Arithmetic progressions in sumsets and $L^p$-almost-periodicity

Izabella Łaba

Joint work with Ernie Croot and Olof Sisask
Edinburgh, June 2011
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**Bourgain 1990: Sumsets contain long arithmetic progressions**

Let $A, B \subset \{1, \ldots, N\}$, $|A| = \alpha N$, $|B| = \beta N$, then $A + B$ contains an arithmetic progression of length at least

$$\exp(c(\alpha \beta \log N)^{1/3} - C \log \log N).$$
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As above, but the length of the progression is at least

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Best to date if \(|A| \approx |B|\). Proof: Fourier analysis again ("hereditary non-uniformity").
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The result is nontrivial if

$$\alpha \beta > C \frac{(\log \log N)^2}{\log N}.$$ 

In particular, both $\alpha$ and $\beta$ need to be greater than $C(\log \log N)^2 / \log N$. 

Sanders 2008: alternative proof of Green’s result
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Croot-Sisask 2010: almost periodicity via probabilistic sampling
Progressions of length at least

\[ \frac{1}{2} \exp \left( c \left( \frac{\alpha \log N}{\log(4/\beta)} \right)^{1/4} \right). \]

Better than Green if \( \beta \) very small (suffices if \( \beta \geq \exp(-(\log N)^c) \)).
Main result

Croot-Łaba-Sisask 2011
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- A much simpler proof of Green’s result.

Again, suffices if $\beta \geq \exp(-\log N c)$. Proofs use almost periodicity and Fourier analysis.
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\[ \frac{1}{2} \exp \left( c \left( \frac{\alpha \log N}{\log^3(2/\beta)} \right)^{1/2} - \log(\beta^{-1} \log N) \right). \]

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- Proofs use almost periodicity and Fourier analysis.
A little notation

- Fourier transform on $\mathbb{Z}_N$:

$$\hat{f}(\xi) = \frac{1}{N} \sum_{x=0}^{N-1} f(x) e^{-2\pi i \xi x / N}$$

- Convolution:

$$f \ast g(x) = \sum_{y=0}^{N-1} f(x-y) g(y)$$

- Useful formulas:

$$f(x) = \sum_{\xi=0}^{N-1} \hat{f}(\xi) e^{2\pi i \xi x / N}, \quad \hat{f} \ast \hat{g}$$
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Bohr sets

For $\Gamma \subset \mathbb{Z}_N$, $\delta > 0$, define

$$\text{Bohr}(\Gamma, \delta) = \{x : |e^{-2\pi i \xi x/N} - 1| \leq \delta, \text{ all } \xi \in \Gamma\}.$$ 

($\delta$ - radius, $d = |\Gamma|$ - rank of Bohr set.)
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- Can be thought of as an approximate orthogonal complement of $\Gamma$.
- Bohr sets have size at least $(C\delta)^d N$.
- Bohr sets contain arithmetic progressions of length at least $c\delta N^{1/d}$. 
Almost periodicity: general framework

General framework

- Let $f = 1_A * 1_B$. Then $f$ is supported on $A + B$, so it suffices to prove that $\text{supp } f$ contains a long AP.

Bohr sets contain long APs, hence it suffices to prove that $\text{supp } f$ contains a large enough Bohr set $T$.

This follows easily if we can prove that $f$ is almost periodic with periods in $T$, in the sense that $\| f(\cdot + t) - f(\cdot) \|_p$ is small for some $p < \infty$ and all $t \in T$.

The main issue is to prove the almost periodicity.
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- The main issue is to prove the almost periodicity.
Almost periodicity: Bourgain's argument

Let $0 < \epsilon < 1$, $p \geq 2$, and let $f = 1_A * 1_B$. Then there exists a Bohr set $T$ with

$$d \leq Cp^2 \epsilon^{-2} \log(1/\epsilon), \quad \rho = c \epsilon^2 / p$$

such that for all $t \in T$,

$$\|f(x + t) - f(x)\|_{L^p(x)} \leq \epsilon \|\hat{f}\|_1 = \epsilon \sqrt{\alpha \beta}.$$  

