A fractal analog of the regular value theorem

Alex Iosevich

University of Rochester, New York

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- Question: Can we say something about a fractal dimension of

$$\{x \in \mathcal{E} : \phi(x) = y\}$$

  if $\mathcal{E}$ is a set of a given Hausdorff dimension?

Some definitions:

- Given $E, F \subset \mathbb{R}^d$, $d \geq 2$, $\phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^m$, define

$$S_t^{\phi}(E \times F) = \{(x, y) \in E \times F : \phi(x, y) = t\}.$$

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- Recall that $E \subset \mathbb{R}^d$ is said to be Ahlfors-David regular if there exists a Borel measure μ, supported on E and $C > 0$ such that for all $x \in E$,

$$C^{-1}\delta^{\alpha} \leq \mu(B_{\delta}(x)) \leq C\delta^{\alpha}$$

for every $\delta > 0$, where α is the Hausdorff dimension of E and $B_{\delta}(x)$ is the ball of radius δ centered at x.

Generalized Radon transform (GRT):

- Given $f : \mathbb{R}^d \to \mathbb{R}$, define the generalized Radon transform associated to $\phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^m$ by

$$T_{\phi,t}f(x) = \int_{\{y:\phi(x,y)=t\}} f(y)\psi(x,y)d\sigma_{x,t}(y),$$

where $d\sigma_{x,t}$ is the Lebesgue measure on the set $\{y : \phi(x,y) = t\}$ and ψ is a smooth cut-off function.

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- Our first result establishes a dimensional inequality purely in terms of L^2-Sobolev bounds of such operators.

Theorem (Eswarathasan, A.I., Taylor)

Let $E, F \subset \mathbb{R}^d$, $d \geq 2$, be compact and Ahlfors-David regular. Suppose that

$$T_{\phi,t} : L^2(\mathbb{R}^d) \to L^2_s(\mathbb{R}^d)$$

with constants uniform in $t \in T$, for some $s > 0$, and assume that

$$\frac{dim_{\mathcal{H}}(E) + dim_{\mathcal{H}}(F)}{2} > d - s.$$

Then for $t \in T$,

$$\overline{dim}_{\mathcal{M}}(S_t^{\phi}(E \times F)) \leq dim_{\mathcal{H}}(E) + \dim_{\mathcal{H}}(F) - m.$$

- We now consider a concrete application of this results which allows us to establish connections between the problem at hand and several related problems in geometric measure theory and geometric combinatorics.

Phong-Stein Condition

Definition

$\phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ satisfies the Phong-Stein rotational curvature condition at t if

$$det \begin{pmatrix} 0 & \nabla_x \phi \\ -(\nabla_y \phi)^T & \frac{\partial^2 \phi}{dx_i dy_j} \end{pmatrix} \neq 0$$

on the set $\{(x,y) \in B \times B : \phi(x,y) = t\}$.

- Recall that the Phong-Stein condition guarantees that

$$T_{\phi,t} : L^2(\mathbb{R}^d) \to L^2_{\frac{d-1}{2}}(\mathbb{R}^d).$$

Phong-Stein Corollary

Corollary (Eswarathasan, A.I., Taylor)

Suppose that $\phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ *satisfies the Phong-Stein rotational curvature condition at* t. *Then*

$$\overline{dim}_{\mathcal{M}}(S_t^{\phi}(E \times F)) \leq dim_{\mathcal{H}}(E) + dim_{\mathcal{H}}(F) - 1$$

holds under the assumption that

$$dim_{\mathcal{H}}(E) + dim_{\mathcal{H}}(F) > d + 1.$$

Moreover, the result is **sharp** *in the sense that there exist sets* E, F *with* $dim_{\mathcal{H}}(E) + dim_{\mathcal{H}}(F) < d + 1$, *and a function* ϕ *satisfying the Phong-Stein rotational curvature assumption such that*

$$\overline{dim}_{\mathcal{M}}(S_t^{\phi}(E \times F)) > dim_{\mathcal{H}}(E) + dim_{\mathcal{H}}(F) - 1$$

Key Estimate

- The following estimate is the key to the proof:

$$\mu \times \mu\{(x, y) \in E \times E : t \leq \phi(x, y) \leq t + \varepsilon\}$$

$$\lesssim \epsilon \int \int |x - y|^{-s} d\mu(x) d\mu(y) \text{ if } s > \frac{d+1}{2}.$$

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- This estimate can be viewed as a continuous analog of the Szemeredi-Trotter incidence theorem in geometric combinatorics. This theorem says that the number of incidences between n points and n lines (or, more generally, n curves satisfying some intersection hypotheses) in the plane is $\leq Cn^{\frac{4}{3}}$.

