Arithmetic progressions in sumsets and L^p-almost-periodicity

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Joint work with Ernie Croot and Olof Sisask
Edinburgh, June 2011

Arithmetic progressions in sumsets

Let A, B be finite subsets of $\mathbb{Z}$, $A + B = \{a + b : \ a \in A, b \in B\}$.
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Bourgain 1990: Sumsets contain long arithmetic progressions

Let $A, B \subset \{1, \ldots, N\}$, $|A| = \alpha N$, $|B| = \beta N$, then $A + B$ contains an arithmetic progression of length at least

$$\exp(c(\alpha\beta \log N)^{1/3} - C \log\log N).$$

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Proof: Almost periodicity, via Fourier analysis. More details later.

Arithmetic progressions in sumsets - continued

Green 2002: improvement of Bourgain's exponent

As above, but the length of the progression is at least

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Best to date if $|A| \approx |B|$. Proof: Fourier analysis again ("hereditary non-uniformity").

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The result is nontrivial if

$$\alpha\beta > C\frac{(\log\log N)^2}{\log N}.$$

In particular, both α and β need to be greater than $C(\log\log N)^2/\log N$.

Arithmetic progressions in sumsets - continued

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Croot-Sisask 2010: almost periodicity via probabilistic sampling

Progressions of length at least

$$\frac{1}{2}\exp\left(c\left(\frac{\alpha\log N}{\log(4/\beta)}\right)^{1/4}\right).$$

Better than Green if β very small (suffices if $\beta \geq \exp(-(\log N)^c)$.

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Croot-Łaba-Sisask 2011

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- Proofs use almost periodicity and Fourier analysis.

A little notation

- Fourier transform on $\mathbb{Z}_N$:

$$\widehat{f}(\xi) = \frac{1}{N} \sum_{x=0}^{N-1} f(x) e^{-2\pi i \xi x / N}$$

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- Useful formulas:

$$f(x) = \sum_{\xi=0}^{N-1}\widehat{f}(\xi)e^{2\pi i\xi x/N}, \;\; \widehat{f * g} = \widehat{f}\widehat{g}$$

L^p norms

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- For $\Gamma \subset \mathbb{Z}_N$, $\delta > 0$, define

$$\mathrm{Bohr}(\Gamma, \delta) = \{x : \ |e^{-2\pi i \xi x/N} - 1| \leq \delta, \ \text{all } \xi \in \Gamma\}.$$

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- Bohr sets contain arithmetic progressions of length at least $c\delta N^{1/d}$.

Almost periodicity: general framework

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- This follows easily if we can prove that f is *almost periodic* with periods in T, in the sense that $\|f(\cdot + t) - f(\cdot)\|_p$ is small for some $p < \infty$ and all $t \in T$.

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- The main issue is to prove the almost periodicity.

Almost periodicity: Bourgain's argument

Bourgain 1990.

Let $0 < \epsilon < 1$, $p \geq 2$, and let $f = \mathbf{1}_A * \mathbf{1}_B$. Then $\exists$ Bohr set T with

$$d \leq Cp^2\epsilon^{-2}\log(1/\epsilon), \rho = c\epsilon^2/p$$

such that for all $t \in T$,

$$\|f(x+t) - f(x)\|_{L^p(x)} \leq \epsilon\|\widehat{f}\|_1 = \epsilon\sqrt{\alpha\beta}.$$

(The last equality is by Parseval.)

Bourgain's argument

- $f(x) = \sum_{\xi=0}^{N-1} \widehat{f}(\xi) e^{2\pi i \xi x/N} =: f_1 + f_2 + f_3$, according to size of Fourier coefficients (small, medium, large)

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- The almost periodic part (corresponding to large Fourier coefficients) determines the Bohr set T. In fact, if

$$\Gamma = \{\xi : \ |\widehat{f}(\xi)| \geq c\},$$

then we can take $T = \mathrm{Bohr}(\Gamma, \rho)$ (for appropriate $c, \rho > 0$).

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  then we can take $T = \text{Bohr}(\Gamma, \rho)$ (for appropriate $c, \rho > 0$).
- The medium and small coefficients contribute negligible errors.

Almost periodicity: the probabilistic approach

Croot-Sisask 2010.

Let $f = \mathbf{1}_A * \mathbf{1}_B / |B|$. Then $\exists$ a set T (not necessarily a Bohr set) of size $|T| \geq (c\beta)^{Cp/\epsilon^2}$ such that for $t \in T$,

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- Simple probabilistic proof uses random sampling.
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- Works also for non-abelian groups.
- A key part of Sanders's proof of Roth's theorem with density $c(\log\log N)^5 / \log N$.

Long APs in $A + B$, B small

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- Use $kT = T + \cdots + T$ instead. Almost periodicity with periods in T implies almost periodicity with periods in kT, by iteration (with worse constants). But kT has more structure, in particular contains long APs.

Long APs in $A+B$, B small

Croot-Łaba-Sisask 2011

- Exponent $1/4$ improved to $1/2$:

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- Uses an idea from Sanders's paper: almost periodicity with periods in $kT - kT$ can in fact be bootstrapped to almost periodicity with period in a Bohr set. (The rank estimate comes from a theorem of Chang.)

A new proof of Green's result

Croot-Łaba-Sisask 2011

Revisit Bourgain's approach via almost periodicity, but with better estimates on the size of the Bohr set T of periods, therefore on the length of the AP contained in it. This recovers Green's result, with a much simpler proof.

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The almost periodicity result

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- Let $\gamma(x)$ random variable, $\gamma(x) = e^{2\pi i \xi x/N}$ with probability $\widehat{f}(\xi)$ (hence $\mathbb{E}\gamma = f$).

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- Let $\gamma(x)$ random variable, $\gamma(x) = e^{2\pi i \xi x/N}$ with probability $\widehat{f}(\xi)$ (hence $\mathbb{E}\gamma = f$).
- $g = (\gamma_1 + \cdots + \gamma_k)/k$, γ_j are iid copies of γ.

Proof of almost periodicity via Fourier sampling

Marcinkiewicz-Zygmund inequality

$$
\begin{aligned}
\mathbb{E}|g(x) - f(x)|^p &\leq \frac{(Cp)^{p/2}}{k^{p/2}} \mathbb{E}\Big(\frac{1}{k}\sum_j |\gamma_j(x) - f(x)|^2\Big)^{p/2} \\
&\leq \frac{(Cp)^{p/2}}{k^{p/2}} \mathbb{E}|\gamma(x) - f(x)|^p \\
&\leq \frac{(Cp)^{p/2} 2^p}{k^{p/2}} \\
&\leq C\epsilon^p \text{ if } k = \lfloor cp/\epsilon^2 \rfloor .
\end{aligned}
$$

Thank you!