(The last equality is by Parseval.)
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Bourgain’s argument

\[ f(x) = \sum_{\xi = 0}^{N-1} \hat{f}(\xi) e^{2\pi i \xi x / N} =: f_1 + f_2 + f_3, \]

according to size of Fourier coefficients (small, medium, large).
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The almost periodic part (corresponding to large Fourier coefficients) determines the Bohr set \( T \). In fact, if

\[ \Gamma = \{ \xi : |\hat{f}(\xi)| \geq c \}, \]

then we can take \( T = \text{Bohr}(\Gamma, \rho) \) (for appropriate \( c, \rho > 0 \)).
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- \( f(x) = \sum_{\xi=0}^{N-1} \hat{f}(\xi)e^{2\pi i \xi x/N} =: f_1 + f_2 + f_3 \), according to size of Fourier coefficients (small, medium, large)
- The almost periodic part (corresponding to large Fourier coefficients) determines the Bohr set \( T \). In fact, if
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  \Gamma = \{ \xi : |\hat{f}(\xi)| \geq c \},
  \]
  then we can take \( T = \text{Bohr}(\Gamma, \rho) \) (for appropriate \( c, \rho > 0 \)).
- The medium and small coefficients contribute negligible errors.
Almost periodicity: the probabilistic approach

Croot-Sisask 2010.
Let \( f = 1_A * 1_B / |B| \). Then \( \exists \) a set \( T \) (not necessarily a Bohr set) of size \( |T| \geq (c\beta)^{Cp}/\epsilon^2 \) such that for \( t \in T \),
\[
\| f(x + t) - f(x) \|_{L^p(x)} \leq \epsilon \| f \|_{L^p(x)} = \epsilon \alpha^{1/p}.
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- Works also for non-abelian groups.
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- Simple probabilistic proof uses random sampling.
- Better than Bourgain’s estimate if $\beta$ is small.
- Works also for non-abelian groups.
- A key part of Sanders’s proof of Roth’s theorem with density $c(\log \log N)^5 / \log N$. 
Long APs in $A + B$, $B$ small

Croot-Sisask 2010

$A + B$ contains progressions of length at least

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- Use $kT = T + \cdots + T$ instead. Almost periodicity with periods in $T$ implies almost periodicity with periods in $kT$, by iteration (with worse constants). But $kT$ has more structure, in particular contains long APs.
Croot–Łaba–Sisask 2011

- Exponent $1/4$ improved to $1/2$:

$$\frac{1}{2} \exp \left( c \left( \alpha \log N \log^3 \left( \frac{2}{\beta} \right)^{1/2} \right) - \log \left( \beta^{-1} \log N \right) \right).$$
Croot-Łaba-Sisask 2011

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\]

- Uses an idea from Sanders’s paper: almost periodicity with periods in \( kT - kT \) can in fact be bootstrapped to almost periodicity with period in a Bohr set. (The rank estimate comes from a theorem of Chang.)
A new proof of Green’s result

Croot-Łaba-Sisask 2011
Revisit Bourgain’s approach via almost periodicity, but with better estimates on the size of the Bohr set \( T \) of periods, therefore on the length of the AP contained in it. This recovers Green’s result, with a much simpler proof.
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The almost periodicity result
Let $f = 1_A \ast 1_B /$. Then $\exists$ a Bohr set $T$ of rank $d \leq Cp/\epsilon^2$, radius $\delta = c\epsilon$ such that for $t \in T,$

$$\|f(x + t) - f(x)\|_{L^p(x)} \leq \epsilon \| \hat{f} \|_1 = \epsilon \sqrt{\alpha \beta}.$$
Proof of almost periodicity via Fourier sampling

Simple probabilistic proof uses random sampling in Fourier space.

\[ f(x) = \sum_{\xi} \hat{f}(\xi) e^{2\pi i \xi x / N} \]

Assume for simplicity that \( \hat{f}(\xi) \geq 0 \), \( \|\hat{f}\|_1 = 1 \).

Let \( \gamma(x) \) random variable, \( \gamma(x) = e^{2\pi i \xi x / N} \) with probability \( \hat{f}(\xi) \) (hence \( E\gamma = f \)).

\[ g = (\gamma_1 + \cdots + \gamma_k) / k, \gamma_j \text{ are iid copies of } \gamma. \]
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- \( g = (\gamma_1 + \cdots + \gamma_k) / k \), \( \gamma_j \) are iid copies of \( \gamma \).
Marcinkiewicz-Zygmund inequality

\[ \mathbb{E}|g(x) - f(x)|^p \leq \frac{(Cp)^{p/2}}{kp^{p/2}} \mathbb{E}\left( \frac{1}{k} \sum_j |\gamma_j(x) - f(x)|^2 \right)^{p/2} \]

\[ \leq \frac{(Cp)^{p/2}}{kp^{p/2}} \mathbb{E}|\gamma(x) - f(x)|^p \]

\[ \leq \frac{(Cp)^{p/2} 2^p}{kp^{p/2}} \]

\[ \leq C \varepsilon^p \text{ if } k = \left\lceil cp/\varepsilon^2 \right\rceil. \]
Thank you!