A fractal analog of the regular value theorem

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Regular Value Theorem

Let $X$ be a smooth manifold of dimension $n$, $Y$ a smooth manifold of dimension $m < n$, and $\phi : X \to Y$ a smooth mapping.

Question: Can we say something about a fractal dimension of $\{x \in E : \phi(x) = y\}$ if $E$ is a set of a given Hausdorff dimension?
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- Question: Can we say something about a fractal dimension of

$$\{ x \in \mathcal{E} : \phi(x) = y \}$$

if $\mathcal{E}$ is a set of a given Hausdorff dimension?
Some definitions:

- Given $E, F \subset \mathbb{R}^d, d \geq 2$, $\phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^m$, define

$$S_t^\phi(E \times F) = \{(x, y) \in E \times F : \phi(x, y) = t\}.$$
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  $$S_t^\phi (E \times F) = \{(x, y) \in E \times F : \phi(x, y) = t\}.$$ 

- Recall that $E \subset \mathbb{R}^d$ is said to be Ahlfors-David regular if there exists a Borel measure $\mu$, supported on $E$ and $C > 0$ such that for all $x \in E$,

  $$C^{-1} \delta^\alpha \leq \mu(B_\delta(x)) \leq C \delta^\alpha$$

  for every $\delta > 0$, where $\alpha$ is the Hausdorff dimension of $E$ and $B_\delta(x)$ is the ball of radius $\delta$ centered at $x$. 
Generalized Radon transform (GRT):

- Given \( f : \mathbb{R}^d \to \mathbb{R} \), define the generalized Radon transform associated to \( \phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^m \) by

\[
T_{\phi, t} f(x) = \int_{\{ y : \phi(x, y) = t \}} f(y) \psi(x, y) d\sigma_{x, t}(y),
\]

where \( d\sigma_{x, t} \) is the Lebesgue measure on the set \( \{ y : \phi(x, y) = t \} \) and \( \psi \) is a smooth cut-off function.
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- Our first result establishes a dimensional inequality purely in terms of \( L^2 \)-Sobolev bounds of such operators.
Theorem (Eswarathasan, A.I., Taylor)

Let $E, F \subset \mathbb{R}^d$, $d \geq 2$, be compact and Ahlfors-David regular. Suppose that

$$T_{\phi, t} : L^2(\mathbb{R}^d) \rightarrow L^2_s(\mathbb{R}^d)$$

with constants uniform in $t \in T$, for some $s > 0$, and assume that

$$\frac{\dim_{\mathcal{H}}(E) + \dim_{\mathcal{H}}(F)}{2} > d - s.$$ 

Then for $t \in T$,

$$\overline{\dim_{\mathcal{M}}(S_t^\phi(E \times F))} \leq \dim_{\mathcal{H}}(E) + \dim_{\mathcal{H}}(F) - m.$$ 

- We now consider a concrete application of this results which allows us to establish connections between the problem at hand and several related problems in geometric measure theory and geometric combinatorics.
Phong-Stein Condition

Definition

\( \phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \) satisfies the Phong-Stein rotational curvature condition at \( t \) if

\[
\det \begin{pmatrix}
0 & \nabla_x \phi \\
-(\nabla_y \phi)^T & \frac{\partial^2 \phi}{dx_i dy_j}
\end{pmatrix} \neq 0
\]
on the set \( \{(x, y) \in B \times B : \phi(x, y) = t\} \).

