Multiparameter singular (and fractional) Radon transforms

Brian Street
based on joint work with E. M. Stein
The Frobenius Theorem

\( M \) a connected manifold. \( \mathcal{D} \subseteq \Gamma(TM) \) a \( C^\infty \) module of vector fields. Assume

\( \mathcal{D} \) is involutive: \( X, Y \in \mathcal{D} \Rightarrow [X, Y] \in \mathcal{D} \).

\( \mathcal{D} \) is locally finitely generated as a \( C^\infty \) module:
\[ \forall x \in M, \exists U \ni x, X_1, \ldots, X_q \in \mathcal{D}, \forall Y \in \mathcal{D}, Y \big| U = q \sum_{j=1}^{q} c_j X_j \big| U, c_j \in C^\infty. \]

Then, \( \forall x \in M, \exists! \) maximal, connected, injectively immersed, submanifold \( L \hookrightarrow M, x \in L, T_y L = \mathcal{D}_y \).

\( L \) is called a "leaf." We say the manifold is "foliated into leaves."

Note: \( \dim \mathcal{D}_x \) not necessarily constant, e.g. \( \langle x \partial_y \rangle \). Points near which \( \dim \mathcal{D}_x \) are not constant are called singular points.
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- $\mathcal{D}$ is locally finitely generated as a $C^\infty$ module: $\forall x \in M$, $\exists U \ni x$, $X_1, \ldots, X_q \in \mathcal{D}$, $\forall Y \in \mathcal{D}$,

$$Y \bigg|_U = \sum_{j=1}^{q} c_j X_j \bigg|_U, \quad c_j \in C^\infty.$$
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The setting

- Given \( \gamma_t(x) = \gamma(t, x) : \mathbb{R}^N_0 \times \mathbb{R}^n_0 \to \mathbb{R}^n \), \( \gamma_0(x) \equiv x \).
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$$Tf(x) = \psi(x) \int f(\gamma_t(x)) K(t) \, dt.$$
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- $\psi \in C_0^\infty(\mathbb{R}^n)$ supported on a small neighborhood of 0.
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- $K(t)$ is a “singular kernel” supported near $t = 0$. 

If $K$ is a “fractional integral kernel,” we instead want smoothing $L^p \to L^{p,s}$, where $L^{p,s}$ denotes a non-isotropic Sobolev space.
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- Question: Given a class of kernels \( K \), what conditions on \( \gamma \) imply \( T : L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n), 1 < p < \infty? \)
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  - If \( K \) is a “fractional integral kernel,” we instead want smoothing \( L^p \to L^p_s \), where \( L^p_s \) denotes a non-isotropic Sobolev space.
  - First “singular kernels” then “fractional kernels” very near singular kernels.
Single-parameter case

\[ Tf (x) = \psi (x) \int f (\gamma t (x)) K (t) \, dt \]

\( K \) is a Calderón-Zygmund kernel.
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\( K \) is a Calderón-Zygmund kernel.

- \( |\partial_t^\alpha K(t)| \lesssim |t|^{-N-|\alpha|} \).
- “Cancellation condition.”
- E.g., \( K(t) = \frac{\eta(t)}{t}, \eta \in C_0^\infty(\mathbb{R}) \).
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Studied by Christ, Nagel, Stein, and Wainger.
Multiparameter case

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\begin{itemize}
  \item Decompose \( \mathbb{R}^N = \mathbb{R}^{N_1} \times \cdots \times \mathbb{R}^{N_\nu} \).
  \item \( |K(t_1, \ldots, t_\nu)| \lesssim |t_1|^{-N_1} \cdots |t_\nu|^{-N_\nu} \) (with similar estimates for derivatives).
  \item “Cancellation condition.”
  \item E.g., \( K(t_1, \ldots, t_\nu) = K_1(t_1) \otimes \cdots \otimes K_\nu(t_\nu) \), each \( K_\mu \) is a Calderón-Zygmund kernel.
\end{itemize}
Single-parameter case

\[ Tf(x) = \psi(x) \int f(\gamma_t(x)) K(t) \, dt, \quad K \text{ a Calderón-Zygmund kernel} \]
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Special Case:

\[ \gamma_t (x) = e^{\sum_{0 < |\alpha| \leq L} t^\alpha X_\alpha x}, \quad X_\alpha \in C^\infty \text{ vector fields.} \]
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C-N-S-W: \( T : L^p \to L^p \) (\( 1 < p < \infty \)) if \( X_\alpha \) satisfy Hörmander’s condition
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C-N-S-W: \( T : L^p \rightarrow L^p \) (\( 1 < p < \infty \)) if \( X_\alpha \) satisfy Hörmander’s condition: i.e., if

\[ X_\alpha, [X_\alpha, X_\beta], [X_\alpha, [X_\beta, X_\delta]], \ldots \]

span the tangent space at each point.
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span the tangent space at each point.