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- An explicit link between discrete and continuous incidence theorems was explored by A.I., H. Jorati and I. Laba in "Geometric Incidence Theorems via Fourier Analysis".

Sharpness Example

- Set $\phi(x,y) = ||x-y||_B$, where $||\cdot||_B$ is the norm induced by B, a symmetric convex body with smooth boundary and non-vanishing Gaussian curvature obtained by glueing the upper and the lower hemisphere of the paraboloid glued together smoothly.

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- Our plan is to construct a set E with $\dim_{\mathcal{H}}(E) = s < \frac{d+1}{2}$ by scaling and thickening the integer lattice.
- We then show that the dimensional inequality fails for $s < \frac{d+1}{2}$ due to the fact that the paraboloid $\{x \in [-1,1]^d : x_d = x_1^2 + \cdots + x_{d-1}^2\}$, dilated by $R^{\frac{d}{d+1}}$ in the first $d-1$ coordinates and by $R^{\frac{2d}{d+1}}$ in the remaining coordinate, contains $\approx R^{d-2+\frac{2}{d+1}}$ lattice points.

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- In the context of averages of Fourier transforms of measures, this example was previously used by Barcelo, Bennett, Carbery, Ruiz and Viela. In the context of discrete incidence theorems, this example was previously used by Valtr.

Combinatorial motivation: Valtr's construction

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- Recall that Szekely's version of the Szemeredi-Trotter incidence theorem says that the number of incidences between the points and parabolas above is $\leq CN^{\frac{4}{3}}$.

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- The single distance conjecture would instantly imply the Erdős distance conjecture, recently proved by Larry Guth and Nets Katz.
- It was conjectured for quite a while that the unit distance conjecture is still valid if the Euclidean distance is replaced by a metric generated by a norm defined by centrally symmetric convex set with a strictly convex boundary. As we shall see in a moment, this is not true.

Valtr's construction (continued)

- Consider an n by n^2 integer grid. Translate the parabola given by the equation $y = x^2$ by every point in the grid. This gives us a family of $N = n^3$ points and N parabolas.

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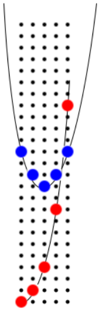

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- Let $\mathcal{P}$ denote the integer lattice points inside $3R\Gamma$. Let $\mathcal{L}$ denote the collection of curves obtained by translating $R\Gamma$ by every lattice point inside.
- It is not difficult to see that $|\mathcal{P}| \approx |\mathcal{L}| \approx R^2$.
- By Szekely's version of the Szemeredi-Trotter incidence theorem, the number of incidences between $\mathcal{P}$ and $\mathcal{L}$ is $\leq CR^{\frac{8}{3}}$. It follows that

$$\#\{R\Gamma \cap \mathbb{Z}^2\} \leq C\frac{R^{\frac{8}{3}}}{R^2} = CR^{\frac{2}{3}}.$$

Sharpness Example: continued

- Define

$$\mu_q(x) = q^{-d} q^{\frac{d^2}{s}} \sum_{a \in \mathbb{Z}^d} \prod_{j=1}^{d} \psi_0\left(\frac{a_j}{q^{\alpha_j}}\right) \psi_0\left(q^{\frac{d}{s}}\left(x_j - \frac{a_j}{q^{\alpha_j}}\right)\right),$$

where ψ_0 is a smooth cut-off,

$$\alpha_j = \frac{d}{d+1}, 1 \leq j \leq d-1 \text{ and } \alpha_d = \frac{2d}{d+1}.$$

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- Let E_q denote the support of μ_q. A calculation shows that

$$\int\int |x-y|^{-s} d\mu_q(x) d\mu_q(y) \approx 1$$

and the set $E = \cap_i E_{q_i}$ is an Ahlfors-David regular set of dimension s if $q_i = 2$ and $q_{i+1} > q_i^i$.

