

Justification of the BORN approximation in potential scattering

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Edinburgh, 9 June 2011

¹Joint Work with J.A Barceló, D. Faraco and A. Ruiz

Scattering Solutions

$$\begin{cases} (\Delta + q + k^2)\psi = 0 \\ \psi = e^{ikx \cdot \theta} + \psi_{out}, \end{cases}$$

Outgoing Sommerfeld Radiation Condition

$$\psi(x, \theta, k) = e^{ikx \cdot \theta} + C|x|^{\frac{1-n}{2}} k^{\frac{n-3}{2}} e^{ik|x|} A(\theta', \theta, k) + o(|x|^{\frac{1-n}{2}}),$$

where $\theta' = x/|x|$.

Scattering Transform (Far field pattern)

$$A(\theta', \theta, k) = \int_{\mathbb{R}^n} e^{-ik\theta' \cdot y} q(y) \psi(y, k, \theta) dy.$$

Basic idea: $A(\theta', \theta, k) \approx \widehat{q}(k(\theta - \theta'))$

Many different possibilities to approximate q from $A(\theta', \theta, k)$ (Born approximation)

fixed angle data (θ fixed)

$$\widehat{q}_\theta(k(\theta' - \theta)) = A(\theta', \theta, k)$$

Backscattering $\theta' = -\theta$

$$A(\theta, -\theta, k) = \widehat{q}_b(2k\theta)$$

Full data

$$q_B(x) = \int_{S^{n-1}} q_\theta(x) d\theta$$

What information about the actual potential q is contained in the Born approximations, q_θ , q_B or q_b ?

Recovery of singularities

Find the best exponents β, s such that

$$q \in W^{\alpha,r} \Rightarrow q - q_B \in W^{\beta,s}$$

- Fixed angle data: A. Ruiz (2001)
- Backscattering data: Ola, Päivärinta, Serov (2D; 2001), Ruiz , V(2D, 3D;2005) J.M. Reyes (2D, 3D;2007) Melin, Lagergren, Beltita (oddD; 2009)
- Full data: Sommersalo, Päivärinta, Serov (91, 94, 98; 2007). They treat the case of real potentials in the category of Lebesgue and Hölder spaces.

Example: If q is real L^∞ with compact support $q - q_B \in C^\beta$ for $\beta > 0$.

The Theorem (s)

Theorem (Full data, Barceló, Faraco, Ruiz, V. 2010)

Let $q \in L^p(\mathbb{R}^n) \cap W^{\alpha,2}(\mathbb{R}^n)$, $p > n/2$, $\alpha < \frac{n}{2}$, Then
 $q - q_B \in W_{loc}^{\beta,2}$, when $\beta - \alpha < \epsilon(\alpha)$.

$$n=2 \quad \epsilon(\alpha) = 1$$

$$n=3 \quad \epsilon(\alpha) = \min\{1, \frac{1}{2} + \frac{2}{3}\alpha\}$$

In general best β such that if $q \in W^{\alpha,r}(\mathbb{R}^n)$, then
 $q - q_B \in W^{\beta,2}(\mathbb{R}^n)$.

Example $q \in W^{\alpha,3}(\mathbb{R}^3) \Rightarrow \beta < \min\{\alpha + 1, \frac{5}{3}\alpha + \frac{5}{6}\}$

First result for complex potentials but $\overbrace{\left|\frac{1}{2} - \frac{1}{r}\right| \leq \frac{1}{2n}}^{n \geq 3}, \overbrace{\left|\frac{1}{2} - \frac{1}{r}\right| \leq \frac{1}{3}}^{n=2}$

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q_B : High frequency (HF) approximation (adapted for complex potentials)

Direct problem: For $q \in L^p(\mathbb{R}^n)$, complex, $p > \frac{n}{2}$, there exists k_0 such that $\psi(x, \theta, k)$ exists and is unique, for $k \geq k_0$.

Key: Estimates for R_k (**Limiting absorption principles**) :

Agmon-Hörmander, Kenig-Ruiz-Sogge, Vega-Ruiz and Born Series.
Complex potentials: Mochizuki, Saito, Isozaki, Davies, Greenleaf, Uhlmann.