Rephrase: If the involutive distribution generated by \( \{ X_\alpha \} \) is equal to the entire space of smooth sections of the tangent bundle, then \( T : L^p \rightarrow L^p \).
Enter Frobenius

\[ Tf(x) = \psi(x) \int f \left( e^{\sum_{0<|\alpha|\leq L} t^\alpha x^\alpha} K(t) \right) dt, \quad K \text{ a C-Z kernel} \]

Generalization of C-N-S-W: If the involutive distribution, \( D \), generated by \( \{X_\alpha\} \) is locally finitely generated as a \( C^\infty \) module, then \( T : L^p \to L^p \).
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Generalization of C-N-S-W: If the involutive distribution, \( \mathcal{D} \), generated by \( \{ X_\alpha \} \) is locally finitely generated as a \( C^\infty \) module, then \( T : L^p \to L^p \).

Proof idea: Use \( \mathcal{D} \) to foliate the ambient space into leaves. On each leaf \( \{ X_\alpha \} \) satisfies Hörmander’s condition. Apply C-N-S-W to each leaf.
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Generalization of C-N-S-W: If the involutive distribution, \( D \), generated by \( \{ X_\alpha \} \) is locally finitely generated as a \( C^\infty \) module, then \( T : L^p \to L^p \).

Proof idea: Use \( D \) to foliate the ambient space into leaves. On each leaf \( \{ X_\alpha \} \) satisfies Hörmander’s condition. Apply C-N-S-W to each leaf. More precisely, use the coordinate charts defining the leaves to pull the operator back to each leaf, where C-N-S-W applies. But the classical proofs of the Frobenius theorem don’t give good control on these charts near singular points! The key is a quantitative version of the Frobenius theorem.
Real analyticity

\[ Tf (x) = \psi (x) \int f \left( e^{\sum_{0<|\alpha| \leq L} t^\alpha X_\alpha x} \right) K (t) \, dt, \quad K \text{ a C-Z kernel} \]

Special Case: If \( \{ X_\alpha \} \) are real analytic, then the involutive distribution is automatically finitely generated as a \( C^\infty \) module (Weierstrass preparation) and \( T : L^p \to L^p \).
Now we consider more general $\gamma_t(x)$, $\gamma_0(x) \equiv x$.

$$Tf(x) = \psi(x) \int f(\gamma_t(x)) K(t) \, dt$$

Two conditions on $\gamma$:  

▶ Finite type condition (analogous to the conditions of the Frobenius theorem).
▶ Algebraic condition (vacuous in the single-parameter case).

The finite type condition holds automatically if $\gamma$ is real analytic.
Conditions

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The finite type condition holds automatically if $\gamma$ is real analytic. First: real analytic. Later: $C^\infty$. 
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Tf (x) = \psi (x) \int f (\gamma_t (x)) K(t) \, dt, \quad K \text{ a C-Z kernel}
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Mf (x) = \psi (x) \sup_{0<\delta<1} \int_{|t|\leq 1} |f (\gamma_\delta t (x))| \, dt
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\( T \) and \( M \) are bounded on \( L^p \), \( 1 < p < \infty \).
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$\gamma$ real analytic, $\gamma_0(x) \equiv x$

$$Tf(x) = \psi(x) \int f(\gamma_t(x)) K(t) \, dt, \quad K \text{ a C-Z kernel}$$

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$T$ and $M$ are bounded on $L^p$, $1 < p < \infty$.

Bourgain: $\gamma_t(x) = x + tv(x)$, $v$ a real analytic vector field in $\mathbb{R}^2$
The multiparameter case

\[ K(t_1, \ldots, t_\nu) \text{ a product kernel; e.g.,} \]
\[ K(t_1, \ldots, t_\nu) = K_1(t_1) \otimes \cdots \otimes K_\nu(t_\nu). \]
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\[ Tf(x) = \psi(x) \int f(\gamma t_1, \ldots, t_\nu(x)) K(t_1, \ldots, t_\nu)\,dt_1 \cdots dt_\nu \]

\[ Mf(x) = \psi(x) \sup_{0 < \delta_1, \ldots, \delta_\nu << 1} \int_{|t| \leq 1} |f(\gamma \delta t_1, \ldots, \delta t_\nu(x))|\,dt_1 \cdots dt_\nu \]
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$\mathcal{M}f(x) = \psi(x) \sup_{0<\delta_1, \ldots, \delta_\nu<<1} \int_{|t|\leq 1} |f(\gamma_{\delta_1 t_1, \ldots, \delta_\nu t_\nu}(x))| \, dt_1 \cdots dt_\nu$

$T$ and $\mathcal{M}$ behave differently!
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\( T \) and \( M \) behave differently!