- Recall that the Phong-Stein condition guarantees that

\[
T_{\phi, t} : L^2(\mathbb{R}^d) \to L^{\frac{2}{d-1}}(\mathbb{R}^d).
\]
Corollary (Eswarathasan, A.I., Taylor)

Suppose that \( \phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \) satisfies the Phong-Stein rotational curvature condition at \( t \). Then

\[
\dim_{\mathcal{M}}(S_t^\phi(E \times F)) \leq \dim_{\mathcal{H}}(E) + \dim_{\mathcal{H}}(F) - 1
\]

holds under the assumption that

\[
\dim_{\mathcal{H}}(E) + \dim_{\mathcal{H}}(F) > d + 1.
\]

Moreover, the result is sharp in the sense that there exist sets \( E, F \) with \( \dim_{\mathcal{H}}(E) + \dim_{\mathcal{H}}(F) < d + 1 \), and a function \( \phi \) satisfying the Phong-Stein rotational curvature assumption such that

\[
\dim_{\mathcal{M}}(S_t^\phi(E \times F)) > \dim_{\mathcal{H}}(E) + \dim_{\mathcal{H}}(F) - 1
\]
The following estimate is the key to the proof:

\[
\mu \times \mu \{ (x, y) \in E \times E : t \leq \phi(x, y) \leq t + \varepsilon \} \\
\lesssim \epsilon \int \int |x - y|^{-s} d\mu(x) d\mu(y) \text{ if } s > \frac{d + 1}{2}.
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This estimate can be viewed as a continuous analog of the Szemeredi-Trotter incidence theorem in geometric combinatorics. This theorem says that the number of incidences between \( n \) points and \( n \) lines (or, more generally, \( n \) curves satisfying some intersection hypotheses) in the plane is \( \leq Cn^{\frac{4}{3}} \).
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An explicit link between discrete and continuous incidence theorems was explored by A.I., H. Jorati and I. Laba in ”Geometric Incidence Theorems via Fourier Analysis”.

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Set $\phi(x, y) = ||x - y||_B$, where $|| \cdot ||_B$ is the norm induced by $B$, a symmetric convex body with smooth boundary and non-vanishing Gaussian curvature obtained by glueing the upper and the lower hemisphere of the paraboloid glued together smoothly.
Sharpness Example

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- Our plan is to construct a set $E$ with $\dim_H(E) = s < \frac{d+1}{2}$ by scaling and thickening the integer lattice.

In the context of averages of Fourier transforms of measures, this example was previously used by Barcelo, Bennett, Carbery, Ruiz and Viela. In the context of discrete incidence theorems, this example was previously used by Valtr.
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We then show that the dimensional inequality fails for $s < \frac{d+1}{2}$ due to the fact that the paraboloid $\{x \in [-1, 1]^d : x_d = x_1^2 + \cdots + x_{d-1}^2\}$, dilated by $R^{\frac{d}{d+1}}$ in the first $d-1$ coordinates and by $R^{\frac{2d}{d+1}}$ in the remaining coordinate, contains $\approx R^{d-2+\frac{2}{d+1}}$ lattice points.
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Let us first recall the Erdős unit distance conjecture.
Combinatorial motivation: Valtr’s construction

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- The number of incidences ((points, circle): point ∈ circle) between $N$ points and $N$ circles in the plane is $\leq CN \log(N)$. 

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- Recall that Szekely’s version of the Szemeredi-Trotter incidence theorem says that the number of incidences between the points and parabolas above is \(\leq CN^{\frac{4}{3}}\).
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- The single distance conjecture would instantly imply the Erdős distance conjecture, recently proved by Larry Guth and Nets Katz.

- It was conjectured for quite a while that the unit distance conjecture is still valid if the Euclidean distance is replaced by a metric generated by a norm defined by centrally symmetric convex set with a strictly convex boundary. As we shall see in a moment, this is not true.
Valtr’s construction (continued)

- Consider an $n$ by $n^2$ integer grid. Translate the parabola given by the equation $y = x^2$ by every point in the grid. This gives us a family of $N = n^3$ points and $N$ parabolas.
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Most parabolas are incident to \( \geq \frac{n}{2} \) points, so the total number of incidences is \( \geq cn^4 = cN^{\frac{4}{3}} \).
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Let \( \mathcal{P} \) denote the integer lattice points inside \( 3R\Gamma \). Let \( \mathcal{C} \) denote the collection of curves obtained by translating \( R\Gamma \) by every lattice point inside.
Let $\Gamma$ be a closed convex curve in the plane with origin in its interior.

