

Variations on the Carleson-Hunt theorem

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Includes joint work with Nazarov, Seeger, Tao, Thiele, Wright

The Carleson-Hunt theorem

$$f : [0, 1) \rightarrow \mathbb{C}$$

$$\hat{f}(k) = \int_0^1 f(x) e^{-2\pi i k x} dx$$

$$S[f](N, x) = \sum_{k=-N}^N \hat{f}(k) e^{2\pi i k x}$$

Theorem (Carleson-Hunt)

If $f \in L^p$, $p > 1$, then $\lim_{N \rightarrow \infty} S[f](N, x)$ converges for almost every x .

Proven using maximal operator bound:

$$\|S[f](N, x)\|_{L_x^p(\ell_N^\infty)} \leq C_p \|f\|_{L^p}$$

Fourier integral version

$$f : \mathbb{R} \rightarrow \mathbb{C}$$

$$\hat{f}(\eta) = \int_{\mathbb{R}} f(x) e^{-2\pi i \eta x} dx$$

$$S[f](\xi, x) = \int_{-\infty}^{\xi} \hat{f}(\eta) e^{2\pi i \eta x} d\eta$$

Theorem (Carleson, Hunt)

Suppose $1 < p < \infty$. Then

$$\|S[f](\xi, x)\|_{L_x^p(L_\xi^\infty)} \leq C_p \|f\|_{L^p}$$

Implies pointwise a.e. continuity of Fourier integrals for $f \in L^p$.

One direction of improvement

Can the L_ξ^∞ norm in

$$\|S[f](\xi, x)\|_{L_x^p(L_\xi^\infty)} \leq C_p \|f\|_{L^p}$$

be replaced by a (morally) stronger norm?

Goal: Illuminate further properties of the functions $S[f](\cdot, x)$

Variation-norms

$$g : \mathbb{R} \rightarrow \mathbb{C}$$

$$1 \leq r \leq \infty$$

r -variation “norm”:

$$\|g\|_{V^r} := \sup_K \sup_{\xi_0 < \dots < \xi_K} \left(\sum_{j=1}^K |g(\xi_j) - g(\xi_{j-1})|^r \right)^{\frac{1}{r}}$$

Note:

- ▶ $\|g\|_{V^r} \geq \|g\|_{L^\infty}$ provided g has a zero
- ▶ $\|g\|_{V^r} \geq \|g\|_{V^s}$ if $r \leq s$

Quantitative convergence info

If $\xi_0 < \xi_1 < \dots < \xi_K$
and $|g(\xi_j) - g(\xi_{j-1})| > \lambda$ for each j

$$\text{then } K \leq \frac{\|g\|_{V_r}^r}{\lambda^r}$$

i.e. $\leq \frac{\|g\|_{V_r}^r}{\lambda^r}$ “ λ -jumps”

In particular:

$\|g\|_{V_r} < \infty \Rightarrow$ For every y $\lim_{x \rightarrow y^\pm} g(x)$ convergent

Context

$$f : \mathbb{R} \rightarrow \mathbb{C} \quad t \in \mathbb{R}^+$$

$$A_t[f](x) := \frac{1}{2t} \int_{-t}^t f(x+t) dt$$

Standard fact: $\|A_t[f](x)\|_{L_x^p(L_t^\infty)} \leq C_p \|f\|_{L^p}, \quad p > 1$

Theorem

Suppose $r > 2, 1 < p < \infty$. Then

$$\|A_t[f](x)\|_{L_x^p(V_t^r)} \leq C_p \|f\|_{L^p}, \quad 1 < p < \infty$$

- ▶ Discrete – Lépingle
- ▶ Continuous – Bourgain, Jones, Kaufman, Rosenblatt, Wierdl
- ▶ $r > 2$ optimal – Qian
- ▶ Weak estimate at $r = 2$

Why?

Lebesgue differentiation theorem:

$f \in L^p \Rightarrow \lim_{t \rightarrow 0} A_t[f](x)$ convergent for x a.e.

