joint work with Iritani which shows that the genus zero crepant resolution conjecture implies a higher genus version. Iritani discussed his results on the functoriality of toric quantum cohomology under birational morphisms, and showed how the Fourier-Mukai transformations are related to Yongbin Ruan's conjectures on crepant resolutions and flops.

There is much current interest in derived categories. In application to objects of geometry, algebra and mathematical physics, Fourier-Mukai transforms provide an extra level of “quantum symmetry” analogous to some of the more mysterious constructions of the string theorists. This applies to Kontsevich's homological mirror symmetry, to the Broué conjecture in representation theory of finite groups, and to surprisingly many other classes of geometric objects. One of the main areas of advance is the derived version of the McKay correspondence of Bridgeland, King and Reid, which has many current repercussions. Bondal described a new approach to a mirror symmetry for toric orbifolds. Orlov gave a sketch of his proof that the bounded derived category of coherent sheaves on a smooth quasiprojective curve has dimension one. Katzarkov discussed foundations of generalized homological mirror symmetry and its possible relations to birational geometry and questions of rationality.

The unusual combination of different subjects gave a special character to the meeting, with experts in one area getting oversight into other areas. There were talks on holomorphic symplectic manifolds, homological mirror symmetry, orbifold Gromov-Witten theory, explicit birational geometry, higher dimensional Cremona groups, and higher dimensional algebraic geometry.

There were 43 participants, representing a broad spectrum of interests in algebraic geometry. We had 22 lectures, leaving plenty of time for discussion. The workshop reflected the state of the art across a wide cross-section of current interests in algebraic geometry. The historic ICMS building (James Clerk Maxwell's house) lent an intimate atmosphere that encouraged social and scientific interaction. The reactions of the participants, as expressed for example in the questionnaire, underline the high quality of the talks and the success of the meeting.

Below is some of the workshop feedback. This is genuinely representative, there were no negative comments on the academic programme whatsoever.

A selection of people's highlights of the workshop:

- The very good quality of talks.
- I liked most talks by Misha Verbitsky, Aleksei Bondal, and JunMuk Hwang.
- The high quality of research presented.
- Discussions with several colleagues were stimulating. Some of them I met first time. Some of them I met before but didn't have chance to have close discussion because usual conferences are of large size. The workshop was of small size and more intimate atmosphere.
- Talks by Alexeev, Coates, Iritani and Hwang.
- I don't think there was a particular highlight of the workshop. Most of the talks were at a very high level of quality both in exposition and mathematics. Also, the topics were reaching in quite different directions.
- Bringing together the recent progress in birational geometry, Gromov-Witten theory and stacky approach.

- Talks by Orlov and Kaledin, Birkar's lecture.
- The best part was the diversity of the topics covered. I had the opportunity to listen to talks on subjects in a wide range of areas, and to meet people I have not met before.
- I appreciated very much the talks by Cascini, Alexeev and Corti.
- Very hot (mathematically) and diverse topics.
- The fact that it was so well attended, both by established algebraic geometers, postdocs and PhD students. I had interesting mathematical (and nonmathematical) conversations with lots of people.
- Either the talk by Corti on the work of Lazic (that aims to provide a more conceptual proof of finite generation of the canonical ring), or the talk by Orlov on his results extending Rouquier's work on dimensions of derived categories.
- The majority of talks were very interesting for me. I especially appreciated the talks of Alexeev, Bondal, Hwang, Iritani, Katzarkov, and Prohorov.
- Discussions with Coates and Iritani.

Comments on the overall academic value of the workshop:

- Excellent.
- It was very good. There were excellent talks and the workshop gathered experts from different fields of research.
- Very good. The talks were good, I had great mathematical discussions with the participants, and I enjoyed being in Edinburgh.
- Great. It is necessary to have this kind of workshop to maintain high quality research in the UK.
- Extremely valuable. Several of the participants had not been able to take much part in last year's Warwick symposium, and it was very useful to plug their recent work into our experience.
- Was quite high. It was good that the talks were not in one narrow field but spread over many active areas in algebraic geometry. This way, I actually had an opportunity to learn new and interesting things.
- Certain topics like classification of Fano varieties are important, but not often discussed as extensively as in the workshop. It was a very good chance to learn about recent developments in the topics of the workshop.
- Outstanding.

- I thought it was very strong. A number of nice results were announced and work in progress was sketched. I had several very useful conversations with experts at the conference. Extremely good. Very high standard.
- The overall academic value of workshop was great.
- Very good. Most of the talks were very interesting, and the quality rate was higher than at the most conferences I go to.
- I think it was just great.
- Very high - world class researchers of the highlighted fields had a chance to bring each other brand new ideas in a very friendly and comfortable atmosphere.

Comments about the interactions with other participants:

Did the workshop help develop/sustain contacts likely to result in new research?

- Yes it did, it gave me an opportunity to talk to people that I do not often get a chance to hear.
- Yes, in particular with Coates and Iritani.
- Most likely!
- I haven't started any project yet. But discussions with some people may lead to a result in the future.
- Yes, with both the talks and discussions with a number of participants.
- I got some contact with M. Verbitsky in particular and met S. Lamy for the first time. It gave me the opportunity to discuss with C. Birkar with whom I have a project in common. And also with I. Cheltsov and T. de Fernex with whom I hope to develop some collaboration in the future.
- Yes. Especially talks of Katzarkov and Prokhorov.
- Yes. In particular, I had several discussions with M. Reid which allow me to improve arguments in my paper (in preparation).