Sharpness Example: The End

- Given $x \in E_q$, the number of balls of radius $q^{-\frac{d}{s}}$ needed to cover $\{y \in E_q : ||x-y||_B = 1\}$ is $\gtrsim q^{\frac{d(d-1)}{d+1}}$. It follows that the number of balls of radius $q^{-\frac{d}{s}}$ needed to cover $S_t^{\phi}(E_q, E_q)$ is

$$\gtrsim q^d \cdot q^{\frac{d(d-1)}{d+1}} = q^{\frac{2d^2}{d+1}} = \left(q^{-\frac{d}{s}}\right)^{-\frac{2ds}{d+1}}.$$

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- It follows that the upper Minkowski dimension of $S_t^{\phi}(E, E)$ is at least $\frac{2ds}{d+1}$. This number is greater than $2s-1$ if $s < \frac{d+1}{2}$.

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- It follows that the upper Minkowski dimension of $S_t^{\phi}(E, E)$ is at least $\frac{2ds}{d+1}$. This number is greater than $2s - 1$ if $s < \frac{d+1}{2}$.

- If the sphere is used instead of the paraboloid in this setup, one only gets sharpness for $s < \frac{d}{2}$. This is because

$$\#\{RS^{d-1} \cap \mathbb{Z}^d\} \leq C_\epsilon R^{d-2+\epsilon}.$$

Key Estimate → Conclusion:

- $$\epsilon^{dim_{\mathcal{H}}(E)+dim_{\mathcal{H}}(F)}\epsilon^{-\overline{dim}_{\mathcal{M}}(S_t^{\phi}(E\times F))} \lessapprox \mu^E \times \mu^F \left\{ \left(S_t^{\phi}(E \times F) \right)^{\epsilon} \right\}$$

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- $$\leq \mu^F \times \mu^F \{(x,y) \in E^{\epsilon} \times F^{\epsilon} : t - c_d\epsilon \leq \phi(x,y) \leq t + c_d\epsilon\} \lesssim \epsilon.$$

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- It follows that
$$\overline{dim}_{\mathcal{M}}(S_t^{\phi}(E \times F)) \leq dim_{\mathcal{H}}(E) + dim_{\mathcal{H}}(F) - 1.$$

Sketch of the Proof

- Key Estimate:

$$\mu^E \times \mu^F\{(x,y) \in E \times E : t \leq \phi(x,y) \leq t+\varepsilon\} \lesssim \varepsilon$$

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$$= \sum_{j,k} \int_t^{t+\epsilon} < \mu_j^E, T_{\phi,r}(\mu_k^F) > dr$$

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- $$\lesssim \epsilon \sum_j ||\mu_j^E||_2 ||T_{\phi,r}(\mu_j^F)||_2 \text{ (using orthogonality)}$$

Sketch of the Proof (continued)

- $$\lesssim \sum_j 2^{-j\frac{d-1}{2}} \cdot ||\mu_j^E||_2 \cdot ||\mu_j^F||_2 \text{ (using Sobolev bounds for GRT)}$$

Sketch of the Proof (continued)

- $$\lesssim \sum_j 2^{-j\frac{d-1}{2}} \cdot ||\mu_j^E||_2 \cdot ||\mu_j^F||_2 \text{ (using Sobolev bounds for GRT)}$$

- $$\lesssim \sum_j 2^{-j\frac{d-1}{2}} 2^{j\left(d-\frac{dim_{\mathcal{H}}(E)+dim_{\mathcal{H}}(F)}{2}\right)} \text{ (using dimensional assumptions).}$$

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- The geometric series converges if

$$dim_{\mathcal{H}}(E) + dim_{\mathcal{H}}(F) > d + 1 \text{ as claimed.}$$

Concluding questions and remarks

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- If one wishes to study k-tuples of points with l relations between then, one is naturally led to l-linear analogs of generalized Radon transforms. It would be nice to develop such a theory.