Standard proof ($1 < p < \infty$):

Obvious convergence for dense subclass of L^p
+
Maximal operator bound: $\|A_t[f](x)\|_{L^p_x(L^\infty_t)} \leq C\|f\|_{L^p}$
+
Approximation argument

Using variational bound instead of maximal operator bound gives convergence directly without resorting to prior known convergence on dense subclass.

Especially interesting for transfer to ergodic theory

Variation-norm Carleson-Hunt

Theorem (RO, Seeger, Tao, Thiele, Wright)

Suppose $r > 2$ and $r' < p < \infty$. Then

$$\|S[f](\xi, x)\|_{L_x^p(V_\xi^r)} \leq C_p \|f\|_{L^p}.$$

We also obtain a R.W.T. estimate at $p = r'$

- ▶ $r > 2$, $p > r'$ are necessary conditions
- ▶ Also gives corresponding theorem for Fourier series on the circle, via transference

Key: Incorporate Lépingle's bound into a proof of Carleson's theorem.

Original Motivation:

Estimates for 1d Schrödinger/Dirac equations

Transference from linear to nonlinear works for $r < 2$

Linear estimates hold for $r > 2$

Q: Endpoint at $r = 2$???

L^q multiplier norms

$$m : \mathbb{R} \rightarrow \mathbb{C}$$

$$\|m\|_{M^q} = \sup_{\|g\|_{L^q}=1} \|(m\hat{g})^\vee\|_{L^q}$$

Immediate:

$$\|m\|_{M^2} = \|m\|_{L^\infty}$$

Marcinkiewicz:

$$\|m\|_{M^q} \leq C_q \|m\|_{V^1} + \|m\|_{L^\infty} \quad 1 < q < \infty$$

Coifman, Rubio de Francia, and Semmes:

$$\|m\|_{M^q} \leq C_{q,r} (\|m\|_{V^r} + \|m\|_{L^\infty}) \quad \left| \frac{1}{q} - \frac{1}{2} \right| < \frac{1}{r}, \quad r \geq 2$$

C/R/S + V.C.

Corollary

Suppose $|\frac{1}{q} - \frac{1}{2}| < \min(\frac{1}{p'}, \frac{1}{2})$. Then

$$\|S[f](\xi, x)\|_{L_x^p(M_\xi^q)} \leq C_{p,q} \|f\|_{L^p}$$

Note: $(S[f](\cdot, x)\hat{g})^\vee(y) \stackrel{m}{=} \int g(y-t)f(x+t)\frac{1}{t} dt$

- ▶ Connects to H.T. along a vector field depending on one variable
- ▶ Connects to extensions of Return Times Theorem

Truncations of S

$$S[f](\xi, \cdot) \stackrel{m}{=} f * e^{2\pi i \xi x} \frac{1}{x}$$

$$\phi \in C_c^\infty(\mathbb{R})$$

$$S_k[f](\xi, \cdot) = f * e^{2\pi i \xi x} \frac{1}{x} \phi(2^{-k} x)$$

Well-known:

$$\text{C.H.} \Rightarrow \|S_k[f](\xi, x)\|_{L_x^p(L_\xi^\infty(\ell_k^\infty))} \leq C_p \|f\|_{L^p}$$

Same trick:

$$\text{V.C.} \Rightarrow \|S_k[f](\xi, x)\|_{L_x^p(L_\xi^\infty(V_k^{r'}))} \leq C_p \|f\|_{L^p} \quad r > 2, p > r'$$

$$\text{V.C.} \Rightarrow \|S_k[f](\xi, x)\|_{L_x^p(\ell_k^\infty(V_\xi^{r'}))} \leq C_p \|f\|_{L^p} \quad r > 2, p > r'$$

Gives reproof of Wiener-Wintner for H.T.