Let $\mathcal{P}$ denote the integer lattice points inside $3R\Gamma$. Let $\mathcal{L}$ denote the collection of curves obtained by translating $R\Gamma$ by every lattice point inside.

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Let $\Gamma$ be a closed convex curve in the plane with origin in its interior.

Let $P$ denote the integer lattice points inside $3R\Gamma$. Let $L$ denote the collection of curves obtained by translating $R\Gamma$ by every lattice point inside.

It is not difficult to see that $|P| \approx |L| \approx R^2$.

By Szekely’s version of the Szemeredi-Trotter incidence theorem, the number of incidences between $P$ and $L$ is $\leq CR^8_3$. It follows that

$$\#\{R\Gamma \cap \mathbb{Z}^2\} \leq C \frac{R^8_3}{R^2} = CR^2_3.$$
Define

\[ \mu_q(x) = q^{-d} q^{\frac{d^2}{s}} \sum_{a \in \mathbb{Z}^d} \prod_{j=1}^{d} \psi_0 \left( \frac{a_j}{q^{\alpha_j}} \right) \psi_0 \left( q^{\frac{d}{s}} \left( x_j - \frac{a_j}{q^{\alpha_j}} \right) \right), \]

where \( \psi_0 \) is a smooth cut-off,

\[ \alpha_j = \frac{d}{d+1}, 1 \leq j \leq d - 1 \text{ and } \alpha_d = \frac{2d}{d+1}. \]
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Let \( E_q \) denote the support of \( \mu_q \). A calculation shows that

\[ \int \int \left| x - y \right|^{-s} d\mu_q(x) d\mu_q(y) \approx 1 \]

and the set \( E = \cap_i E_{q_i} \) is an Ahlfors-David regular set of dimension \( s \) if \( q_i = 2 \) and \( q_{i+1} > q_i \).
Given $x \in E_q$, the number of balls of radius $q^{-\frac{d}{s}}$ needed to cover
\[ \{ y \in E_q : \|x - y\|_B = 1 \} \] is \( \gtrsim q^{\frac{d(d-1)}{d+1}} \). It follows that the number of balls of radius $q^{-\frac{d}{s}}$ needed to cover $S^\phi_t(E_q, E_q)$ is

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\gtrsim q^d \cdot q^{\frac{d(d-1)}{d+1}} = q^{\frac{2d^2}{d+1}} = (q^{\frac{d}{s}})^{\frac{2ds}{d+1}}.
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It follows that the upper Minkowski dimension of $S^\phi_t(E, E)$ is at least
$\frac{2ds}{d+1}$. This number is greater than $2s - 1$ if $s < \frac{d+1}{2}$. 
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It follows that the upper Minkowski dimension of $S^\phi_t (E, E)$ is at least\[ \frac{2ds}{d+1}. \] This number is greater than $2s - 1$ if $s < \frac{d+1}{2}$.

If the sphere is used instead of the paraboloid in this setup, one only
gets sharpness for $s < \frac{d}{2}$. This is because
\[ \#\{RS^{d-1} \cap \mathbb{Z}^d\} \leq C_\epsilon R^{d-2+\epsilon}. \]
Key Estimate → Conclusion:

\[ \epsilon^{\dim_H(E) + \dim_H(F)} \epsilon^{-\dim_M(S^\phi_t(E \times F))} \leq \mu^E \times \mu^F \left\{ (S^\phi_t(E \times F))^\epsilon \right\} \]
Key Estimate \rightarrow \text{Conclusion:}

\[ \epsilon \text{dim}_H(E) + \text{dim}_H(F) \leq \text{dim}_M(S_t^\phi(E \times F)) \leq \mu^E \times \mu^F \left\{ \left( S_t^\phi(E \times F) \right)^\epsilon \right\} \]