Neumann-Born series

$$A(\theta', \theta, k) = \int_{\mathbb{R}^n} e^{-ik\theta' \cdot y} q(y) \psi(y, k, \theta) dy.$$

$$(\Delta + q + k^2)\psi = 0; \quad \psi = e^{ikx \cdot \theta} + \psi_{out}$$

$$(-\Delta - k^2)(\psi_{out}) = q\psi, \quad R_+(k^2) = \lim_{\epsilon \rightarrow 0} (-\Delta - k^2 + i\epsilon)^{-1}$$

$$\psi = (I - R_+(k^2)(q \cdot))^{-1}(e^{ikx \cdot \theta}) = e^{ikx \cdot \theta} + \sum_{j=1}^{\infty} (R_+(k^2)(q \cdot))^j(e^{ikx \cdot \theta})$$

$$A(\theta', \theta, k) = \widehat{q}(k(\theta' - \theta)) + \underbrace{\sum_{j=1}^{\infty} \int_{\mathbb{R}^n} e^{-ik\theta' \cdot y} q(y) (R_+(k^2)(q \cdot (\cdot)))^j(e^{ik\theta \cdot}) dy}_{Q_{j+1}}$$

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$$\chi A(\theta', \theta, k) = \chi \widehat{q}(k(\theta' - \theta)) + \underbrace{\sum_{j=1}^{\infty} \chi \int_{\mathbb{R}^n} e^{-ik\theta' \cdot y} q(y) (R_+(k^2)(q \cdot (\cdot)))^j(e^{ik\theta \cdot \cdot})}_{Q_{j+1}}$$

$$q(x) = \int e^{ix\xi} \widehat{q}(\xi) d\xi \approx \int e^{ix\xi} \chi(\xi) \widehat{q}(\xi) d\xi.$$

Change of variable, $\xi = k(\theta' - \theta)$, averaging in θ variable

$$q_B(x) = \int_{S^{n-1} \times (0, \infty) \times S^{n-1}} e^{ikx(\theta' - \theta)} A(\theta', \theta, k) \underbrace{\chi(|\theta' - \theta|^2 k^{n-1}) dk d\theta' d\theta}_{d\mu}$$

$$q - q_B = \sum_{j=2}^{\infty} Q_{j,B}$$

$$Q_{j+1,B} = \int_{S^{n-1} \times (0, \infty) \times S^{n-1} \times \mathbb{R}^n} e^{ikx(\theta' - \theta)} q(y) \chi(R_k(q))^j(y) d\mu dy$$

- $j > 2$ Averaging estimates of Ruiz (2001) for fixed angle
- $j = 2$ (Quadratic Term) \Rightarrow special treatment

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Q_2 as a bilinear Fourier multiplier

$$Q_{2,B} = \int_{S^{n-1} \times (0, \infty) \times S^{n-1} \times \mathbb{R}^n} e^{ikx(\theta' - \theta)} e^{-ik\theta' y} q(y) R_k(q(\cdot) e^{ik\theta(\cdot)}) d\mu dy$$

$$R_k(qe^{ik\theta(\cdot)})(y) = G_k \star f = \int G_k(y - z) q(z) e^{ikz} dz$$

$$F(y, z) = \int_{S^{n-1} \times (0, \infty) \times S^{n-1}} e^{iky\theta'} e^{-ikz\theta} G_k(y, z) d\mu$$

$$\begin{aligned} Q_{2,B}(q)(x) &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} F(x - y, x - z) q(y) q(z) dy dz \\ &= \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{i(\xi + \eta)} m(\xi, \eta) \widehat{q}(\xi) \widehat{q}(\eta) d\xi d\eta \end{aligned}$$

where,

$$m(\xi, \eta) = \frac{C(n)}{|\xi||\eta|} \left(\frac{\xi}{|\xi|} \cdot \frac{\eta}{|\eta|} + i \sqrt{1 - \left(\frac{\xi}{|\xi|} \cdot \frac{\eta}{|\eta|} \right)^2} \right) = Q_{\Re} + iQ_{\Im}$$

The real part

Has product structure!