Theorem (Lacey, Terwilleger)

Suppose (X, μ, T) is a measure preserving system and $f \in L^p(X)$, $p > 1$. Then there is a set $X_f \subset X$ of full measure such that for every $x \in X_f$ and every ξ we have

$$\lim_{N \rightarrow \infty} \sum'_{n=-N}^N \frac{1}{n} f(T^n(x)) e^{i\xi n} \quad \text{convergent}$$

Maximal L^q multiplier norms

$$m_k : \mathbb{Z} \times \mathbb{R} \rightarrow \mathbb{C}$$

$$\|m_k\|_{M^{q,*}} = \sup_{\|g\|_{L^q}=1} \|(m_k \hat{g})^\vee(x)\|_{L_x^q(\ell_k^\infty)}$$

Theorem (Demeter, Lacey, Tao, Thiele)

Suppose $1 < p < \infty$. Then

$$\|S_k[f](\xi, x)\|_{L_x^p(M_{\xi,k}^{2,*})} \leq C_p \|f\|_{L^p}.$$

Also with $\frac{1}{t}$ replaced by more general kernels.

Key: Incorporate strengthened version of “Bourgain type maximal multiplier lemma” into a proof of Carleson’s theorem.

Return times theorem

Corollary

Suppose (X, μ, T) is a m.p.s and $f \in L^p(X)$, $p > 1$. Then there is a set $X_f \subset X$ of full measure such that for every $x \in X_f$ and every second m.p.s. system (Y, ν, S) and $g \in L^q(Y)$, $q \leq 2$

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(T^n(x))g(S^n y) \quad \text{convergent for a.e. } y \in Y.$$

Extends Bourgain's original range of exponents $\frac{1}{p} + \frac{1}{q} \leq 1$

Corollary

Same as above, but

$$\lim_{N \rightarrow \infty} \sum_{n=-N}^N \frac{1}{n} f(T^n(x))g(S^n(y)) \quad \text{convergent for a.e. } y \in Y.$$

Further advances

Can the exponents in R.T. be pushed to the full range $\frac{1}{p} + \frac{1}{q} < \frac{3}{2}$?

Theorem (Demeter)

Yes for averages, provided an appropriate L^p extension of Bourgain type maximal multiplier lemma. Also strong evidence for such lemma.

Key: Avoided having to prove Carleson-type theorem by treating averages differently from Hilbert transforms.

Theorem (Nazarov, RO, Thiele)

The appropriate L^p extension of Bourgain type maximal multiplier lemma holds.

Some remaining questions

Do we have $M^{q,*}$ -Carleson in the full range $\frac{1}{p} + \frac{1}{q} < \frac{3}{2}$?

(M^q -Carleson in this range from V.C. + C.R.S.)

Let

$$\|m_k\|_{M^{q,r}} = \sup_{\|g\|_{L^q}=1} \|(m_k \hat{g})^\vee(x)\|_{L^q_x(V_k^r)}$$

Do we have $M^{q,r}$ Carleson?

The answer to both is yes, at least for the Walsh model of the problem.

Walsh model

Consider the operation \oplus defined on \mathbb{R}^+ as follows: Each digit in the binary expansion of $x \oplus y$ is given by adding the corresponding digits in the binary expansions of x and y modulo 2.

$$f : \mathbb{R}^+ \rightarrow \mathbb{R}$$

\hat{f} = The \oplus -Fourier transform of f .

Miracle: $\hat{1}_{[0,1)} = 1_{[0,1)}$

$$S[f](\xi, x) = (1_{[0,\xi)} \hat{f})^\vee$$

$S_k[f](\xi, x)$ = truncation of wave-packet sum which gives $S[f](\xi, x)$

Then

Theorem (RO)

Suppose $1 < p < \infty$, $q < 2$, $\frac{1}{p} + \frac{1}{q} < \frac{3}{2}$, $r > 2$. Then

$$\|S_k[f](\xi, x)\|_{L_x^p(M_{\xi,k}^{q,r})} \leq C_{p,q,r} \|f\|_{L^p}$$

Along the way, also obtain:

Theorem (RO)

Suppose $1 < p < \infty$, $r > 2$. Then:

$$\|S_k[f](\xi, x)\|_{L_x^p(L_\xi^\infty(V_k^r))} \leq C_{p,r} \|f\|_{L^p}$$