E.g. \( \gamma_{t_1,t_2}(x) = x - t_1 t_2, \, x, t_1, t_2 \in \mathbb{R}. \)
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\[ K(t_1, \ldots, t_\nu) \text{ a product kernel; e.g., } K(t_1, \ldots, t_\nu) = K_1(t_1) \otimes \cdots \otimes K_\nu(t_\nu). \]

\[ \mathcal{T}f(x) = \psi(x) \int f(\gamma_{t_1, \ldots, t_\nu}(x)) K(t_1, \ldots, t_\nu) \, dt_1 \cdots dt_\nu \]

\[ \mathcal{M}f(x) = \psi(x) \sup_{0 < \delta_1, \ldots, \delta_\nu < 1} \int |f(\gamma_{\delta_1 t_1, \ldots, \delta_\nu t_\nu}(x))| \, dt_1 \cdots dt_\nu \]

\( \mathcal{T} \) and \( \mathcal{M} \) behave differently!

E.g. \( \gamma_{t_1, t_2}(x) = x - t_1 t_2, \ x, t_1, t_2 \in \mathbb{R}. \)

Fact: There exist product kernels \( K(t_1, t_2) \) such that \( \mathcal{T} \) is not bounded on \( L^2 \) (Nagel-Wainger)

Fact: \( \mathcal{M} \) is bounded on \( L^p \) (\( 1 < p \leq \infty \)).
The Maximal Result

\( \gamma \) real analytic, \( \gamma_0 (x) \equiv x \)

\[ Mf (x) = \psi (x) \sup_{0<\delta_1,\ldots,\delta_{\nu}<<1} \int_{|t| \leq 1} \left| f \left( \gamma \delta_1 t_1, \ldots, \delta_{\nu} t_{\nu} (x) \right) \right| \, dt_1 \cdots dt_{\nu} \]

\( M \) is bounded on \( L^p \) (\( 1 < p \leq \infty \)).
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\mathcal{M} f (x) = \psi (x) \sup_{0 < \delta_1, \ldots, \delta_\nu << 1} \int_{|t| \leq 1} |f (\gamma_{\delta_1 t_1, \ldots, \delta_\nu t_\nu} (x))| \; dt_1 \cdots dt_\nu
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\( \mathcal{M} \) is bounded on \( L^p \) \((1 < p \leq \infty)\).
Christ: Special case on nilpotent Lie groups.
Toward singular integrals

\[ Tf (x) = \psi (x) \int f (\gamma_{t_1, \ldots, t_\nu} (x)) K (t_1, \ldots, t_\nu) \, dt_1 \cdots dt_\nu \]
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\[ Tf (x) = \psi (x) \int f (\gamma_{t_1, \ldots, t_\nu} (x)) K (t_1, \ldots, t_\nu) \, dt_1 \cdots dt_\nu \]

Special case

\[ \gamma_{t_1, \ldots, t_\nu} (x) = \exp \left( \sum_{0 < |\alpha_1| + \cdots + |\alpha_\nu| \leq L} t_1^{\alpha_1} \cdots t_\nu^{\alpha_\nu} X_{\alpha_1, \ldots, \alpha_\nu} \right) x, \]

\( X_{\alpha_1, \ldots, \alpha_\nu} \) real analytic vector fields.
Toward singular integrals

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\(X_{\alpha_1, \ldots, \alpha_{\nu}}\) real analytic vector fields.
We call \(\alpha = (\alpha_1, \ldots, \alpha_{\nu})\) a \textit{pure power} if \(|\alpha_\mu| \neq 0\) for precisely one \(\mu\). Otherwise, we call it a \textit{non-pure power}. 
The algebraic condition

\[ \gamma_{t_1, \ldots, t_\nu}(x) = \exp \left( \sum_{0 < |\alpha_1| + \cdots + |\alpha_\nu| \leq L} t_1^{\alpha_1} \cdots t_\nu^{\alpha_\nu} X_{\alpha_1, \ldots, \alpha_\nu} \right) x, \]

In this special case the "algebraic condition" on \( \gamma \) becomes

- \( \forall \) non-pure powers \( \beta, X_\beta \in \text{Inv. Dist.} \{ X_\alpha : \alpha \text{ is pure} \} \).
The algebraic condition

\[ \gamma_{t_1,\ldots,t_\nu}(x) = \exp \left( \sum_{0 < |\alpha_1| + \cdots + |\alpha_\nu| \leq L} t_1^{\alpha_1} \cdots t_\nu^{\alpha_\nu} X_{\alpha_1,\ldots,\alpha_\nu} \right) x, \]