$Q_{\mathbb{R}}(f, f) \approx \sum_{i=1}^n R_i(l_1(f))R_i(l_2(f))$
 $1 < r < n, \beta < \min\{\alpha + 1 + n(\frac{1}{s} - \frac{1}{r}), 2(1 + \alpha) + n(\frac{1}{s} - \frac{2}{r})\}$. Then,

$$\|Q(f_1, f_2)\|_{W_{loc}^{\beta, s}} \leq C \|f_1\|_{W^{\alpha, r}} \|f_2\|_{W^{\alpha, r}} \quad (1)$$

Properties of Riesz potentials and transforms, Generalized Leibniz Rule, Sobolev embedding..

The imaginary part

$$m_{\Im} = \sqrt{1 - \left(\frac{\xi}{|\xi|} \frac{\eta}{|\eta|}\right)^2} = \left|\sin\left(\frac{\xi}{|\xi|}, \frac{\eta}{|\eta|}\right)\right|.$$

- Bad news. No product structure. It is singular when $\xi \parallel \pm \eta$.
 n dimensional.
- Good news. Rotation invariant, continuous. The singular set is very regular.

The real part

Has product structure!

$Q_{\mathbb{R}}(f, f) \approx \sum_{i=1}^n R_i(l_1(f))R_i(l_2(f))$
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Properties of Riesz potentials and transforms, Generalized Leibniz Rule, Sobolev embedding..

The imaginary part

$$m_{\mathfrak{S}} = \sqrt{1 - \left(\frac{\xi}{|\xi|} \frac{\eta}{|\eta|}\right)^2} = \left| \sin\left(\frac{\xi}{|\xi|}, \frac{\eta}{|\eta|}\right) \right|.$$

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Proof I: Whitney Decomposition

Decompose the phase space in cones (angular Whitney cones with respect to the singular set)

$$\mathbb{R}^n \times \mathbb{R}^n \setminus L = \bigcup_{k=1}^{\infty} \bigcup_{i=1}^{2n2^{k(n-1)}} \bigcup_{j \in r(i;k)} \overbrace{\bar{Q}_{i,k} \times \bar{Q}_{j,k}}^{Q_{i,j,k}}.$$

If $(\xi, \eta) \in Q_{i,j,k}$, then $\angle(\xi, \eta) \sim 2^{-k}$ or -2^{-k} .

$$\mathcal{Q}_{\mathfrak{S}}(q, q)(x) = \sum_{k=1}^{\infty} \sum_{i=1}^{2n2^{k(n-1)}} \sum_{j \in r(i;k)} T_{i,j,k}(q_{i,k}, q_{j,k}),$$

where we define

$$T_{i,j,k}(f, g)(x) = \int \int e^{i\pi x(\xi+\eta)} m(\xi, \eta) \frac{\widehat{f}(\xi)}{|\xi|} \frac{\widehat{g}(\eta)}{|\eta|} \chi_{Q_{i,j,k}} d\xi d\eta$$

and

$$\widehat{q_{i,k}} = \widehat{q} \chi_{Q_{i,k}}, \widehat{q_{j,k}} = \widehat{q} \chi_{Q_{j,k}}.$$

Proof 2: The bricks

2D: In $Q_{i,k} \times Q_{j,k}$, either

$$\left| \sin \left(\frac{\xi}{|\xi|}, \frac{\eta}{|\eta|} \right) \right| = \frac{\xi_1}{|\xi|} \frac{\eta_2}{|\eta|} - \frac{\xi_2}{|\xi|} \frac{\eta_1}{|\eta|}$$

or

$$\left| \sin \left(\frac{\xi}{|\xi|}, \frac{\eta}{|\eta|} \right) \right| = \frac{\xi_2}{|\xi|} \frac{\eta_1}{|\eta|} - \frac{\xi_1}{|\xi|} \frac{\eta_2}{|\eta|}.$$

Moreover, by rotation, we can assume that $\frac{|\xi_1|}{|\xi|} \leq 2^{-k}$ and $\frac{|\eta_1|}{|\eta|} \leq 2^{-k}$.

Higher dimensions: write

$$\sqrt{1 - \left(\frac{\xi}{|\xi|} \frac{\eta}{|\eta|} \right)^2}$$

as an infinite sum of terms with product structure and small L^∞ norm.