\[ \leq \mu^F \times \mu^F \{(x, y) \in E^\epsilon \times F^\epsilon : t - c_d \epsilon \leq \phi(x, y) \leq t + c_d \epsilon \} \lesssim \epsilon. \]
Key Estimate → Conclusion:

\[ \epsilon^{\dim_{\mathcal{H}}(E) + \dim_{\mathcal{H}}(F)} e^{-\dim_{\mathcal{M}}(S_t^\phi(E \times F))} \lesssim \mu^E \times \mu^F \left\{ \left( S_t^\phi(E \times F) \right)^\epsilon \right\} \]

\[ \leq \mu^F \times \mu^F \left\{ (x, y) \in E^\epsilon \times F^\epsilon : t - c_d \epsilon \leq \phi(x, y) \leq t + c_d \epsilon \right\} \lesssim \epsilon. \]

\[ \dim_{\mathcal{M}}(S_t^\phi(E \times F)) \leq \dim_{\mathcal{H}}(E) + \dim_{\mathcal{H}}(F) - 1. \]
Sketch of the Proof

Key Estimate:

\[ \mu^E \times \mu^F \{(x, y) \in E \times E : t \leq \phi(x, y) \leq t + \varepsilon \} \lesssim \varepsilon \]
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LHS = \sum_{j,k} \int \int_{\{y : t \leq \phi(x, y) \leq t + \varepsilon\}} \psi(x, y) d\mu_k^E(y) d\mu_j^F(x)
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\[
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Key Estimate:

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\[ LHS = \sum_{j,k} \int \int_{\{y : t \leq \phi(x,y) \leq t + \varepsilon\}} \psi(x, y) d\mu^E_k(y) d\mu^F_j(x) \]

\[ = \sum_{j,k} \int_t^{t+\varepsilon} \langle \mu^E_j, T_{\phi,r}(\mu^F_k) \rangle dr \]

\[ \lesssim \varepsilon \sum_j \|\mu^E_j\|_2 \|T_{\phi,r}(\mu^F_j)\|_2 \quad \text{(using orthogonality)} \]
Sketch of the Proof (continued)

\[ \lesssim \sum_j 2^{-j \frac{d-1}{2}} \cdot \| \mu_j^E \|_2 \cdot \| \mu_j^F \|_2 \text{ (using Sobolev bounds for GRT)} \]
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\sum_j 2^{-j\frac{d-1}{2}} \cdot \|\mu_j^E\|_2 \cdot \|\mu_j^F\|_2 \quad (\text{using Sobolev bounds for GRT})
\]

\[
\sum_j 2^{-j\frac{d-1}{2}} 2^j \left( d - \frac{\dim_H(E) + \dim_H(F)}{2} \right) \quad (\text{using dimensional assumptions}).
\]
Sketch of the Proof (continued)

\[ \sim \sum_j 2^{-j \frac{d-1}{2}} \cdot \| \mu^E_j \|_2 \cdot \| \mu^F_j \|_2 \text{ (using Sobolev bounds for GRT)} \]

\[ \sim \sum_j 2^{-j \frac{d-1}{2}} 2^j \left( d - \frac{\text{dim}_\mathcal{H}(E) + \text{dim}_\mathcal{H}(F)}{2} \right) \text{ (using dimensional assumptions)} . \]

The geometric series converges if

\[ \text{dim}_\mathcal{H}(E) + \text{dim}_\mathcal{H}(F) > d + 1 \text{ as claimed.} \]
Concluding questions and remarks

- Is there a better positive result in this context for the sphere than the one for the paraboloid?
Concluding questions and remarks

- Is there a better positive result in this context for the sphere than the one for the paraboloid?

- If one wishes to study $k$-tuples of points with $l$ relations between then, one is naturally led to $l$-linear analogs of generalized Radon transforms. It would be nice to develop such a theory.