In this special case the “algebraic condition” on \( \gamma \) becomes

\begin{itemize}
  \item \( \forall \) non-pure powers \( \beta \), \( X_\beta \in \text{Inv. Dist.} \left\{ X_\alpha : \alpha \text{ is pure} \right\} \).
  \item Scale invariant version of the above. \( \forall \) non-pure \( \beta \), \( \forall \delta \in [0,1]^\nu \),
    \[ \delta_1^{\beta_1} \cdots \delta_\nu^{\beta_\nu} X_\beta \in \text{Inv. Dist.} \left\{ \delta_1^{\alpha_1} \cdots \delta_\nu^{\alpha_\nu} X_\alpha : \alpha \text{ is pure} \right\}, \]
    “uniformly” in \( \delta \).
\end{itemize}

Then, \( T : L^p \to L^p \).
The algebraic condition

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    \item \( \forall \) non-pure powers \( \beta, X_\beta \in \text{Inv. Dist.} \{ X_\alpha : \alpha \text{ is pure} \}. \)
    \item Scale invariant version of the above. \( \forall \) non-pure \( \beta, \forall \delta \in [0,1]_\nu, \)
        \[ \delta_1^{1|\beta_1|} \cdots \delta_\nu^{1|\beta_\nu|} X_\beta \in \text{Inv. Dist.} \{ \delta_1^{1|\alpha_1|} \cdots \delta_\nu^{1|\alpha_\nu|} X_\alpha : \alpha \text{ is pure} \}, \]
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\end{itemize}

Then, \( T : L^p \rightarrow L^p. \)

In the single parameter case, every power is pure, and this condition is vacuous.
The $C^\infty$ case

$\gamma_t (x) \in C^\infty$, $\gamma_0 (x) \equiv x$. 

$Tf (x) = \psi (x) \int f (\gamma_{t_1}, \ldots, t_\nu (x)) K (t_1, \ldots, t_\nu) \ dt_1 \cdots dt_\nu$

$W (t, x) = \frac{d}{d\epsilon} \bigg|_{\epsilon=1} \gamma_{\epsilon t} \circ \gamma_t^{-1} (x) \in T_x \mathbb{R}^n$
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$(\text{Finite type})$ $\mathcal{D}_\delta := \text{Inv. Dist.} \left\{ \delta_1^{\alpha_1} \cdots \delta_\nu^{\alpha_\nu} X_{\alpha_1,\ldots,\alpha_\nu} \right\}$ is locally finitely generated as a $C^\infty$ module, “uniformly in $\delta,”$

and $W(\delta_1 t_1, \ldots, \delta_\nu t_\nu) \in \mathcal{D}_\delta$ “uniformly in $\delta.”$
The $C^\infty$ case

$\gamma_t (x) \in C^\infty$, $\gamma_0 (x) \equiv x.$

$$Tf (x) = \psi (x) \int f (\gamma_{t_1, \ldots, t_\nu} (x)) K (t_1, \ldots, t_\nu) \, dt_1 \cdots dt_\nu$$

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► (Finite type) $D_\delta := \text{Inv. Dist.} \left\{ \delta_{\alpha_1}^{1} \cdots \delta_{\alpha_\nu}^{\nu} X_{\alpha_1, \ldots, \alpha_\nu} \right\}$ is locally finitely generated as a $C^\infty$ module, “uniformly in $\delta$,” and $W (\delta_1 t_1, \ldots, \delta_\nu t_\nu) \in D_\delta$ “uniformly in $\delta$.”

► (Algebraic) “Uniformly in $\delta$,” $\forall$ non-pure $\beta$,

$\delta_1^{\beta_1} \cdots \delta_\nu^{\beta_\nu} X_\beta \in \text{Inv. Dist.} \left\{ \delta_{\alpha_1}^{1} \cdots \delta_{\alpha_\nu}^{\nu} X_\alpha : \alpha \text{ is pure} \right\}.$
Fractional integrals

We assume the same conditions on $\gamma$.

$$Tf (x) = \psi (x) \int f (\gamma_{t_1, \ldots, t_\nu} (x)) K (t_1, \ldots, t_\nu) \, dt_1 \cdots dt_\nu$$

Let $K$ be a fractional integral kernel.
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$$|K (t_1, \ldots, t_\nu)| \lesssim |t_1|^{-N_1+\delta_1} \cdots |t_\nu|^{-N_\nu+\delta_\nu} \quad (\text{with similar estimates for derivatives}).$$
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Define non-isotropic Sobolev spaces $L^p_{s_1, \ldots, s_\nu}$ so that each $X_{\alpha_1, \ldots, \alpha_\nu}$ is a differential operator of “order” $(|\alpha_1|, \ldots, |\alpha_\nu|)$. 
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Theorem: For \( 1 < p < \infty \), \( \exists \delta_0 = \delta_0 (p) \), for all \( \delta \in [0, 1]^\nu \) with \( |\delta| \leq \delta_0 (p) \), \( T : L^p \to L^p_{\delta} \) for all fractional integral kernels of order \( \delta \).
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Past work in the single-parameter case: Cuccagna, Greenblatt