Key Proposition

Let $0 < \alpha < \beta < \alpha + 1$, $1 < r < n$, $1 < s < \infty$. Suppose further that $\beta < \min\{\alpha + 1 + n(\frac{1}{s} - \frac{1}{r}), 2(1 + \alpha) + n(\frac{1}{s} - \frac{2}{r})\}$

$$\|T_{i,j,k}(f, g)\|_{W_{loc}^{\beta,s}} \leq C 2^{-k\varpi} \|f\|_{W^{\alpha,r}} \|g\|_{W^{\alpha,r}}, \quad (2)$$

where

$$n=2 \quad \varpi < 1/2 + 1/r'$$

$$n=3 \quad \varpi < \frac{2}{\rho}, \quad \rho = \max\{r, r'\}.$$

Note: Other results for $r > n$ involving L^t .

Proof 3: Angular Littlewood-Paley Decomposition

$$\left\| \sum_i \sum_{j \in r(i;k)} T_{i,j,k}(q_{i,k}, q_{j,k}) \right\|_{W_{loc}^{\beta,s}} \leq C 2^{-k\varpi} \left(\sum_i \|q_{i,k}\|_{W^{\alpha,r}}^2 \right).$$

Let $q_{i,k}$ as above $\alpha \geq 0, 1 < r < \infty$. Then, for all $\epsilon > 0$,

$$\left(\sum_i \|q_{i,k}\|_{W^{\alpha,r}}^2 \right)^{\frac{1}{2}} \leq C_\epsilon 2^{\lambda(r)k + \epsilon k} \|q\|_{W^{\alpha,r}},$$

where, for $r > 2$, $\lambda(r) = (n-1)(\frac{1}{2} - \frac{1}{r})$ and for $r \leq 2$, $\lambda(r) = (n-1)(\frac{1}{r} - \frac{1}{2})$.

$$\|Q_{\mathfrak{S}}(q, q)\|_{W^{\beta,s}} \leq C_\epsilon \|q\|_{W^{\alpha,r}} 2 \sum_k 2^{k(2\lambda - \varpi) + \epsilon}.$$

Theorem (Pure derivative recovery, 2D, 3D Ruiz-V, Reyes)

Let $q \in L^p(\mathbb{R}^n) \cap W^{\alpha,2}(\mathbb{R}^n)$, $p > n/2$, $\alpha \geq 0$. Then $q - q_{back} \in W_{loc}^{\beta,2}$, when $\beta - \alpha < 1/2$.

Neumann-Born series:

$$\chi \widehat{q}(2k\theta) = \chi A(\theta, -\theta, k) - \underbrace{\sum_{j=1}^{\infty} \chi \int_{\mathbb{R}^n} e^{-ik\theta \cdot y} q(y) (R_+(k^2)(q \cdot (\cdot)))^j (e^{ik\theta \cdot}) dy}_{Q_{j+1}}$$

In 2D (Reyes, 2007) $q_3 \in W_{loc}^{\beta,2}$ when $\beta - \alpha < 1$.

Then, by Sobolev embedding $q_3 \in \Lambda^\beta$ if $\beta < \alpha$ (Hölder regularity)

Hölder regularity for the bilinear term in 2D?

In 3D (Ruiz-V, 2005) only $q_3 \in W_{loc}^{\beta,2}$ when $\beta - \alpha < 1/2$.

Quadratic (bilinear) Term

$$\widehat{Q_2(q)}(\eta) = p.v. \int_{\mathbf{R}^n} \frac{\hat{q}(\eta - \xi)\hat{q}(\xi)}{\xi \cdot (\eta - \xi)} d\xi + \frac{i\pi}{|\eta|} \int_{\Gamma(\eta)} \hat{q}(\eta - \xi)\hat{q}(\xi) d\sigma(\xi),$$
$$:= P(q) + iS(q)$$

where $\Gamma(\eta)$ is the sphere centered at $\eta/2$ and radius $|\eta|/2$.

(2005) 2D: If $q \in W^{\alpha,2}$ with $\alpha > 0$ and compactly supported, then $S \in \Lambda^\beta$, $\beta < \alpha$ and P is continuous.

P is reduced to a Hilbert transform on S^1 .

Theorem (Barceló, Faraco, Ruiz and V. 2011)

Let $q \in W^{\alpha,2}$ and such that $|x|q(x) \in L^p$ for some $p < 2$. Then $Q_2(q) \in \Lambda^\beta$ for any $\beta < \alpha$.

- Corollary: For compactly supported q , $q - q_{back} \in \Lambda^\beta$
- Useful for inverse scattering in elasticity system (second order Riesz transforms of the potential involved): limiting absorption principles by Barceló, Folch-Gabayet, Pérez-Esteve, Ruiz and Vilela (2010).
- (2D) gain in integrability as gain of one derivative
- Main difficulties in the proof: the central singular term in P , non compactness of the support.

Estimate for $S(q)$: We consider the spherical term

$$\widehat{Q(q)}(\eta) = \frac{i\pi}{|\eta|} \int_{\Gamma(\eta)} \hat{q}(\eta - \xi) \hat{q}(\xi) d\sigma_{\eta}(\xi) \chi(\eta). \quad (3)$$

For any $0 < \beta < \alpha$ the following a priori estimate holds for any $p \in (1, 2)$

$$\|Q(q)\|_{\Lambda^{\beta}} \leq C(\|q\|_{L^p}^2 + \|q\|_{W^{\alpha,2}}^2).$$

Estimate of the principal value

We split down $\widehat{P(q)}(\eta)$:

$$\widehat{P(q)}(\eta) = \widehat{P_0}(\eta) + \sum_{j=1}^{N(\eta)} \widehat{P}_j(\eta) + \widehat{P_\infty}(\eta),$$

where

$$\widehat{P}_j(\eta) = \int_{\Gamma_j} \frac{\hat{q}(\eta - \xi)\hat{q}(\xi)}{\xi \cdot (\eta - \xi)} d\xi \chi(\eta),$$

$\Gamma_j = \Gamma_j(\eta) = \{\xi \in \mathbb{R}^2 : 2^{-j-2}|\eta| < \|\xi - \eta/2\| - |\eta|/2\| < 2^{-j-1}|\eta|, |\xi| \geq |\eta - \xi|\}$ and $N(\eta) = \log_2 |\eta|$ and

$$\widehat{P_\infty}(\eta) = p.v. \int_{\Gamma_\infty} \frac{\hat{q}(\eta - \xi)\hat{q}(\xi)}{\xi \cdot (\eta - \xi)} d\xi \chi(\eta),$$

for

$\Gamma_\infty = \Gamma_\infty(\eta) = \{\xi \in \mathbb{R}^2 : \|\xi - \eta/2\| - |\eta|/2\| < 1/4, |\xi| \geq |\eta - \xi|\}$.

Estimate of \widehat{P}_∞

Using symmetry w.r.t. the singular set $\Gamma(\eta)$ to get rid of P.V.

Then, by Calderón-Hajlasz estimate

$|f(x) - f(y)| \leq C(M(\nabla f)(x) + M(\nabla f)(y))|x - y|$ reduces to

Lemma

We have the a priori estimates

$$\|\widehat{J(f, g)}\|_{L^1(\langle \cdot \rangle^\beta)} \leq C \|g\|_{W^{\alpha, 2}}^2 (\|\hat{f}\|_{L^{p'}} + \|f\|_{L^2}),$$

for any $p' > 2$, or

$$\|\widehat{J(f, g)}\|_{L^1(\langle \cdot \rangle^\beta)} \leq C \|g\|_{W^{\alpha, 2}}^2 \|\hat{f}\|_{L^{p'}},$$

for p' such that $0 < 1 - \frac{2}{p'} < \alpha - \beta$, where

$$\widehat{J(f, g)}(\eta) = \int_{\Gamma_0^+} \frac{|\hat{f}(\eta - \Phi_\eta(\xi))| |\hat{g}(\xi)|}{|\eta|} d\xi \chi(\eta),$$

- Backscattering 2D: Hölder estimates Q_3 under decay condition of q
- Backscattering 3D: Hölder estimates in quadratic and cubic terms, even compactly supported case (in progress)
- Backscattering general D: iterative procedure to control the general term in the series (done when q is more regular by Lagergren, Beltita and Melin in odd dimension, evolution wave equation ultrahyperbolic operator)
- Partial sets of data between backscattering and full data.